Definition 1

this)

A 0-1 sequence *S* is bitonic if it can be written as the concatenation of subsequences S_1 and S_2 such that either

- \blacktriangleright S₁ is monotonically increasing and S₂ monotonically decreasing, or
- \triangleright S₁ is monotonically decreasing and S₂ monotonically increasing.

Note, that this just defines bitonic 0-1 sequences. Bitonic sequences are defined differently.

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Bitonic Merger

If we feed a bitonic 0-1 sequence S into the network on the right we obtain two bitonic sequences S_T and S_B s.t.

- **1.** $S_B \leq S_T$ (element-wise)
- **2.** S_B and S_T are bitonic

Proof:

- ▶ assume wlog. *S* more 1's than 0's.
- assume for contradiction two 0s at same comparator $(i, j = i + 2^d)$
 - everything 0 btw i and j means we have more than 50% zeros (≠).
 - all 1s btw. i and j means we have less than 50% ones (2).
 - ▶ 1 btw. *i* and *j* and elsewhere means S is not bitonic (\neq).





Bitonic Merger: $(n = 2^d)$ • comparators: $C(n) = 2C(n/2) + n/2 \Rightarrow C(n) = O(n \log n)$. • depth: $D(n) = D(n/2) + 1 \Rightarrow D(d) = O(\log n)$. Bitonic Sorter: $(n = 2^d)$ • comparators: $C(n) = 2C(n/2) + O(n \log n) \Rightarrow$ $C(n) = O(n \log^2 n)$. • depth: $D(n) = D(n/2) + \log n \Rightarrow D(n) = \Theta(\log^2 n)$.

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Odd-Even Merge

How to merge two sorted sequences? $A = (a_1, a_2, ..., a_n), B = (b_1, b_2, ..., b_n), n$ even.

 $\mathbf{n} = (u_1, u_2, \dots, u_n), \mathbf{n} = (v_1, v_2, \dots, v_n), \mathbf{n} \in \mathbf{n}$

Split into odd and even sequences:

 $A_{\text{odd}} = (a_1, a_3, a_5, \dots, a_{n-1}), A_{\text{even}} = (a_2, a_4, a_6, \dots, a_n)$ $B_{\text{odd}} = (b_1, b_3, b_5, \dots, b_{n-1}), B_{\text{even}} = (b_2, b_4, b_6, \dots, b_n)$

Let

 $X = merge(A_{odd}, B_{odd})$ and $Y = merge(A_{even}, B_{even})$

Then

 $S = (x_1, \min\{x_2, y_1\}, \max\{x_2, y_1\}, \min\{x_3, y_2\}, \dots, y_n)$

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Theorem 2

There exists a sorting network with depth $O(\log n)$ and $O(n \log n)$ comparators.

