Technische Universität München
Fakultät für Informatik
Lehrstuhl für Effiziente Algorithmen
Prof. Dr. Ernst W. Mayr
Chris Pinkau

Summer Term 2015
Problem Set 04
May 4, 2015

Complexity Theory

Due date: May 11, 2015 **before** *class!*

Problem 1 (10 Points)

Prove the following two claims.

1. $\mathcal{P} \subseteq \mathcal{NP} \cap \text{co}\mathcal{NP}$.
2. If $\mathcal{P} = \mathcal{NP}$ then $\mathcal{NP} = \text{co}\mathcal{NP}$.

Problem 2 (10 Points)

Recall a *Cook reduction* (i.e., the kind of reduction Stephen Cook used in his original paper to prove that SAT is $\mathcal{NP}$-complete): A language A is Cook reducible to a language B if there is a polynomial-time algorithm that can decide membership in A by using an oracle for B. An oracle is a subroutine that can decide membership in B in $\mathcal{O}(1)$ time.
Show that the language

$$\textsc{Sat} = \{\varphi : \varphi \text{ is a satisfiable boolean formula}\}$$

is Cook reducible to the language

$$\textsc{Tautology} = \{\varphi : \varphi \text{ is a tautology, i.e., every truth assignment satisfies it}\}.$$

Problem 3 (10 Points)

Consider a graph $G = (V, E)$. Recall the following definitions from the lecture:

- A *clique* is defined as a subset $V' \subseteq V$ of vertices such that the induced subgraph of V' is complete, i.e. all vertices in V' are pairwise connected with edges.
  Let $\textsc{Clique} = \{(G, k) : \text{the graph } G \text{ has a clique of } k \text{ vertices}\}$.
- An *independent set* is defined as a subset $V' \subseteq V$ of vertices such that no two vertices of V' are connected by an edge.
  Let $\textsc{Indset} = \{(G, k) : \text{the graph } G \text{ has an independent set of } k \text{ vertices}\}$.

Show the following:

1. $\textsc{Indset} \preceq_m^p \textsc{Clique}$,
2. $\textsc{Clique} \preceq_m^p \textsc{Indset}$,
3. $3\textsc{Sat} \preceq_m^p \textsc{Clique}$, and give the reduction,
4. $\textsc{Clique}$ is $\mathcal{NP}$-complete.

Problem 4 (10 Points)

Consider the problem of *map coloring*: Can you color a map with k different colors, such that no pair of adjacent countries has the same color?

1. Describe the map coloring problem as a proper graph problem and redefine the language k-Colorability = {Maps that are colorable with at most k colors}.

2. Show that 2-Colorability is in $\mathcal{P}$.

3. Show that 3-Colorability is $\mathcal{NP}$-complete.