Relaxation for Set Cover

Primal:

 $\begin{array}{c|c} \min & \sum_{i \in I} w_i x_i \\ \text{s.t. } \forall u & \sum_{i: u \in S_i} x_i \ge 1 \\ & x_i \ge 0 \end{array}$

Dual:

$$\begin{array}{c|c}
\max & \sum_{u \in U} \mathcal{Y}_{u} \\
\text{s.t. } \forall i & \sum_{u:u \in S_{i}} \mathcal{Y}_{u} \leq w_{i} \\
\mathcal{Y}_{u} \geq 0
\end{array}$$



Rounding Algorithm:

Let I denote the index set of sets for which the dual constraint is tight. This means for all $i \in I$

$$\sum_{u:u\in S_i} y_u = w_i$$



Lemma 3

The resulting index set is an f-approximation.

Proof:

Every $u \in U$ is covered.

- Suppose there is a *u* that is not covered.
- This means $\sum_{u:u\in S_i} y_u < w_i$ for all sets S_i that contain u.
- But then y_u could be increased in the dual solution without violating any constraint. This is a contradiction to the fact that the dual solution is optimal.



Proof:

$$\sum_{i \in I} w_i = \sum_{i \in I} \sum_{u: u \in S_i} y_u$$
$$= \sum_u |\{i \in I : u \in S_i\}| \cdot y_u$$
$$\leq \sum_u f_u y_u$$
$$\leq f \sum_u y_u$$
$$\leq f \operatorname{cost}(x^*)$$
$$\leq f \cdot \operatorname{OPT}$$



Let I denote the solution obtained by the first rounding algorithm and I' be the solution returned by the second algorithm. Then

 $I\subseteq I'$.

This means I' is never better than I.

- Suppose that we take S_i in the first algorithm. I.e., $i \in I$.
- This means $x_i \ge \frac{1}{f}$.
- Because of Complementary Slackness Conditions the corresponding constraint in the dual must be tight.
- ► Hence, the second algorithm will also choose *S*_{*i*}.

