Repetition: Primal Dual for Set Cover

Primal Relaxation:

$$\begin{array}{llrcl}
\min & & \sum_{i=1}^{k} w_i x_i & & \\
\text{s.t.} & \forall u \in U & \sum_{i:u\in S_i} x_i & \geq & 1 \\
 & \forall i \in \{1,\dots,k\} & x_i & \geq & 0
\end{array}$$

Dual Formulation:

$$\begin{array}{llrcl}
\max & & \sum_{u\in U} y_u & & \\
\text{s.t.} & \forall i \in \{1,\dots,k\} & \sum_{u:u\in S_i} y_u & \leq & w_i \\
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If we would also fulfill dual slackness conditions

$$y_e > 0 \Rightarrow \sum_{j : e \in S_j} x_j = 1$$

then the solution would be optimal!!!

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This is sufficient to show that the solution is an f-approximation.

Suppose we have a primal/dual pair

$$\begin{array}{llrcl} \min & & \sum_j c_j x_j & & \\ \text{s.t.} & \forall i & \sum_{j:} a_{ij} x_j & \geq & b_i \\ & \forall j & x_j & \geq & 0 \end{array}$$

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and solutions that fulfill approximate slackness conditions:

$$x_j > 0 \Rightarrow \sum_i a_{ij} y_i \geq \frac{1}{\alpha} c_j$$

$$y_i > 0 \Rightarrow \sum_j a_{ij} x_j \leq \beta b_i$$

Then

$$\sum_j c_j x_j$$

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$$\underbrace{\boxed{\sum_j c_j x_j}}_{\text{primal cost}}$$

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right hand side of j-th dual constraint

$$\boxed{\sum_j \boxed{c_j} x_j}$$

primal cost

Then

$$\underset{\overset{\uparrow}{\boxed{\text{primal cost}}}}{\boxed{\sum_j c_j x_j}} \le \alpha \sum_j \left(\sum_i a_{ij} y_i \right) x_j$$

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dual objective

Feedback Vertex Set for Undirected Graphs

- Given a graph $G = (V, E)$ and non-negative weights $w_v \geq 0$ for vertex $v \in V$.

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- Given a graph $G = (V, E)$ and non-negative weights $w_v \geq 0$ for vertex $v \in V$.
- Choose a minimum cost subset of vertices s.t. every cycle contains at least one vertex.

We can encode this as an instance of Set Cover

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- Each vertex can be viewed as a set that contains some cycles.
- However, this encoding gives a Set Cover instance of non-polynomial size.
- The $O(\log n)$-approximation for Set Cover does not help us to get a good solution.

Let C denote the set of all cycles (where a cycle is identified by its set of vertices)

Let $\mathcal{C}$ denote the set of all cycles (where a cycle is identified by its set of vertices)

Primal Relaxation:

$$\begin{array}{llrcl} \min & & \sum_v w_v x_v & & \\ \text{s.t.} & \forall C \in \mathcal{C} & \sum_{v \in C} x_v & \geq & 1 \\ & \forall v & x_v & \geq & 0 \end{array}$$

Dual Formulation:

$$\begin{array}{llrcl} \max & & \sum_{C \in \mathcal{C}} y_C & & \\ \text{s.t.} & \forall v \in V & \sum_{C : v \in C} y_C & \leq & w_v \\ & \forall C & y_C & \geq & 0 \end{array}$$

If we perform the previous dual technique for Set Cover we get the following:

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- Start with $x = 0$ and $y = 0$
- While there is a cycle C that is not covered (does not contain a chosen vertex).
    - Increase y_C until dual constraint for some vertex v becomes tight.
    - set $x_v = 1$.

Then

$$\sum_v w_v x_v$$

Then

$$\sum_v w_v x_v = \sum_v \sum_{C:v\in C} y_C x_v$$

Then

$$\begin{aligned}\sum_v w_v x_v &= \sum_v \sum_{C:v\in C} y_C x_v \\ &= \sum_{v\in S} \sum_{C:v\in C} y_C\end{aligned}$$

where S is the set of vertices we choose.

Then

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where S is the set of vertices we choose.

If every cycle is short we get a good approximation ratio, but this is unrealistic.

Algorithm 1 FeedbackVertexSet

```
y ← 0
x ← 0
while exists cycle $C$ in $G$ do
    increase $y_C$ until there is $v \in C$ s.t. $\sum_{C:v\in C} y_C = w_v$
    $x_v = 1$
    remove $v$ from $G$
    repeatedly remove vertices of degree 1 from $G$
```

Idea:

Always choose a short cycle that is not covered. If we always find a cycle of length at most α we get an α-approximation.

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Observation:

For any path P of vertices of degree 2 in G the algorithm chooses at most one vertex from P.

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If we always choose a cycle for which the number of vertices of degree at least 3 is at most α we get a 2α-approximation.

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Theorem 2

In any graph with no vertices of degree 1, there always exists a cycle that has at most $\mathcal{O}(\log n)$ *vertices of degree 3 or more. We can find such a cycle in linear time.*

This means we have

$$y_C > 0 \Rightarrow |S \cap C| \le \mathcal{O}(\log n) \enspace .$$

Primal Dual for Shortest Path

Given a graph $G = (V, E)$ with two nodes $s, t \in V$ and edge-weights $c : E \to \mathbb{R}^+$ find a shortest path between s and t w.r.t. edge-weights c.

$$\begin{array}{llrcl} \min & & \sum_e c(e)x_e & & \\ \text{s.t.} & \forall S \in S & \sum_{e:\delta(S)} x_e & \geq & 1 \\ & \forall e \in E & x_e & \in & \{0,1\} \end{array}$$

Here $\delta(S)$ denotes the set of edges with exactly one end-point in S, and $S = \{S \subseteq V : s \in S, t \notin S\}$.

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Primal Dual for Shortest Path

The Dual:

$$\begin{array}{llrcl} \max & & \sum_S y_S & & \\ \text{s.t.} & \forall e \in E & \sum_{S: e \in \delta(S)} y_S & \leq & c(e) \\ & \forall S \in \mathcal{S} & y_S & \geq & 0 \end{array}$$

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Primal Dual for Shortest Path

We can interpret the value y_S as the width of a moat surounding the set S.

Each set can have its own moat but all moats must be disjoint.

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Algorithm 1 PrimalDualShortestPath

```
$y \leftarrow 0$
$F \leftarrow \emptyset$
while there is no $s$-$t$ path in $(V, F)$ do
    Let $C$ be the connected component of $(V, F)$ containing $s$
    Increase $y_C$ until there is an edge $e' \in \delta(C)$ such that $\sum_{S: e' \in \delta(S)} y_S = c(e')$.
    $F \leftarrow F \cup \{e'\}$
Let $P$ be an $s$-$t$ path in $(V, F)$
return $P$
```

Lemma 3

At each point in time the set F forms a tree.

Proof:

- In each iteration we take the current connected component from (V, F) that contains s (call this component C) and add some edge from $\delta(C)$ to F.
- Since, at most one end-point of the new edge is in C the edge cannot close a cycle.

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$$\sum_{e\in P} c(e)$$

$$\sum_{e\in P} c(e) = \sum_{e\in P} \sum_{S:e\in\delta(S)} y_S$$

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Hence, we find a shortest path.

If S contains two edges from P then there must exist a subpath P' of P that starts and ends with a vertex from S (and all interior vertices are not in S).

When we increased y_S, S was a connected component of the set of edges F' that we had chosen till this point.

$F' \cup P'$ contains a cycle. Hence, also the final set of edges contains a cycle.

This is a contradiction.

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Steiner Forest Problem:

Given a graph $G = (V, E)$, together with source-target pairs $s_i, t_i, i = 1, \dots, k$, and a cost function $c : E \to \mathbb{R}^+$ on the edges. Find a subset $F \subseteq E$ of the edges such that for every $i \in \{1, \dots, k\}$ there is a path between s_i and t_i only using edges in F.

$$\begin{array}{llrcl} \min & & \sum_e c(e)x_e & & \\ \text{s.t.} & \forall S \subseteq V : S \in S_i \text{ for some } i & \sum_{e \in \delta(S)} x_e & \geq & 1 \\ & \forall e \in E & x_e & \in & \{0,1\} \end{array}$$

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$$\begin{array}{llrcl} \max & & \sum_{S\,:\,\exists i \text{ s.t. } S \in \mathcal{S}_i} y_S & & \\ \textbf{s.t.} & \forall e \in E & \sum_{S: e\in\delta(S)} y_S & \leq & c(e) \\ & & y_S & \geq & 0 \end{array}$$

The difference to the dual of the shortest path problem is that we have many more variables (sets for which we can generate a moat of non-zero width).

Algorithm 1 FirstTry

```
$y \leftarrow 0$
$F \leftarrow \emptyset$
while not all $s_i$-$t_i$ pairs connected in $F$ do
    Let $C$ be some connected component of $(V, F)$ such that $|C \cap \{s_i, t_i\}| = 1$ for some $i$.
    Increase $y_C$ until there is an edge $e' \in \delta(C)$ s.t. $\sum_{S \in S_i : e' \in \delta(S)} y_S = c_{e'}$
    $F \leftarrow F \cup \{e'\}$
return $\bigcup_i P_i$
```

$$\sum_{e \in F} c(e)$$

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If we show that $y_S > 0$ implies that $|\delta(S)\cap F| \le \alpha$ we are in good shape.

However, this is not true:

- Take a complete graph on $k+1$ vertices $v_0, v_1, \ldots, v_k$.

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- The i-th pair is v_0-v_i.

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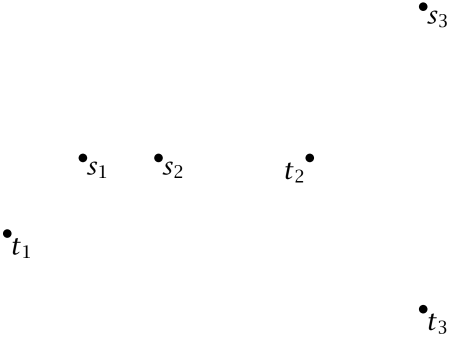
- We only set $y_{\{v_0\}} = 1$. All other dual variables stay 0.
- The final set F contains all edges $\{v_0, v_i\}$, $i = 1, \ldots, k$.
- $y_{\{v_0\}} > 0$ but $|\delta(\{v_0\})\cap F| = k$.

Algorithm 1 SecondTry

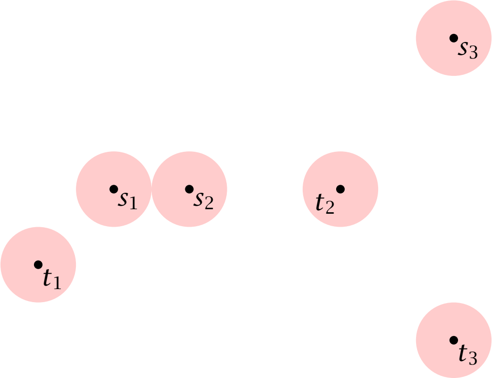
```
$y \leftarrow 0$; $F \leftarrow \emptyset$; $\ell \leftarrow 0$
while not all $s_i$-$t_i$ pairs connected in $F$ do
    $\ell \leftarrow \ell + 1$
    Let $\mathcal{C}$ be set of all connected components $C$ of $(V, F)$ such that $|C \cap \{s_i, t_i\}| = 1$ for some $i$.
    Increase $y_C$ for all $C \in \mathcal{C}$ uniformly until for some edge $e_\ell \in \delta(C')$, $C' \in \mathcal{C}$ s.t. $\sum_{S: e_\ell \in \delta(S)} y_S = c_{e_\ell}$
    $F \leftarrow F \cup \{e_\ell\}$
$F' \leftarrow F$
for $k \leftarrow \ell$ downto 1 do // reverse deletion
    if $F' - e_k$ is feasible solution then
        remove $e_k$ from $F'$
return $F'$
```

The reverse deletion step is not strictly necessary this way. It would also be sufficient to simply delete all unnecessary edges in any order.

Example



Example



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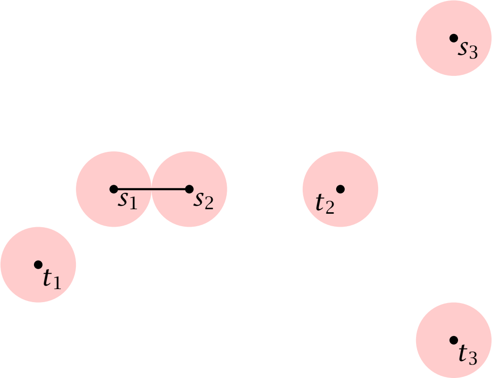

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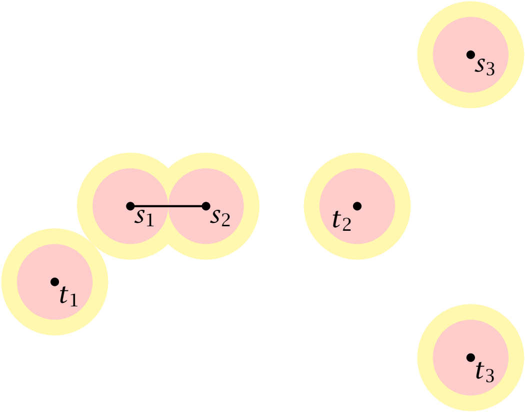

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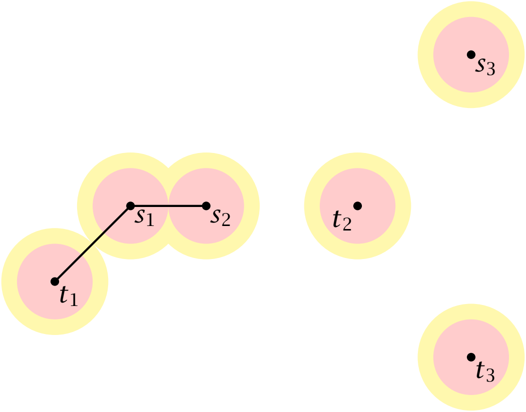

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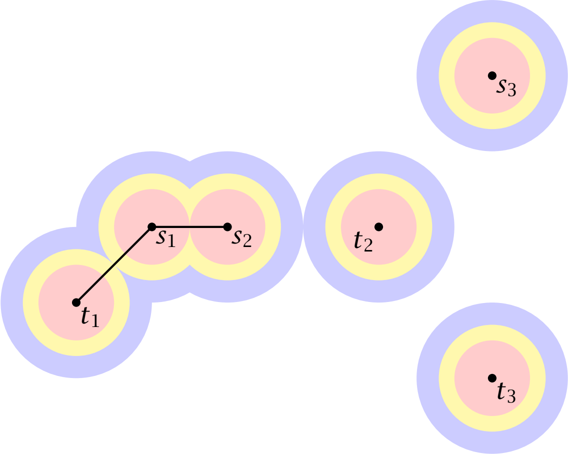

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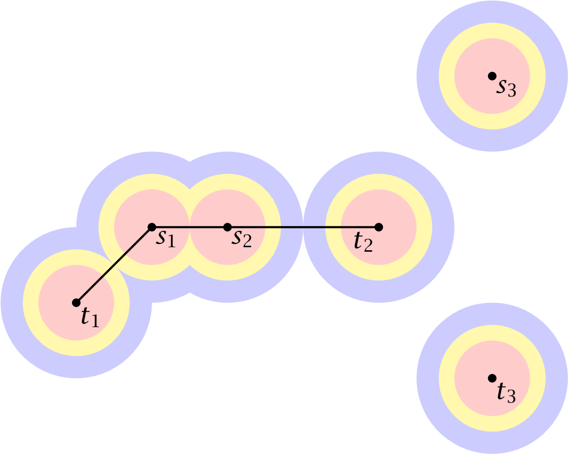

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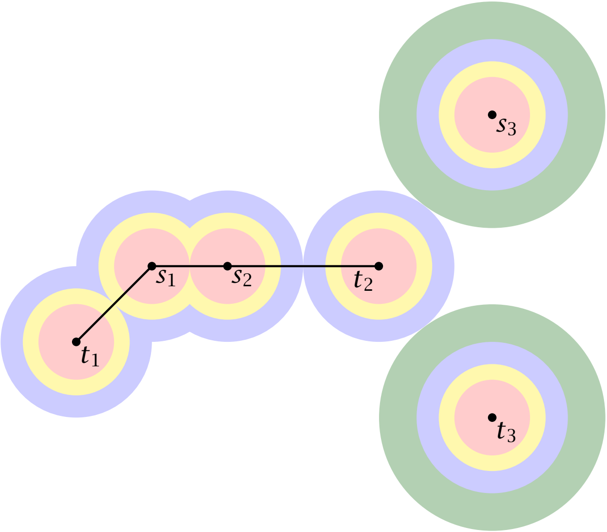

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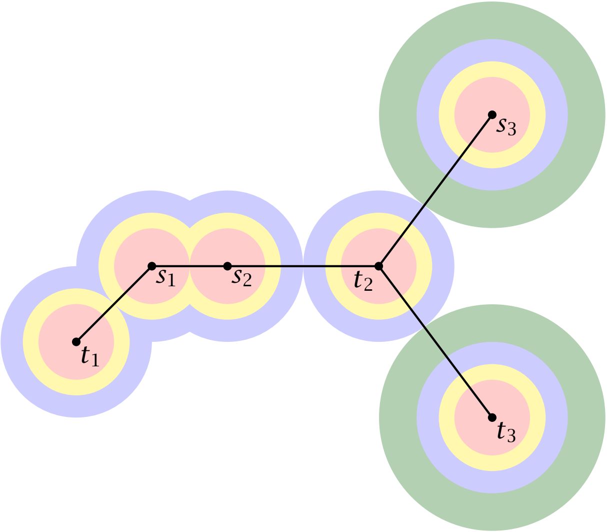

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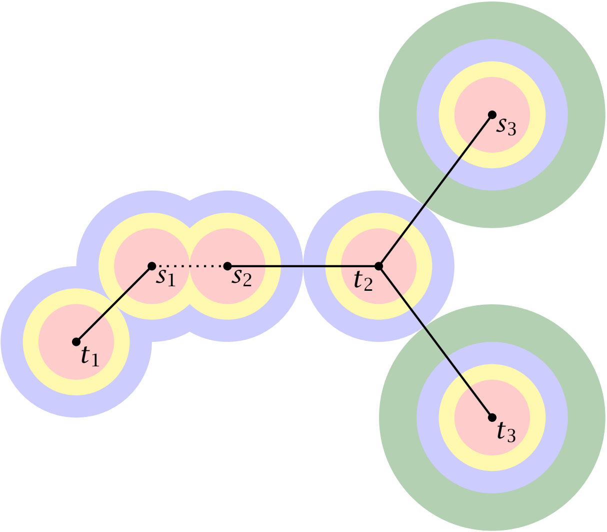

Lemma 4

For any C in any iteration of the algorithm

$$\sum_{C \in C} |\delta(C) \cap F'| \leq 2|C|$$

This means that the number of times a moat from C is crossed in the final solution is at most twice the number of moats.

Proof: later...

$$\sum_{e\in F'} c_e = \sum_{e\in F'}\sum_{S:e\in\delta(S)} y_S = \sum_S |F'\cap\delta(S)|\cdot y_S\ .$$

We want to show that

$$\sum_S |F'\cap\delta(S)|\cdot y_S \le 2\sum_S y_S$$

- In the i-th iteration the increase of the left-hand side is

$$\epsilon\sum_{C\in\mathcal{C}} |F'\cap\delta(C)|$$

  and the increase of the right hand side is $2\epsilon|\mathcal{C}|$.
- Hence, by the previous lemma the inequality holds after the iteration if it holds in the beginning of the iteration.

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Lemma 5

For any set of connected components C in any iteration of the algorithm

$$\sum_{C \in C} |\delta(C) \cap F'| \leq 2|C|$$

Proof:

- At any point during the algorithm the set of edges forms a forest (why?).
- Fix iteration i. e_i is the set we add to F. Let F_i be the set of edges in F at the beginning of the iteration.
- Let $H = F' - F_i$.
- All edges in H are necessary for the solution.

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- Contract all edges in F_i into single vertices V'.
- We can consider the forest H on the set of vertices V'.
- Let $\deg(v)$ be the degree of a vertex $v \in V'$ within this forest.
- Color a vertex $v \in V'$ red if it corresponds to a component from C (an active component). Otw. color it blue. (Let B the set of blue vertices (with non-zero degree) and R the set of red vertices)
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  - But then it must be a red node.