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An extension of binary search with p processors gives that one can find the rank of an element in

$$\log_{p+1}(n) = \frac{\log n}{\log(p+1)}$$

many parallel steps with p processors. (not work-optimal)

This requires a CREW PRAM model. For the EREW model searching cannot be done faster than $\mathcal{O}(\log n - \log p)$ with p processors even if there are p copies of the search key.

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Given two sorted sequences $A = (a_1, \ldots, a_n)$ and $B = (b_1, \ldots, b_n)$, compute the sorted squence $C = (c_1, \ldots, c_n)$.

Definition 1

Let $X = (x_1, \ldots, x_t)$ be a sequence. The rank $\text{rank}(y : X)$ of y in X is

$$\text{rank}(y : X) = |\{x \in X \mid x \le y\}|$$

For a sequence $Y = (y_1, \ldots, y_s)$ we define $\text{rank}(Y : X) := (r_1, \ldots, r_s)$ with $r_i = \text{rank}(y_i : X)$.

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We have already seen a merging-algorithm that runs in time $\mathcal{O}(\log n)$ and work $\mathcal{O}(n)$.

Using the fast search algorithm we can improve this to a running time of $\mathcal{O}(\log\log n)$ and work $\mathcal{O}(n\log\log n)$.

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Merging

Input: $A = a_1, \ldots, a_n$; $B = b_1, \ldots, b_m$; $m \leq n$

1. if $m < 4$ then rank elements of B, using the parallel search algorithm with p processors. Time: $\mathcal{O}(1)$. Work: $\mathcal{O}(n)$.
2. Concurrently rank elements $b_{\sqrt{m}}, b_{2\sqrt{m}}, \ldots, b_m$ in A using the parallel search algorithm with $p = \sqrt{n}$. Time: $\mathcal{O}(1)$. Work: $\mathcal{O}(\sqrt{m} \cdot \sqrt{n}) = \mathcal{O}(n)$

   $j(i) := \text{rank}(b_{i\sqrt{m}} : A)$
3. Let $B_i = (b_{i\sqrt{m}+1}, \ldots, b_{(i+1)\sqrt{m}-1})$; and $A_i = (a_{j(i)+1}, \ldots, a_{j(i+1)})$.

   Recursively compute $\text{rank}(B_i : A_i)$.
4. Let k be index not a multiple of $\sqrt{m}$. $i = \lceil \frac{k}{\sqrt{m}} \rceil$. Then $\text{rank}(b_k : A) = j(i) + \text{rank}(b_k : A_i)$.

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The algorithm can be made work-optimal by standard techniques.

proof on board...

Mergesort

Lemma 2

A straightforward parallelization of Mergesort can be implemented in time $\mathcal{O}(\log n \log\log n)$ and with work $\mathcal{O}(n \log n)$.

This assumes the CREW-PRAM model.

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Mergesort

Let $L[v]$ denote the (sorted) sublist of elements stored at the leaf nodes rooted at v.

We can view Mergesort as computing $L[v]$ for a complete binary tree where the leaf nodes correspond to nodes in the given array.

Since the merge-operations on one level of the complete binary tree can be performed in parallel we obtain time $\mathcal{O}(h \log\log n)$ and work $\mathcal{O}(hn)$, where $h = \mathcal{O}(\log n)$ is the height of the tree.

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Pipelined Mergesort

We again compute $L[v]$ for every node in the complete binary tree.

After round s, $L_s[v]$ is an **approximation** of $L[v]$ that will be improved in future rounds.

For $s \geq 3\,\mathrm{height}(v)$, $L_s[v] = L[v]$.

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In every round, a node v sends $\text{sample}(L_s[v])$ (an approximation of its current list) upwards, and receives approximations of the lists of its children.

It then computes a new approximation of its list.

A node is called active in round s if $s \le 3\,\text{height}(v)$ (this means its list is not yet complete at the start of the round, i.e., $L_{s-1}[v] \neq L[v]$).

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Pipelined Mergesort

Algorithm 11 ColeSort()

```
initialize L_0[v] = A_v for leaf nodes; L_0[v] = ∅ otw.
for s ← 1 to 3 · height(T) do
    for all active nodes v do
        // u and w children of v
        L'_s[u] ← sample(L_{s-1}[u])
        L'_s[w] ← sample(L_{s-1}[w])
        L_s[v] ← merge(L'_s[u], L'_s[w])
```

$$\text{sample}(L_s[v]) = \begin{cases} \text{sample}_4(L_s[v]) & s \le 3\,\text{height}(v) \\ \text{sample}_2(L_s[v]) & s = 3\,\text{height}(v) + 1 \\ \text{sample}_1(L_s[v]) & s = 3\,\text{height}(v) + 2 \end{cases}$$

Pipelined Mergesort

Algorithm 11 ColeSort()

initialize $L_0[v] = A_v$ for leaf nodes; $L_0[v] = \emptyset$ otw.
for $s \leftarrow 1$ **to** $3 \cdot \text{height}(T)$ **do**
    for all active nodes v **do**
        // u and w children of v
        $L'_s[u] \leftarrow \text{sample}(L_{s-1}[u])$
        $L'_s[w] \leftarrow \text{sample}(L_{s-1}[w])$
        $L_s[v] \leftarrow \text{merge}(L'_s[u], L'_s[w])$

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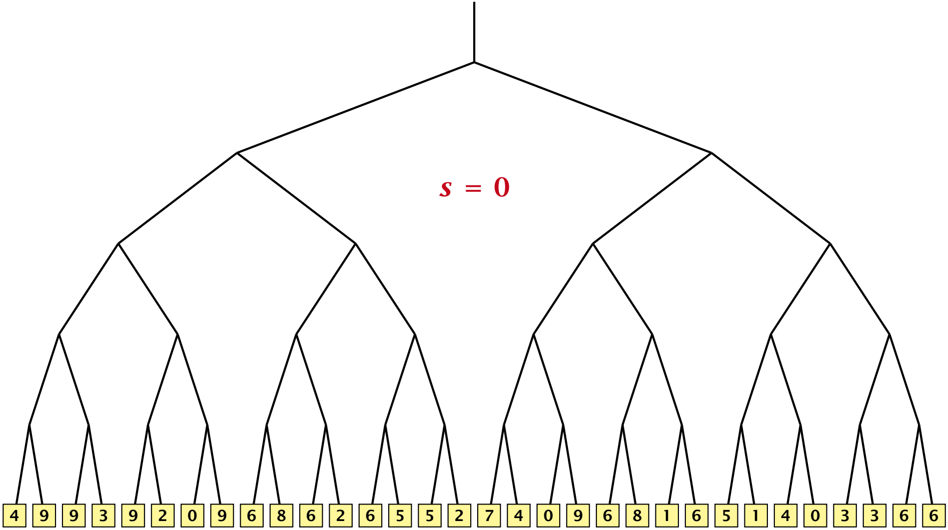
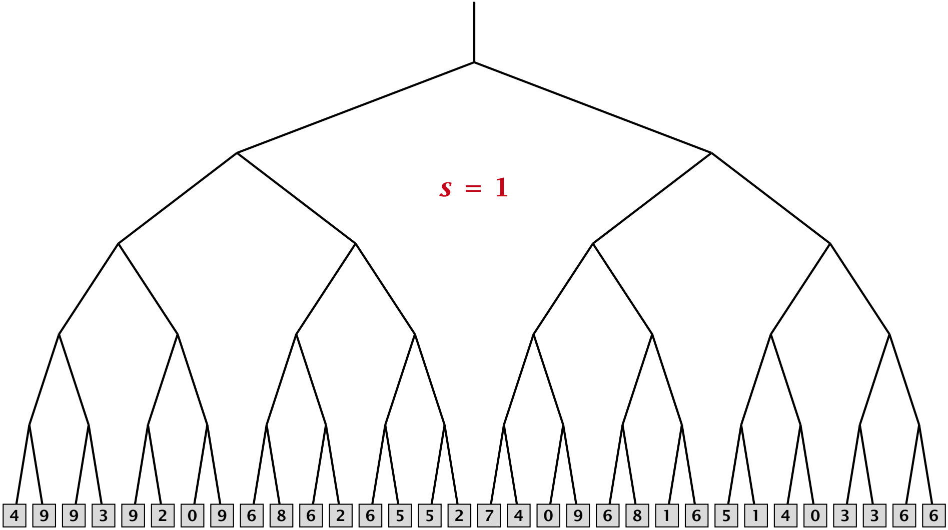
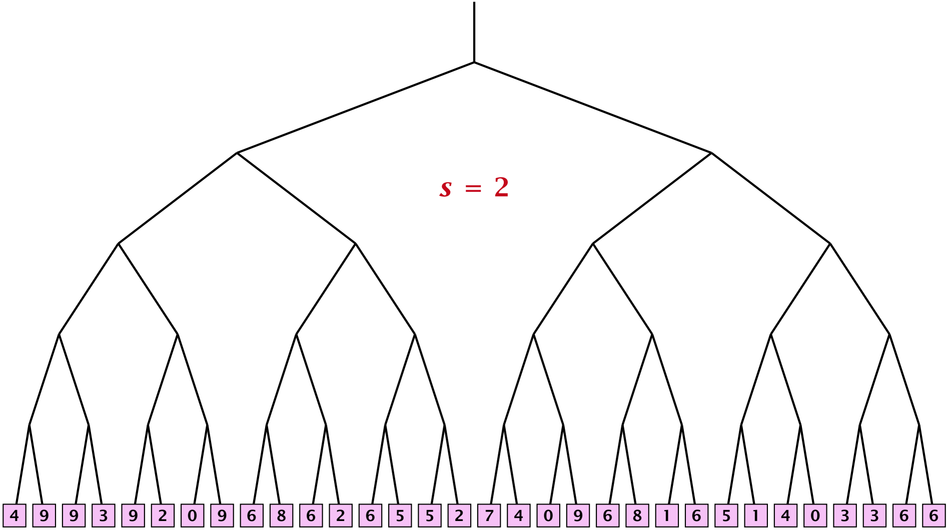
Colesort



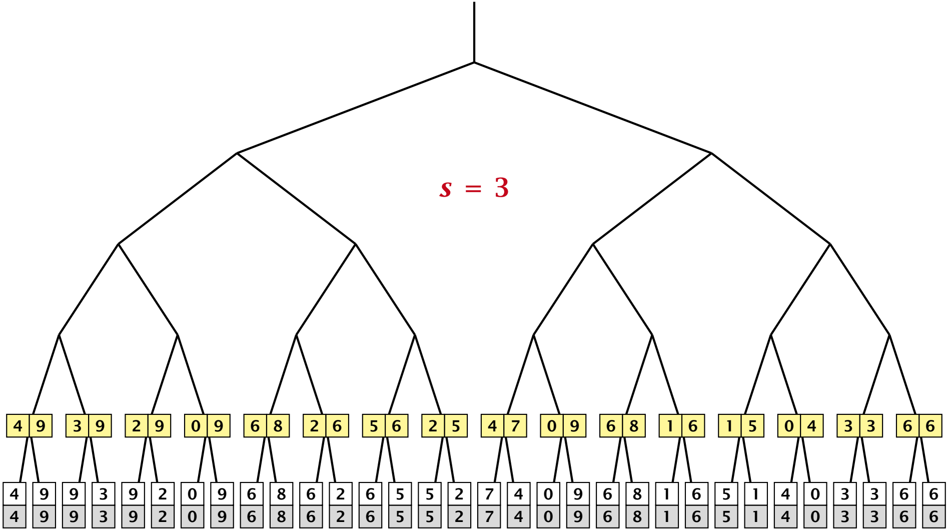
Colesort



Colesort

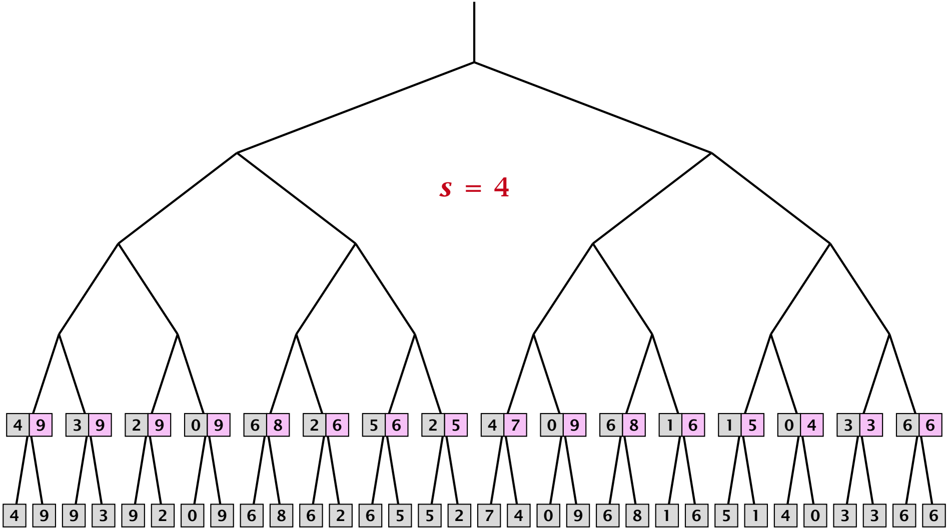


Colesort

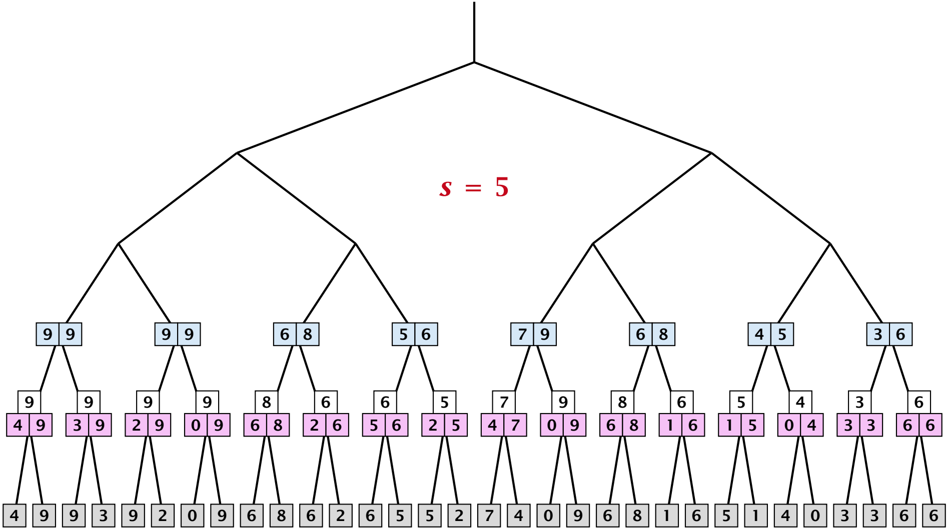


$s = 3$
4 9
3 9
2 9
0 9
6 8
2 6
5 6
2 5
4 7
0 9
6 8
1 6
1 5
0 4
3 3
6 6
4 9 9 3 9 2 0 9 6 8 6 2 6 5 5 2 7 4 0 9 6 8 1 6 5 1 4 0 3 3 6 6
4 9 9 3 9 2 0 9 6 8 6 2 6 5 5 2 7 4 0 9 6 8 1 6 5 1 4 0 3 3 6 6

Colesort

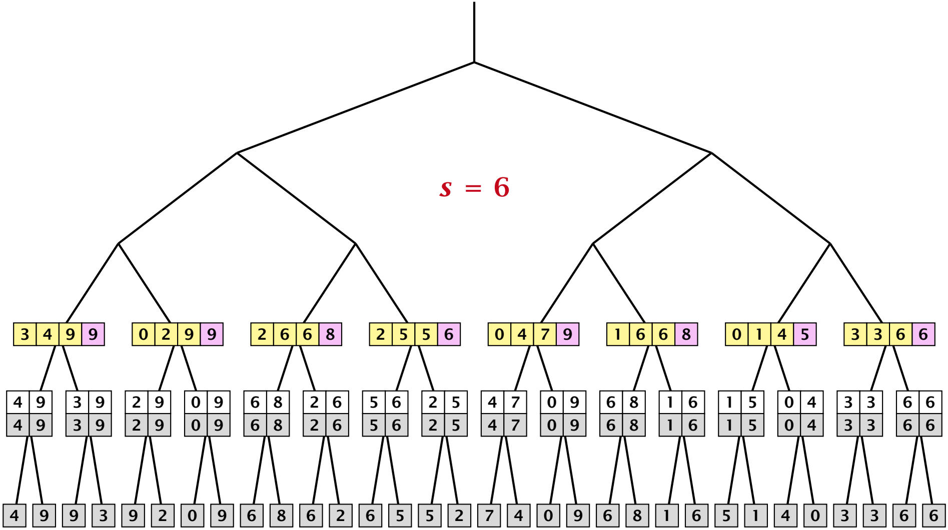

s = 4
4 9 3 9 2 9 0 9 6 8 2 6 5 6 2 5 4 7 0 9 6 8 1 6 1 5 0 4 3 3 6 6
4 9 9 3 9 2 0 9 6 8 6 2 6 5 5 2 7 4 0 9 6 8 1 6 5 1 4 0 3 3 6 6

Colesort

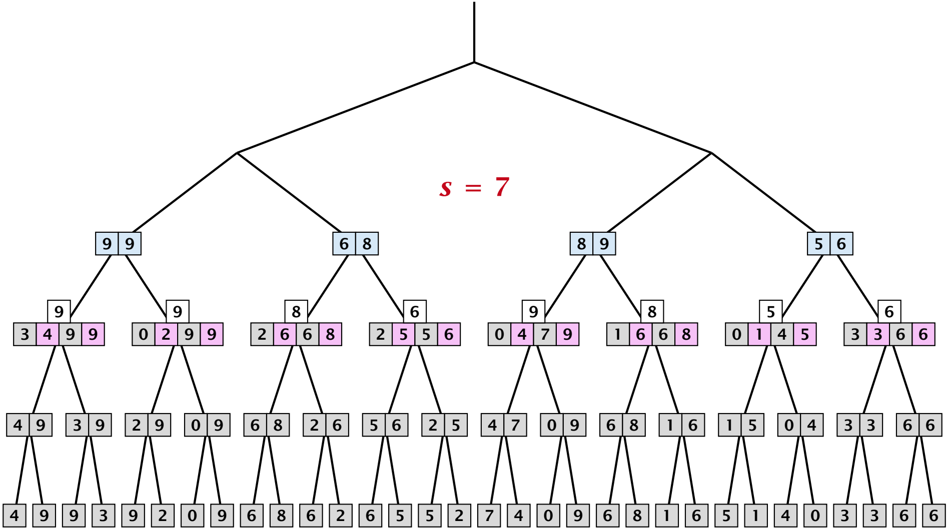


$s = 5$
9 9
9 9
6 8
5 6
7 9
6 8
4 5
3 6
9 9 9 9 8 6 6 5 7 9 8 6 5 4 3 6
4 9 3 9 2 9 0 9 6 8 2 6 5 6 2 5 4 7 0 9 6 8 1 6 1 5 0 4 3 3 6 6
4 9 9 3 9 2 0 9 6 8 6 2 6 5 5 2 7 4 0 9 6 8 1 6 5 1 4 0 3 3 6 6

Colesort

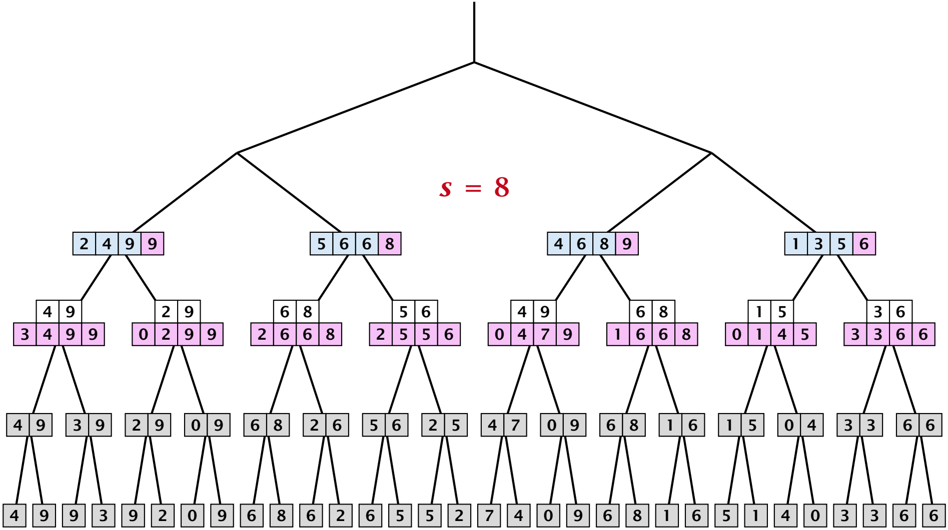

$s = 6$
3 4 9 9
0 2 9 9
2 6 6 8
2 5 5 6
0 4 7 9
1 6 6 8
0 1 4 5
3 3 6 6
4 9 / 4 9
3 9 / 3 9
2 9 / 2 9
0 9 / 0 9
6 8 / 6 8
2 6 / 2 6
5 6 / 5 6
2 5 / 2 5
4 7 / 4 7
0 9 / 0 9
6 8 / 6 8
1 6 / 1 6
1 5 / 1 5
0 4 / 0 4
3 3 / 3 3
6 6 / 6 6
4 9 9 3 9 2 0 9 6 8 6 2 6 5 5 2 7 4 0 9 6 8 1 6 5 1 4 0 3 3 6 6

Colesort



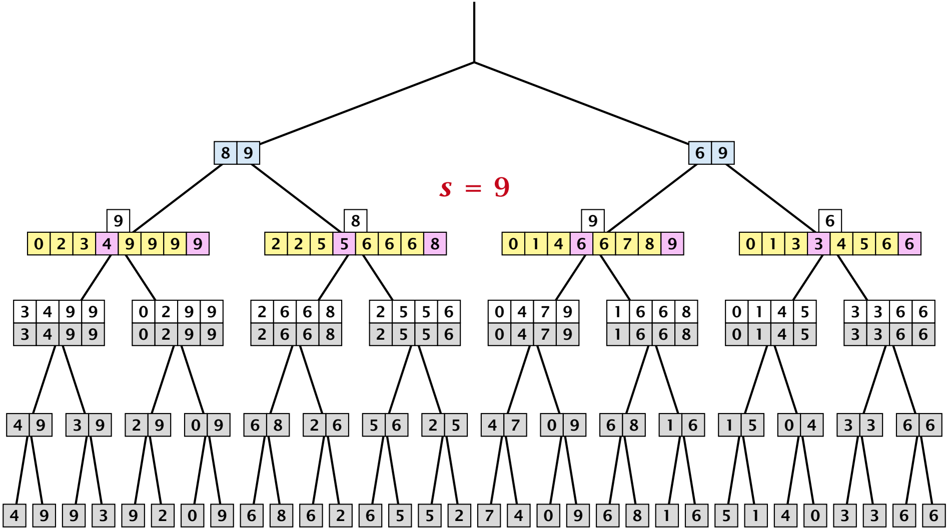
s = 7
9 9
6 8
8 9
5 6
9
9
8
6
9
8
5
6
3 4 9 9
0 2 9 9
2 6 6 8
2 5 5 6
0 4 7 9
1 6 6 8
0 1 4 5
3 3 6 6
4 9
3 9
2 9
0 9
6 8
2 6
5 6
2 5
4 7
0 9
6 8
1 6
1 5
0 4
3 3
6 6
4 9 9 3 9 2 0 9 6 8 6 2 6 5 5 2 7 4 0 9 6 8 1 6 5 1 4 0 3 3 6 6

Colesort

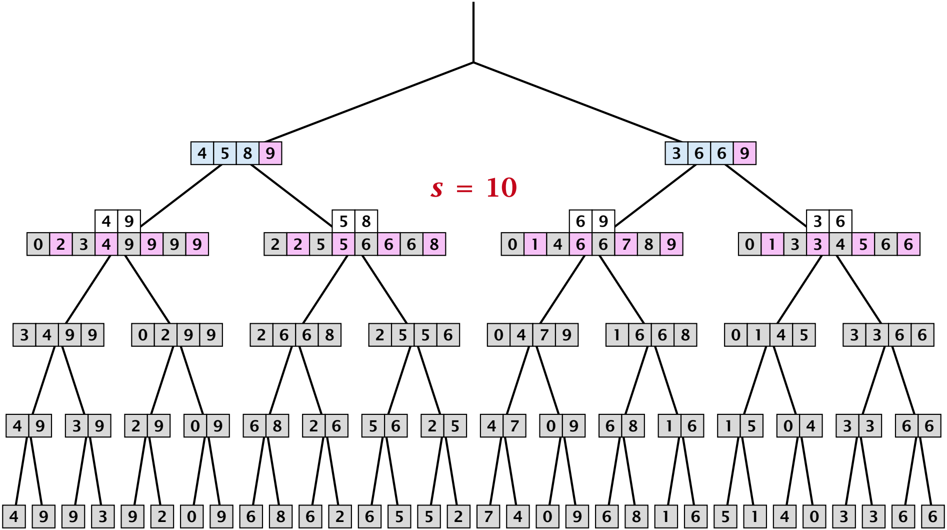


s = 8
2 4 9 9
5 6 6 8
4 6 8 9
1 3 5 6
4 9
3 4 9 9
2 9
0 2 9 9
6 8
2 6 6 8
5 6
2 5 5 6
4 9
0 4 7 9
6 8
1 6 6 8
1 5
0 1 4 5
3 6
3 3 6 6
4 9
3 9
2 9
0 9
6 8
2 6
5 6
2 5
4 7
0 9
6 8
1 6
1 5
0 4
3 3
6 6
4 9 9 3 9 2 0 9 6 8 6 2 6 5 5 2 7 4 0 9 6 8 1 6 5 1 4 0 3 3 6 6

Colesort

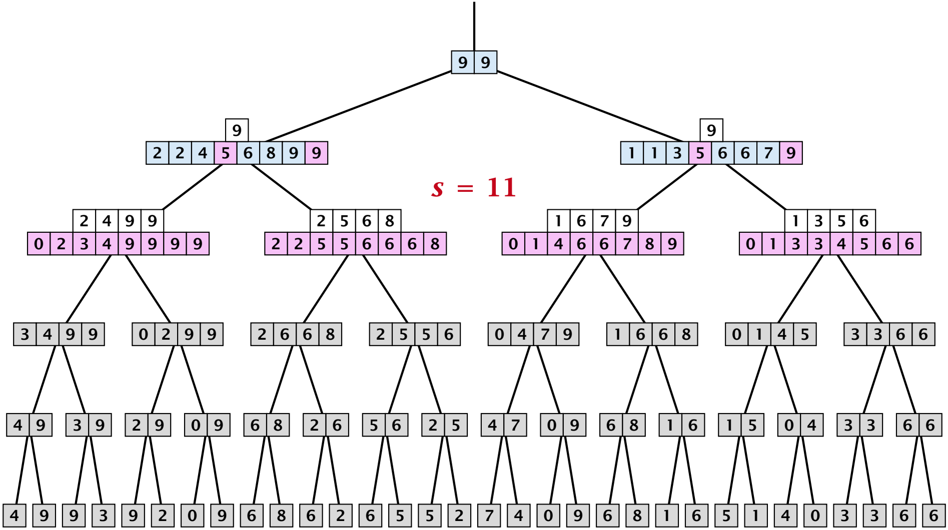


Colesort

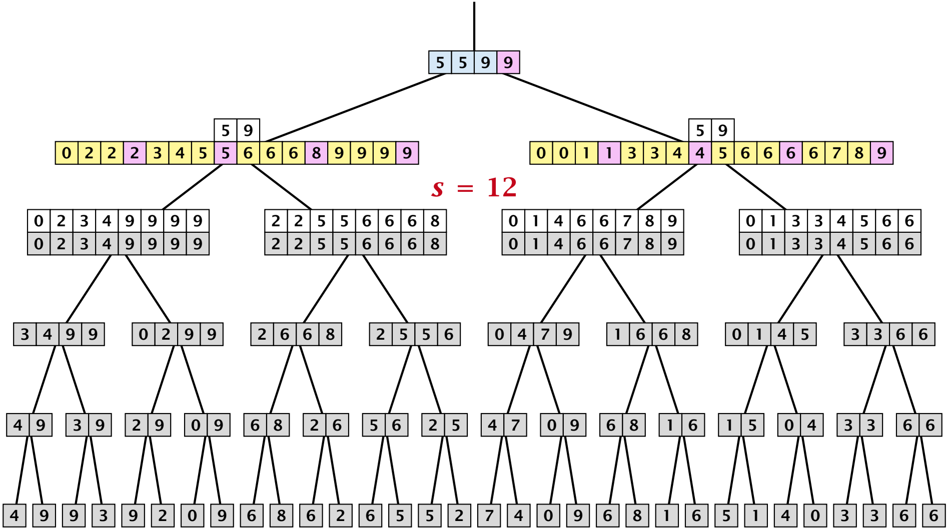


4 5 8 9
3 6 6 9
s = 10
4 9
0 2 3 4 9 9 9 9
5 8
2 2 5 5 6 6 6 8
6 9
0 1 4 6 6 7 8 9
3 6
0 1 3 3 4 5 6 6
3 4 9 9
0 2 9 9
2 6 6 8
2 5 5 6
0 4 7 9
1 6 6 8
0 1 4 5
3 3 6 6
4 9
3 9
2 9
0 9
6 8
2 6
5 6
2 5
4 7
0 9
6 8
1 6
1 5
0 4
3 3
6 6
4 9 9 3 9 2 0 9 6 8 6 2 6 5 5 2 7 4 0 9 6 8 1 6 5 1 4 0 3 3 6 6

Colesort

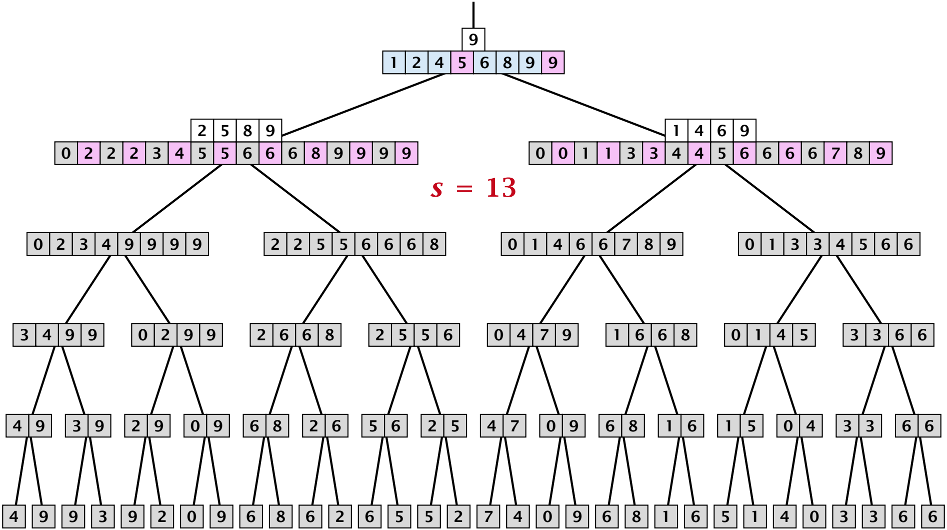

9 9
9
2 2 4 5 6 8 9 9
9
1 1 3 5 6 6 7 9
s = 11
2 4 9 9
0 2 3 4 9 9 9 9
2 5 6 8
2 2 5 5 6 6 6 8
1 6 7 9
0 1 4 6 6 7 8 9
1 3 5 6
0 1 3 3 4 5 6 6
3 4 9 9
0 2 9 9
2 6 6 8
2 5 5 6
0 4 7 9
1 6 6 8
0 1 4 5
3 3 6 6
4 9
3 9
2 9
0 9
6 8
2 6
5 6
2 5
4 7
0 9
6 8
1 6
1 5
0 4
3 3
6 6
4 9 9 3 9 2 0 9 6 8 6 2 6 5 5 2 7 4 0 9 6 8 1 6 5 1 4 0 3 3 6 6

Colesort

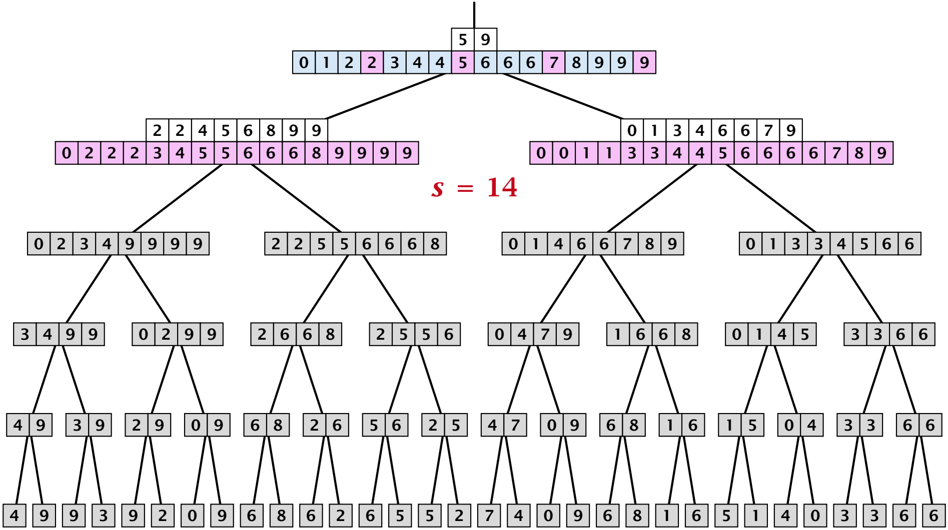


5 5 9 9
5 9
0 2 2 2 3 4 5 5 6 6 6 8 9 9 9 9
5 9
0 0 1 1 3 3 4 4 5 6 6 6 6 7 8 9
$s = 12$
0 2 3 4 9 9 9 9
0 2 3 4 9 9 9 9
2 2 5 5 6 6 6 8
2 2 5 5 6 6 6 8
0 1 4 6 6 7 8 9
0 1 4 6 6 7 8 9
0 1 3 3 4 5 6 6
0 1 3 3 4 5 6 6
3 4 9 9
0 2 9 9
2 6 6 8
2 5 5 6
0 4 7 9
1 6 6 8
0 1 4 5
3 3 6 6
4 9
3 9
2 9
0 9
6 8
2 6
5 6
2 5
4 7
0 9
6 8
1 6
1 5
0 4
3 3
6 6
4 9 9 3 9 2 0 9 6 8 6 2 6 5 5 2 7 4 0 9 6 8 1 6 5 1 4 0 3 3 6 6

Colesort

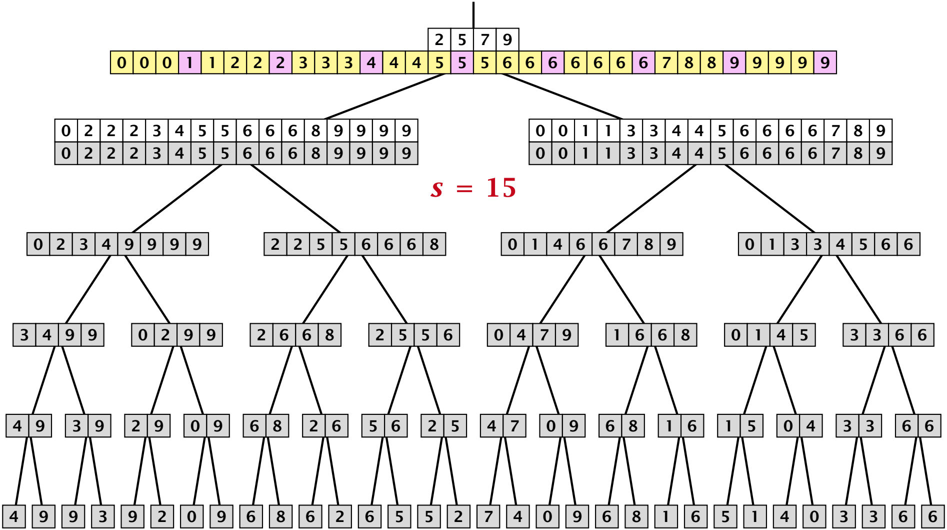

9
1 2 4 5 6 8 9 9
2 5 8 9
0 2 2 2 3 4 5 5 6 6 6 8 9 9 9 9
1 4 6 9
0 0 1 1 3 3 4 4 5 6 6 6 6 7 8 9
s = 13
0 2 3 4 9 9 9 9
2 2 5 5 6 6 6 8
0 1 4 6 6 7 8 9
0 1 3 3 4 5 6 6
3 4 9 9
0 2 9 9
2 6 6 8
2 5 5 6
0 4 7 9
1 6 6 8
0 1 4 5
3 3 6 6
4 9
3 9
2 9
0 9
6 8
2 6
5 6
2 5
4 7
0 9
6 8
1 6
1 5
0 4
3 3
6 6
4 9 9 3 9 2 0 9 6 8 6 2 6 5 5 2 7 4 0 9 6 8 1 6 5 1 4 0 3 3 6 6

Colesort



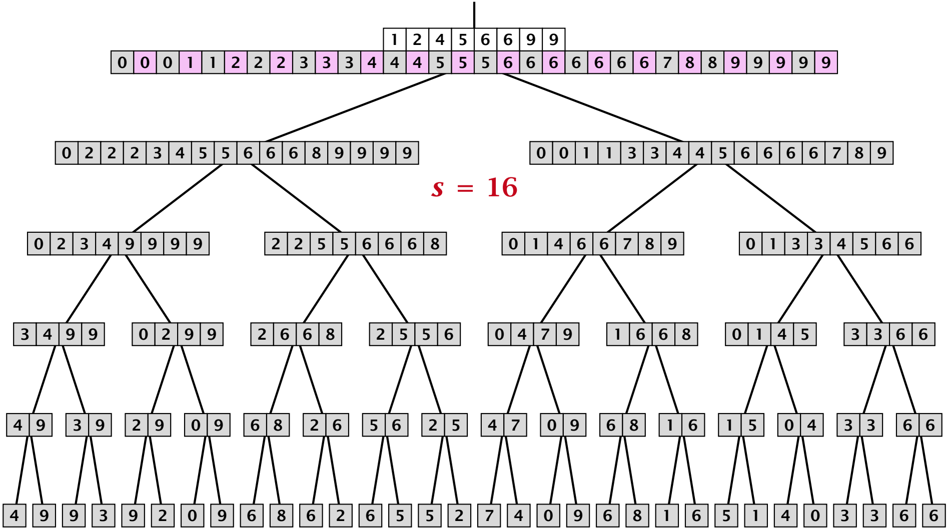
5 9
0 1 2 2 3 4 4 5 6 6 6 7 8 9 9 9
2 2 4 5 6 8 9 9
0 2 2 2 3 4 5 5 6 6 6 8 9 9 9 9
0 1 3 4 6 6 7 9
0 0 1 1 3 3 4 4 5 6 6 6 6 7 8 9
s = 14
0 2 3 4 9 9 9 9
2 2 5 5 6 6 6 8
0 1 4 6 6 7 8 9
0 1 3 3 4 5 6 6
3 4 9 9
0 2 9 9
2 6 6 8
2 5 5 6
0 4 7 9
1 6 6 8
0 1 4 5
3 3 6 6
4 9
3 9
2 9
0 9
6 8
2 6
5 6
2 5
4 7
0 9
6 8
1 6
1 5
0 4
3 3
6 6
4 9 9 3 9 2 0 9 6 8 6 2 6 5 5 2 7 4 0 9 6 8 1 6 5 1 4 0 3 3 6 6

Colesort

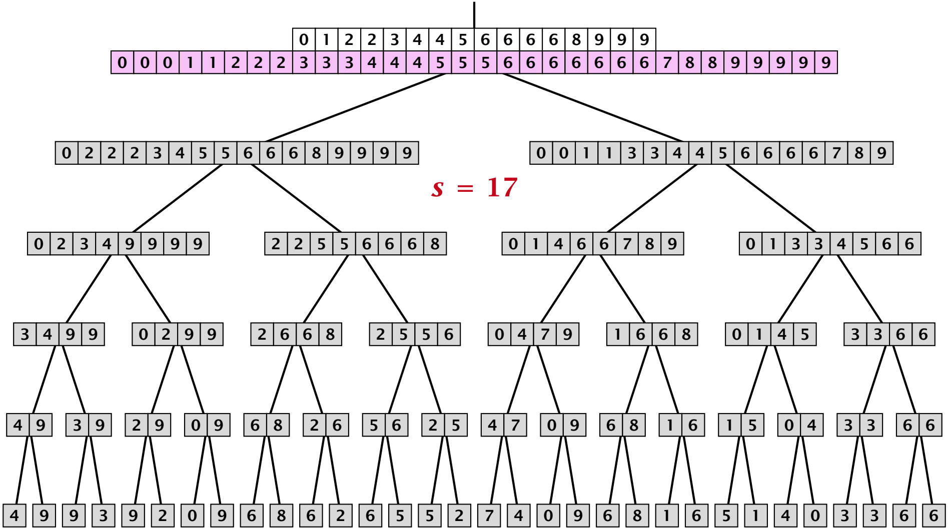


2 5 7 9
0 0 0 1 1 2 2 2 3 3 3 4 4 4 5 5 5 6 6 6 6 6 6 6 7 8 8 9 9 9 9 9
0 2 2 2 3 4 5 5 6 6 6 8 9 9 9 9
0 2 2 2 3 4 5 5 6 6 6 8 9 9 9 9
0 0 1 1 3 3 4 4 5 6 6 6 6 7 8 9
0 0 1 1 3 3 4 4 5 6 6 6 6 7 8 9
s = 15
0 2 3 4 9 9 9 9
2 2 5 5 6 6 6 8
0 1 4 6 6 7 8 9
0 1 3 3 4 5 6 6
3 4 9 9
0 2 9 9
2 6 6 8
2 5 5 6
0 4 7 9
1 6 6 8
0 1 4 5
3 3 6 6
4 9
3 9
2 9
0 9
6 8
2 6
5 6
2 5
4 7
0 9
6 8
1 6
1 5
0 4
3 3
6 6
4 9 9 3 9 2 0 9 6 8 6 2 6 5 5 2 7 4 0 9 6 8 1 6 5 1 4 0 3 3 6 6

Colesort


1 2 4 5 6 6 9 9
0 0 0 1 1 2 2 2 3 3 3 4 4 4 5 5 5 6 6 6 6 6 6 6 7 8 8 9 9 9 9 9
0 2 2 2 3 4 5 5 6 6 6 8 9 9 9 9
0 0 1 1 3 3 4 4 5 6 6 6 6 7 8 9
s = 16
0 2 3 4 9 9 9 9
2 2 5 5 6 6 6 8
0 1 4 6 6 7 8 9
0 1 3 3 4 5 6 6
3 4 9 9
0 2 9 9
2 6 6 8
2 5 5 6
0 4 7 9
1 6 6 8
0 1 4 5
3 3 6 6
4 9
3 9
2 9
0 9
6 8
2 6
5 6
2 5
4 7
0 9
6 8
1 6
1 5
0 4
3 3
6 6
4 9 9 3 9 2 0 9 6 8 6 2 6 5 5 2 7 4 0 9 6 8 1 6 5 1 4 0 3 3 6 6

Colesort



Pipelined Mergesort

Lemma 3

After round $s = 3\,\mathrm{height}(v)$, *the list* $L_s[v]$ *is complete.*

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Proof:

- clearly true for leaf nodes
- suppose it is true for all nodes up to height h;
- fix a node v on level $h+1$ with children u and w
- $L_{3h}[u]$ and $L_{3h}[w]$ are complete by induction hypothesis
- further $\mathrm{sample}(L_{3h+2}[u]) = L[u]$ and $\mathrm{sample}(L_{3h+2}[v]) = L[v]$
- hence in round $3h+3$ node v will merge the complete list of its children; after the round $L[v]$ will be complete

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Pipelined Mergesort

Lemma 4

The number of elements in lists $L_s[v]$ *for active nodes* v *is at most* $\mathcal{O}(n)$.

proof on board...

Definition 5

A sequence X is a c-cover of a sequence Y if for any two consecutive elements α, β from $(-\infty, X, \infty)$ the set $|\{y_i \mid \alpha \le y_i \le \beta\}| \le c$.

Pipelined Mergesort

Lemma 6
$L'_s[v]$ is a 4-cover of $L'_{s+1}[v]$.

If $[a,b]$ fulfills $|[a,b] \cap (A \cup \{-\infty,\infty\})| = k$ we say $[a,b]$ intersects $(-\infty, A, +\infty)$ in k items.

Lemma 7
If $[a,b]$ with $a,b \in L'_s[v] \cup \{-\infty,\infty\}$ intersects $(-\infty, L'_s[v], \infty)$ in $k \geq 2$ items, then $[a,b]$ intersects $(-\infty, L'_{s+1}, \infty)$ in at most $2k$ items.

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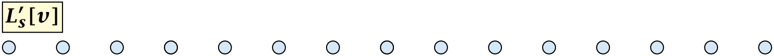
$L'_s[v]$

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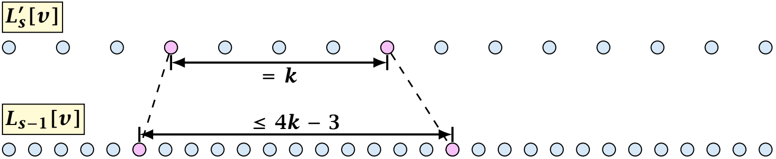
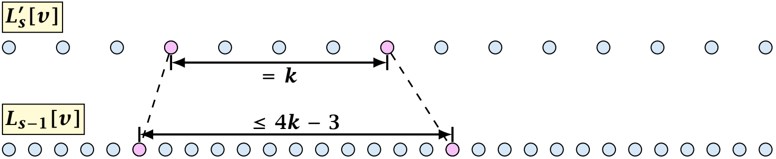


$L'_s[v]$
$= k$
$L_{s-1}[v]$
$\leq 4k - 3$



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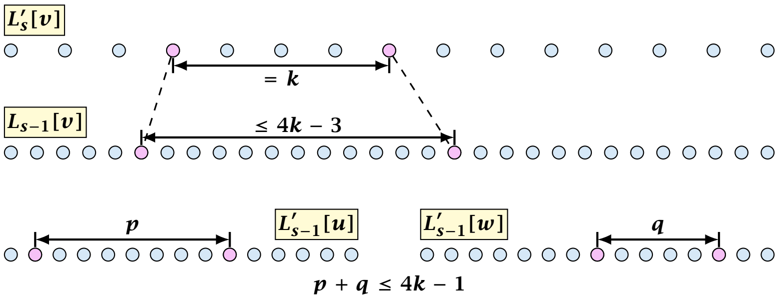
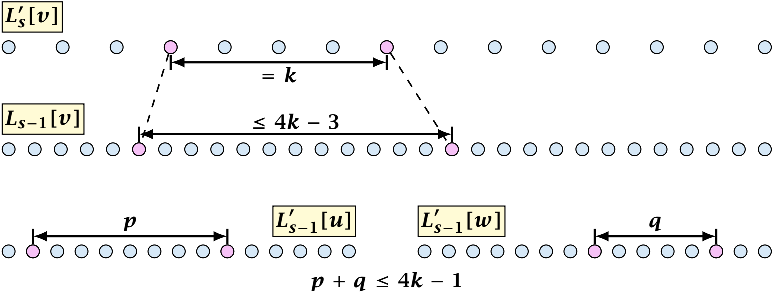


$L'_s[v]$
$= k$
$L_{s-1}[v]$
$\leq 4k - 3$
p
$L'_{s-1}[u]$
$L'_{s-1}[w]$
q
$p + q \leq 4k - 1$



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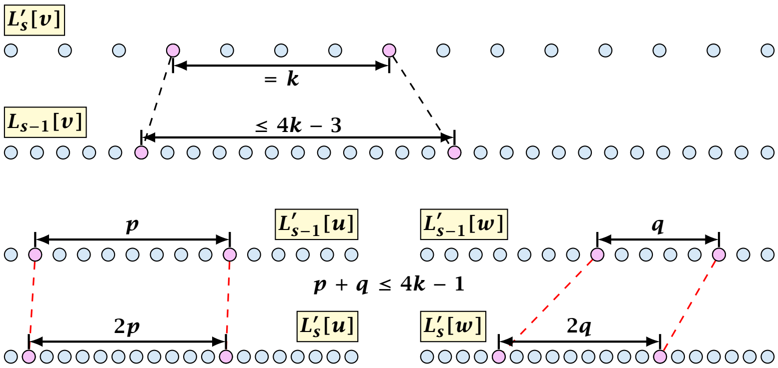

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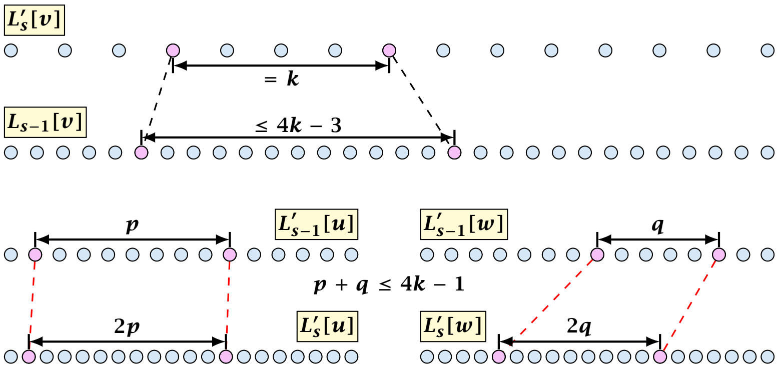
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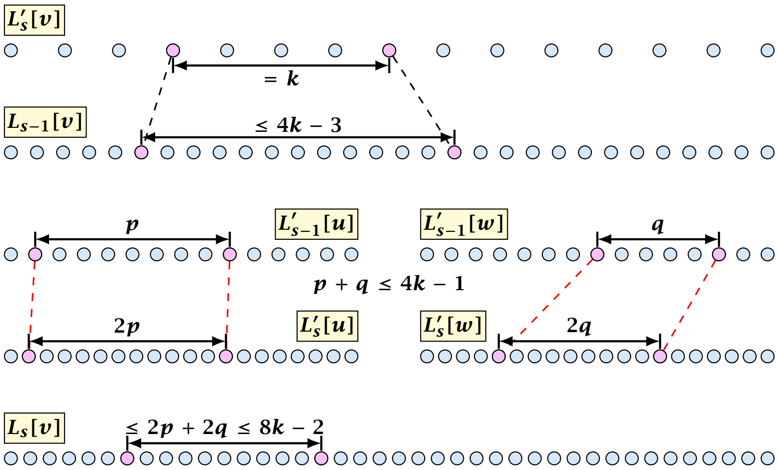
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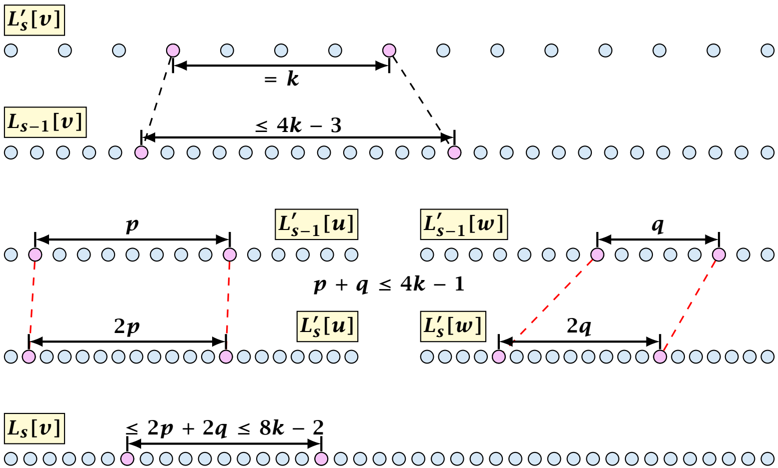

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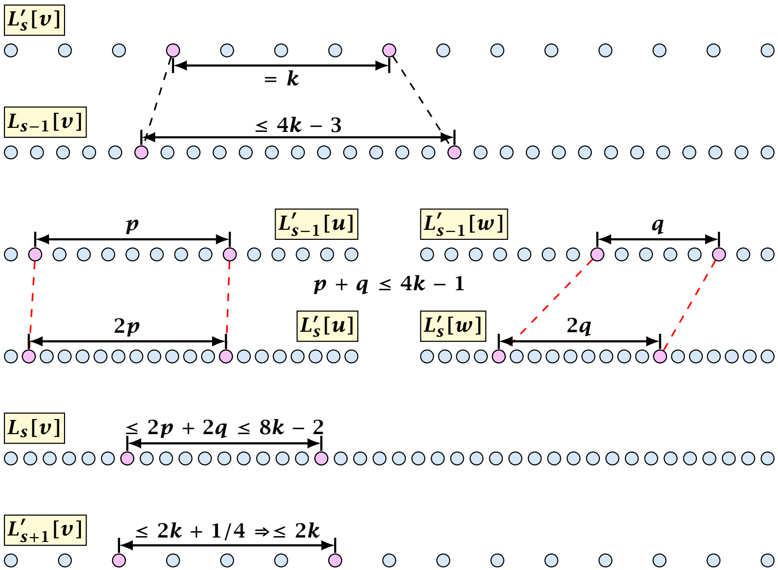
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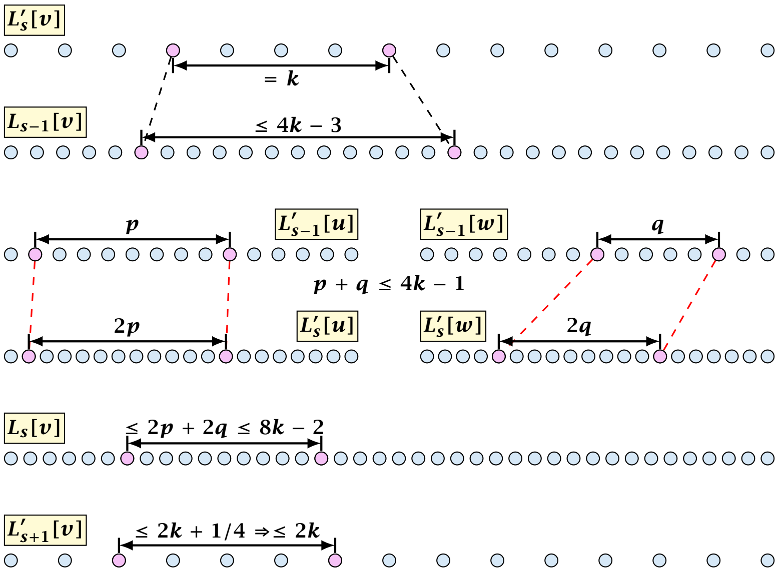

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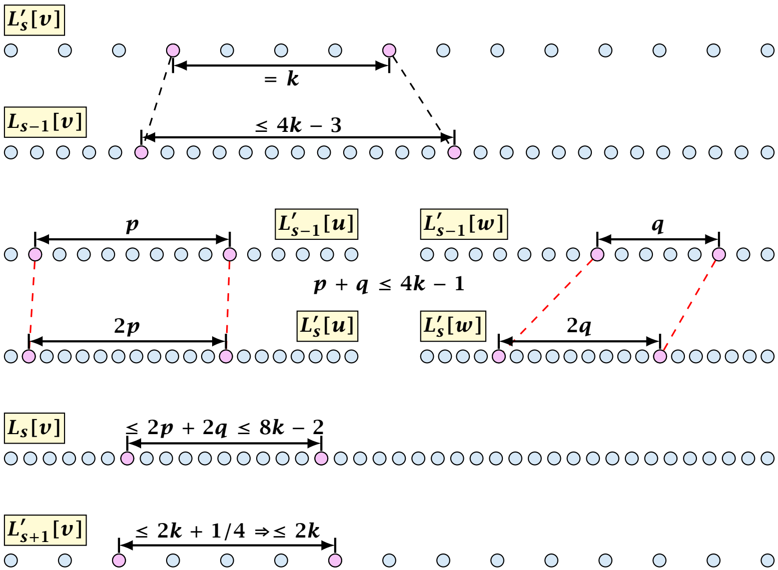

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Merging with a Cover

Lemma 8

Given two sorted sequences A and B. Let X be a c-cover of A and B for constant c, and let $\mathrm{rank}(X:A)$ and $\mathrm{rank}(X:B)$ be known.

We can merge A and B in time $\mathcal{O}(1)$ using $\mathcal{O}(|X|)$ operations.

Merging with a Cover

Lemma 9

Given two sorted sequences A and B. Let X be a c-cover of B for constant c, and let $\text{rank}(A:X)$ and $\text{rank}(X:B)$ be known.

We can compute $\text{rank}(A:B)$ using $\mathcal{O}(|X|+|A|)$ operations.

Merging with a Cover

Lemma 10

Given two sorted sequences A *and* B. *Let* X *be a* c*-cover of* B *for constant* c, *and let* $\mathrm{rank}(A:X)$ *and* $\mathrm{rank}(X:B)$ *be known.*

We can compute $\mathrm{rank}(B:A)$ *using* $\mathcal{O}(|X|+|A|)$ *operations.*

Easy to do with concurrent read. Can also be done with exclusive read but non-trivial.

In order to do the merge in iteration $s + 1$ in constant time we need to know

$$\text{rank}(L_s[v] : L'_{s+1}[u]) \text{ and } \text{rank}(L_s[v] : L'_{s+1}[w])$$

and we need to know that $L_s[v]$ is a 4-cover of $L'_{s+1}[u]$ and $L'_{s+1}[w]$.

Lemma 11

$L_s[v]$ is a 4-cover of $L'_{s+1}[u]$ and $L'_{s+1}[w]$.

- $L_s[v] \supseteq L'_s[u], L'_s[w]$
- $L'_s[u]$ is 4-cover of $L'_{s+1}[u]$
- Hence, $L_s[v]$ is 4-cover of $L'_{s+1}[u]$ as adding more elements cannot destroy the cover-property.

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- Hence, $L_s[v]$ is 4-cover of $L'_{s+1}[u]$ as adding more elements cannot destroy the cover-property.

Analysis

Lemma 12

Suppose we know for every internal node v with children u and w

- $\text{rank}(L'_s[v] : L'_{s+1}[v])$
- $\text{rank}(L'_s[u] : L'_s[w])$
- $\text{rank}(L'_s[w] : L'_s[u])$

We can compute

- $\text{rank}(L'_{s+1}[v] : L'_{s+2}[v])$
- $\text{rank}(L'_{s+1}[u] : L'_{s+1}[w])$
- $\text{rank}(L'_{s+1}[w] : L'_{s+1}[u])$

in constant time and $\mathcal{O}(|L_{s+1}[v]|)$ operations, where v is the parent of u and w.

Given

- $\text{rank}(L'_s[u] : L'_{s+1}[u])$ **(4-cover)**
- $\text{rank}(L'_s[w] : L'_s[u])$
- $\text{rank}(L'_s[u] : L'_s[w])$
- $\text{rank}(L'_s[w] : L'_{s+1}[w])$ **(4-cover)**

Compute

- $\text{rank}(L'_{s+1}[w] : L'_s[u])$
- $\text{rank}(L'_{s+1}[u] : L'_s[w])$

Compute

- $\text{rank}(L'_{s+1}[w] : L'_{s+1}[u])$
- $\text{rank}(L'_{s+1}[u] : L'_{s+1}[w])$

ranks between siblings can be computed easily

Given

- $\text{rank}(L'_s[u] : L'_{s+1}[u])$ **(4-cover)**
- $\text{rank}(L'_s[w] : L'_s[u])$
- $\text{rank}(L'_s[u] : L'_s[w])$
- $\text{rank}(L'_s[w] : L'_{s+1}[w])$ **(4-cover)**

Compute

- $\text{rank}(L'_{s+1}[w] : L'_s[u])$
- $\text{rank}(L'_{s+1}[u] : L'_s[w])$

Compute

- $\text{rank}(L'_{s+1}[w] : L'_{s+1}[u])$
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ranks between siblings can be computed easily

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- $\text{rank}(L'_s[w] : L'_s[u])$
- $\text{rank}(L'_s[u] : L'_s[w])$
- $\text{rank}(L'_s[w] : L'_{s+1}[w])$ **(4-cover)**

Compute

- $\text{rank}(L'_{s+1}[w] : L'_s[u])$
- $\text{rank}(L'_{s+1}[u] : L'_s[w])$

Compute

- $\text{rank}(L'_{s+1}[w] : L'_{s+1}[u])$
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ranks between siblings can be computed easily

Given

- $\text{rank}(L'_s[u] : L'_{s+1}[u])$ (**4-cover** → $\text{rank}(L'_{s+1}[u] : L'_s[u])$)
- $\text{rank}(L'_s[w] : L'_{s+1}[u])$
- $\text{rank}(L'_s[u] : L'_{s+1}[w])$
- $\text{rank}(L'_s[w] : L'_{s+1}[w])$ (**4-cover** → $\text{rank}(L'_{s+1}[w] : L'_s[w])$)

Compute (recall that $L_s[v] = \text{merge}(L'_s[u], L'_s[w])$)

- $\text{rank}(L_s[v] : L'_{s+1}[u])$
- $\text{rank}(L_s[v] : L'_{s+1}[w])$

Compute

- $\text{rank}(L_s[v] : L_{s+1}[v])$ (by adding)
- $\text{rank}(L'_{s+1}[v] : L'_{s+2}[v])$ (by sampling)

Given

- $\text{rank}(L'_s[u] : L'_{s+1}[u])$ (**4-cover** → $\text{rank}(L'_{s+1}[u] : L'_s[u])$)
- $\text{rank}(L'_s[w] : L'_{s+1}[u])$
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- $\mathrm{rank}(L'_s[w] : L'_{s+1}[w])$ (**4-cover** → $\mathrm{rank}(L'_{s+1}[w] : L'_s[w])$)

Compute (recall that $L_s[v] = \mathrm{merge}(L'_s[u], L'_s[w])$)

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