12

Greedy-algorithm:

- start with $f(e) = 0$ everywhere
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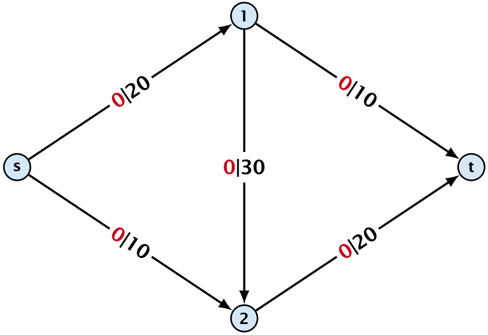

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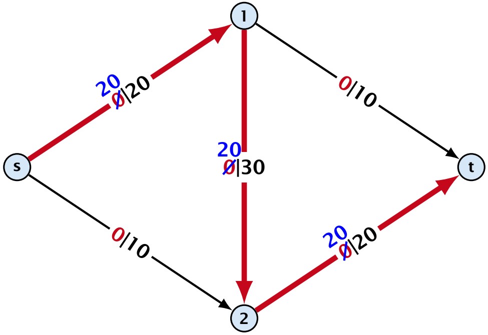

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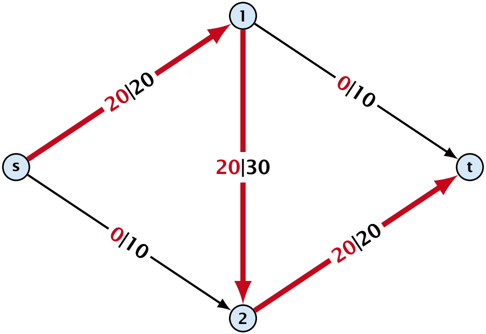

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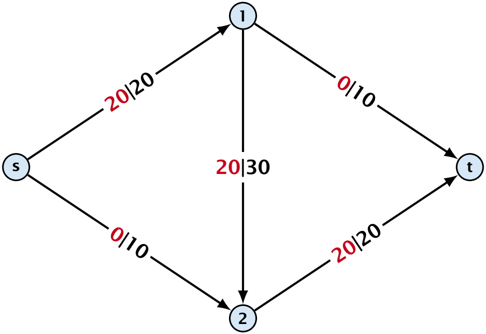

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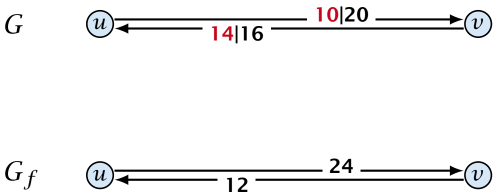

Augmenting Path Algorithm

Definition 1

An augmenting path with respect to flow f, is a path from s to t in the auxiliary graph G_f that contains only edges with non-zero capacity.

Algorithm 46 FordFulkerson($G = (V, E, c)$)

1: Initialize $f(e) \leftarrow 0$ for all edges.
2: **while** $\exists$ augmenting path p in G_f **do**
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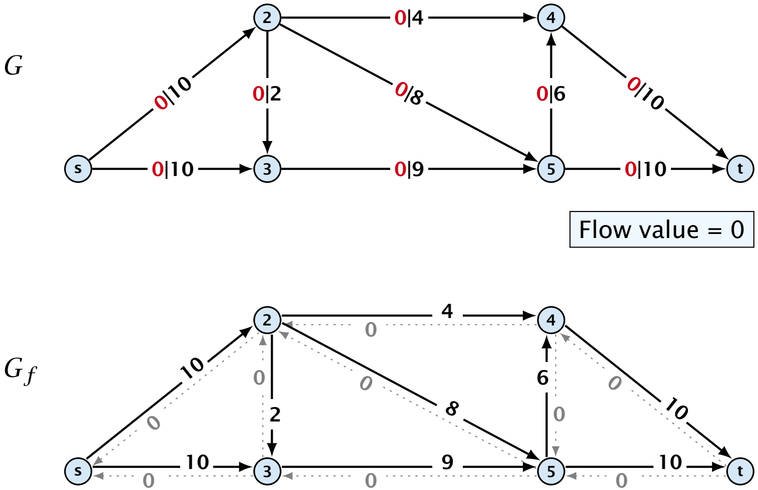

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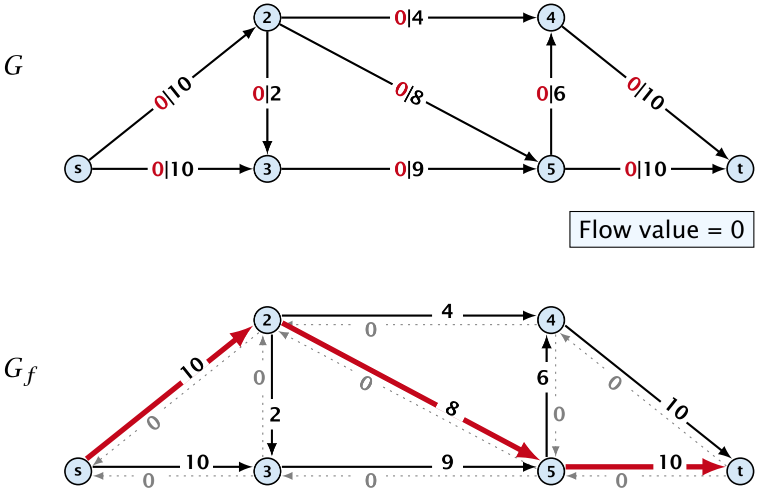

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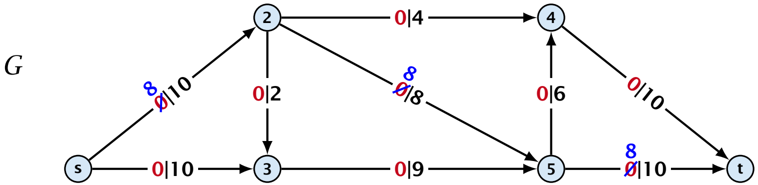
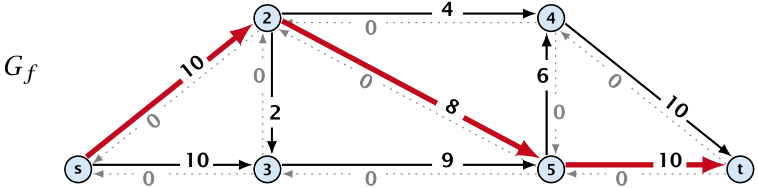


Flow value = 8



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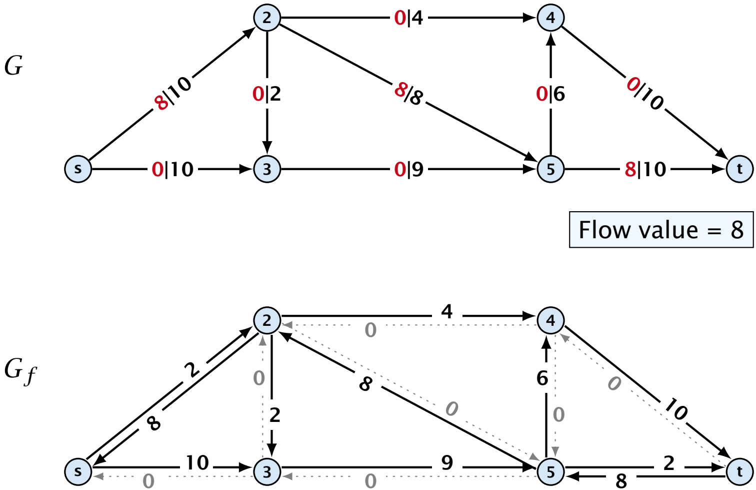

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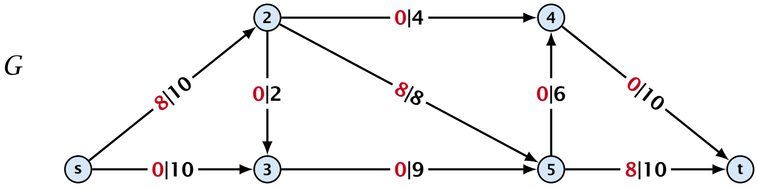

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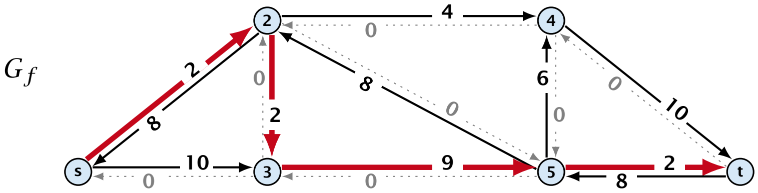

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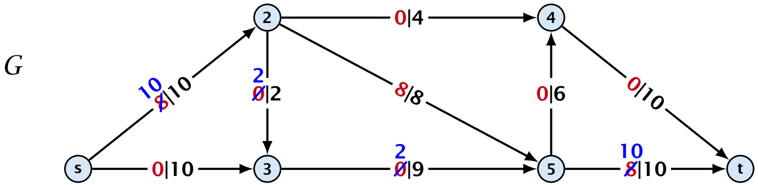

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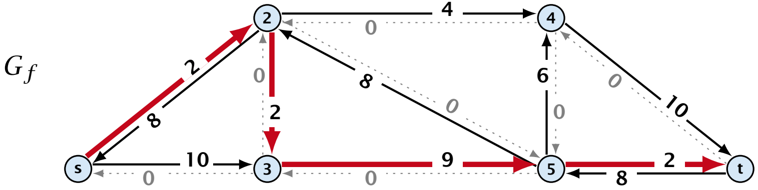
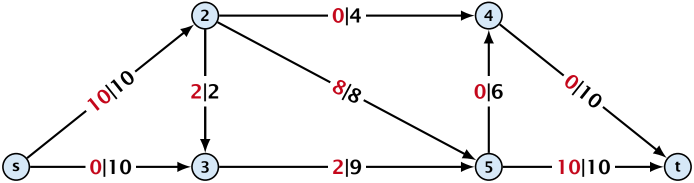


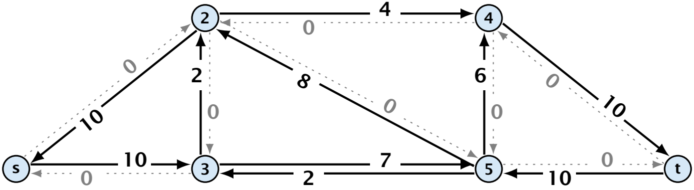
Augmenting Path Algorithm

G



Flow value = 10

G_f



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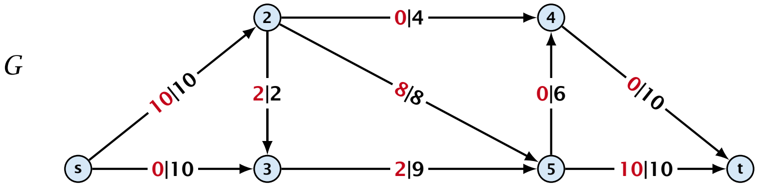

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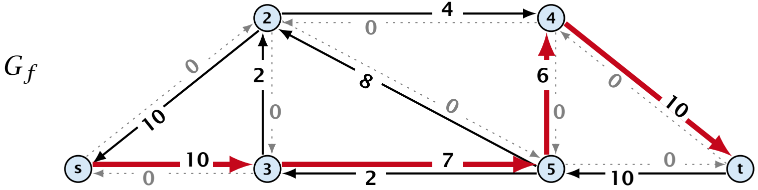

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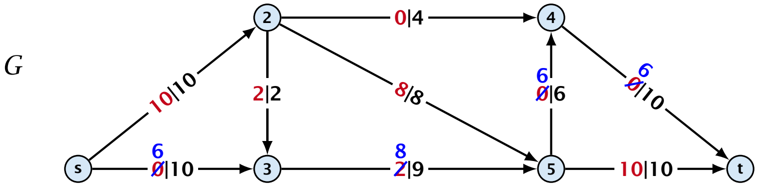

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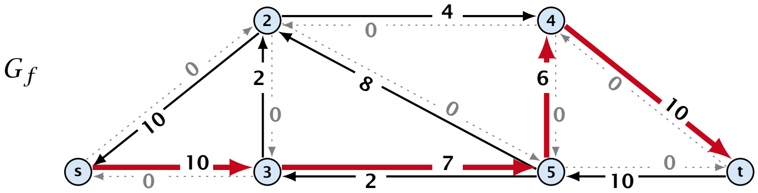

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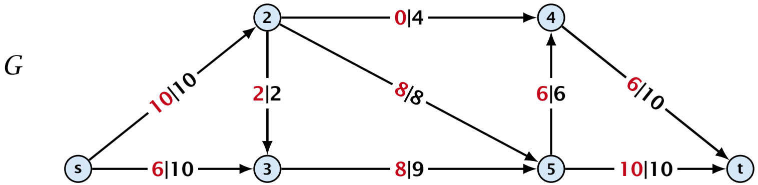

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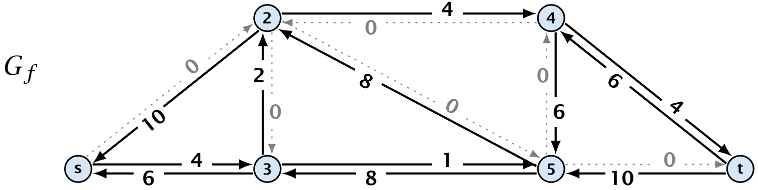

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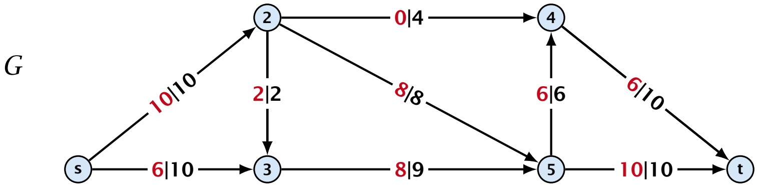

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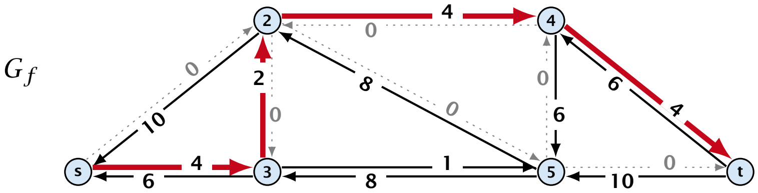

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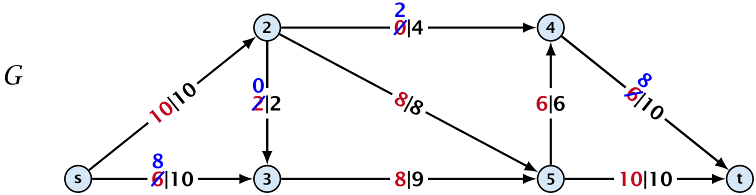
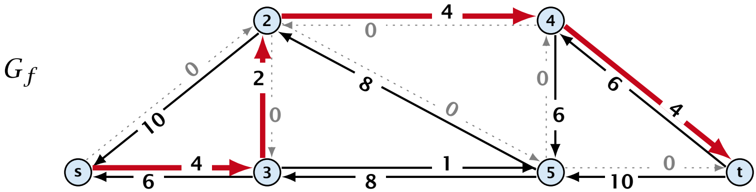


Flow value = 18



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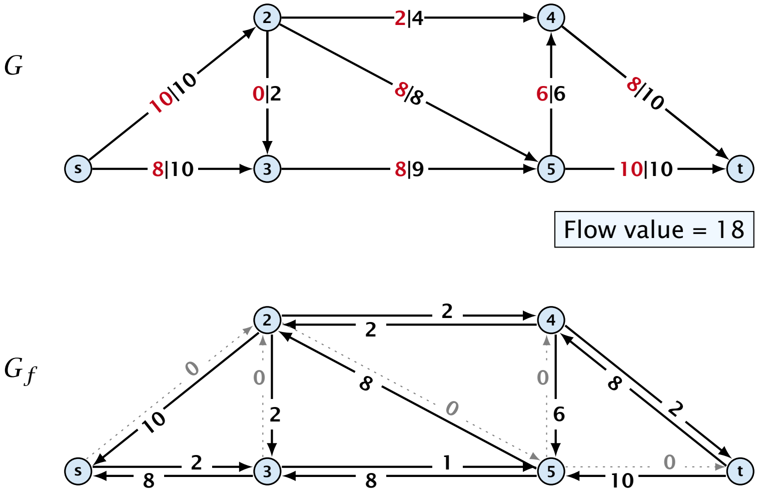

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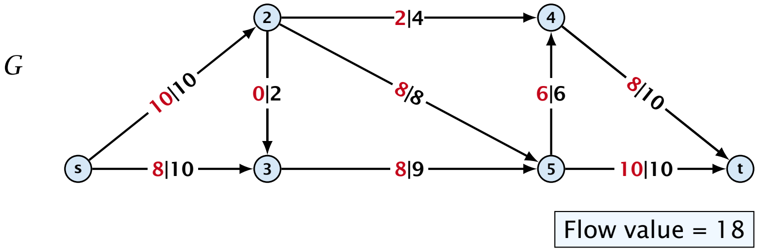

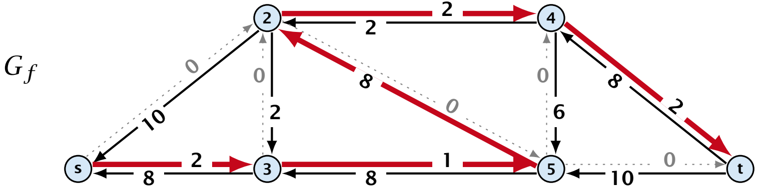

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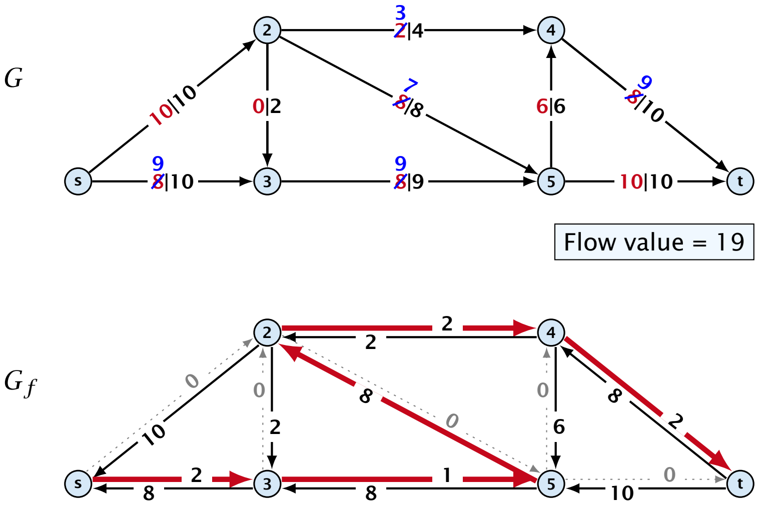

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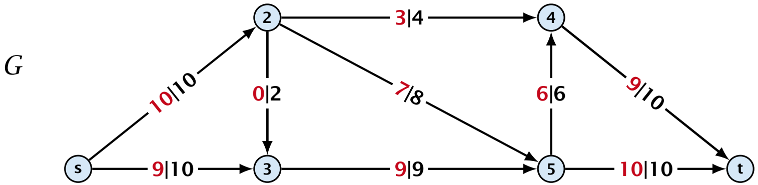

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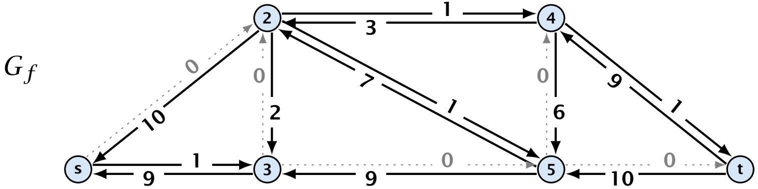

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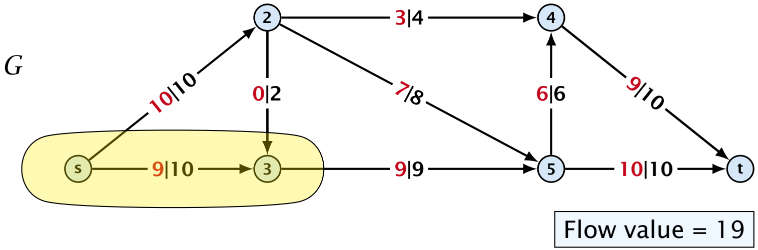

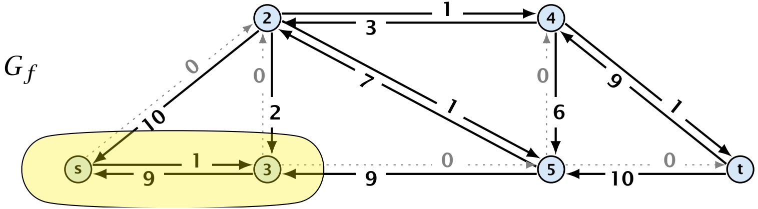

Augmenting Path Algorithm

Theorem 2

*A flow f is a maximum flow **iff** there are no augmenting paths.*

Theorem 3

The value of a maximum flow is equal to the value of a minimum cut.

Proof.

Let f be a flow. The following are equivalent:

1. There exists a cut A, B such that $\text{val}(f) = \text{cap}(A, B)$.
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This finishes the proof.

Here the first equality uses the flow value lemma, and the second exploits the fact that the flow along incoming edges must be 0 as the residual graph does not have edges leaving A.

Analysis

Assumption:
All capacities are integers between 1 and C.

Invariant:
Every flow value $f(e)$ and every residual capacity $c_f(e)$ remains integral troughout the algorithm.

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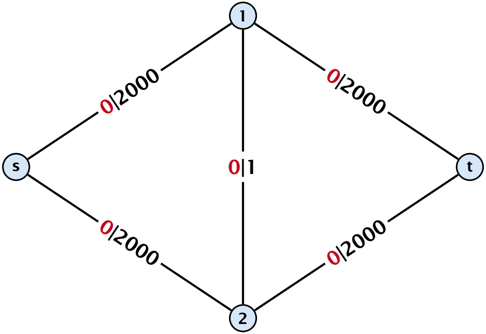
A Bad Input

Problem: The running time may not be polynomial.



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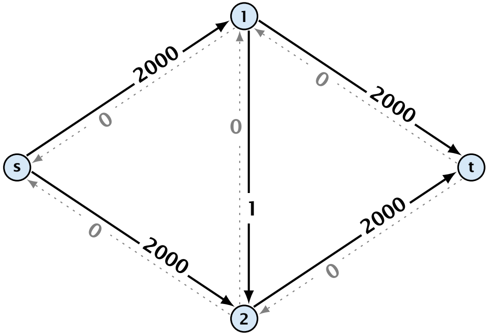

Question:

Can we tweak the algorithm so that the running time is polynomial in the input length?

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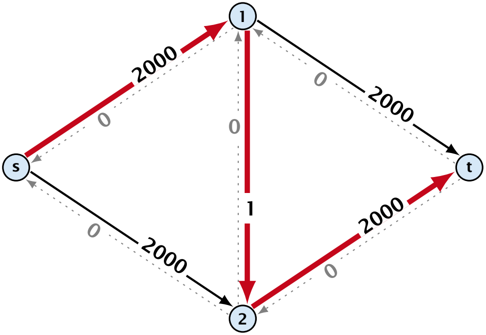

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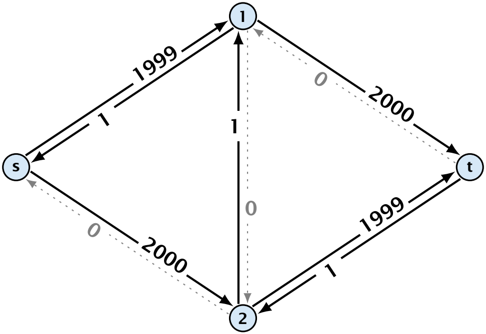

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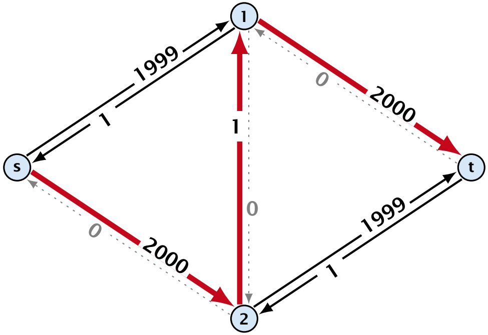

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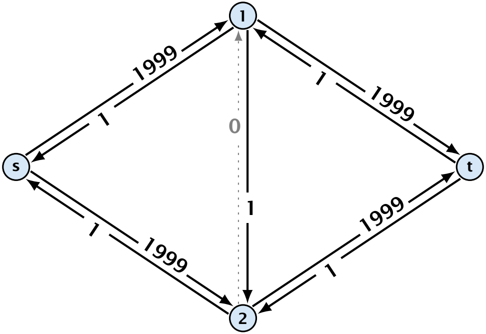

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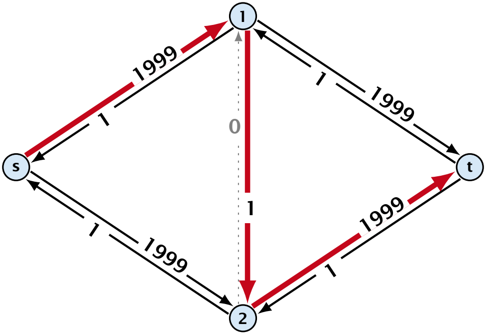

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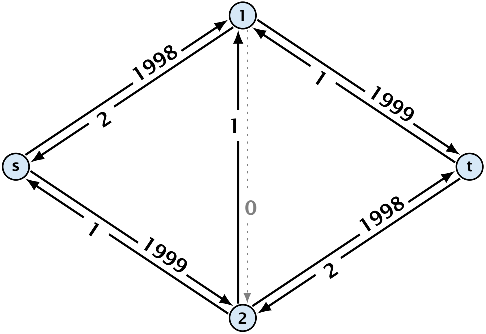

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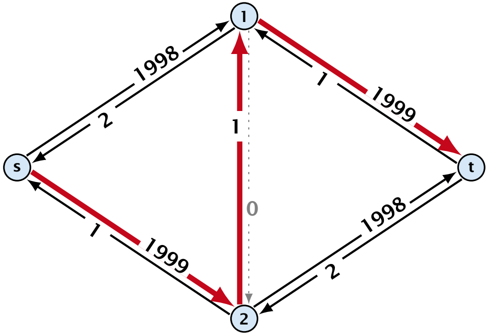

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Can we tweak the algorithm so that the running time is polynomial in the input length?

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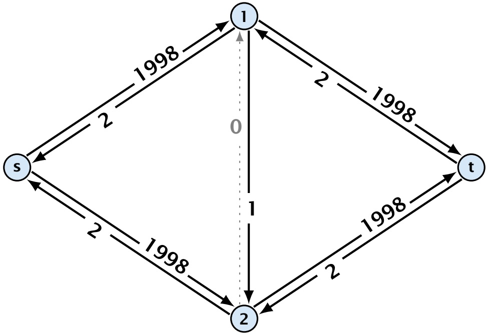
Problem: The running time may not be polynomial.



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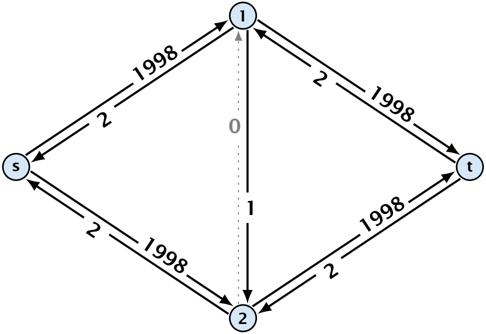

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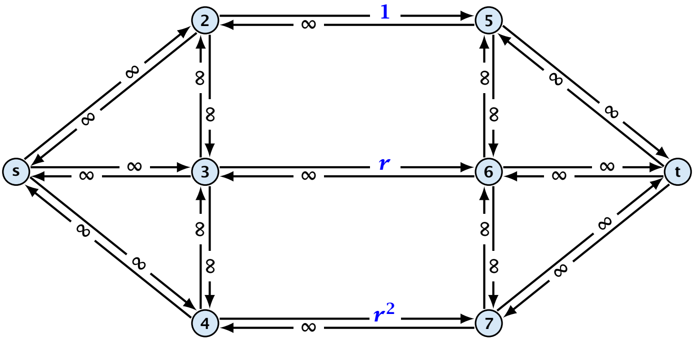
Let $r = \frac{1}{2}(\sqrt{5} - 1)$. Then $r^{n+2} = r^n - r^{n+1}$.



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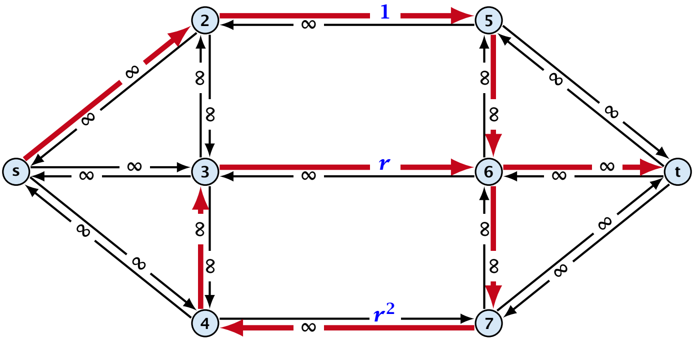

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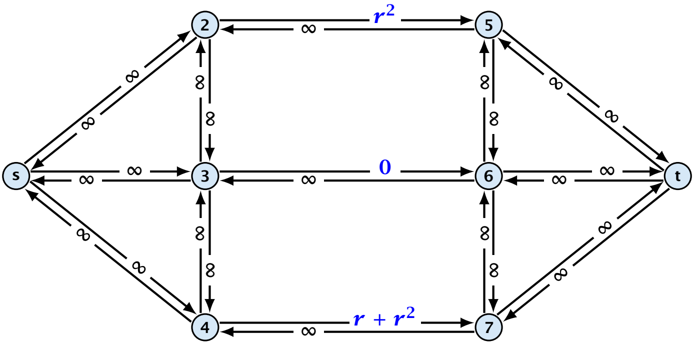

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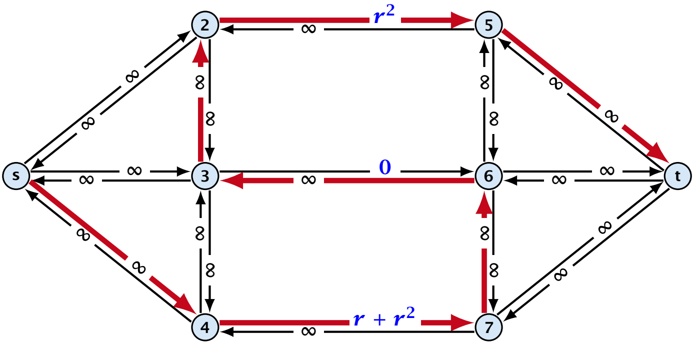

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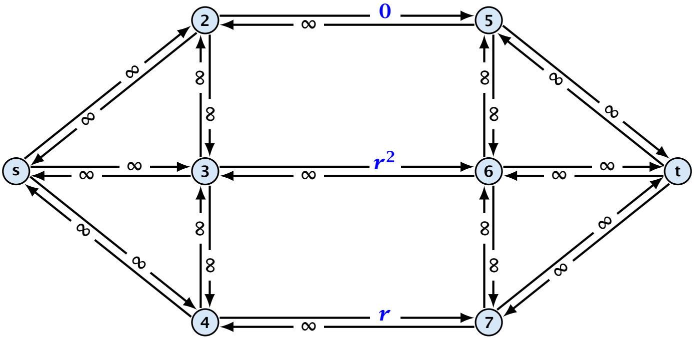

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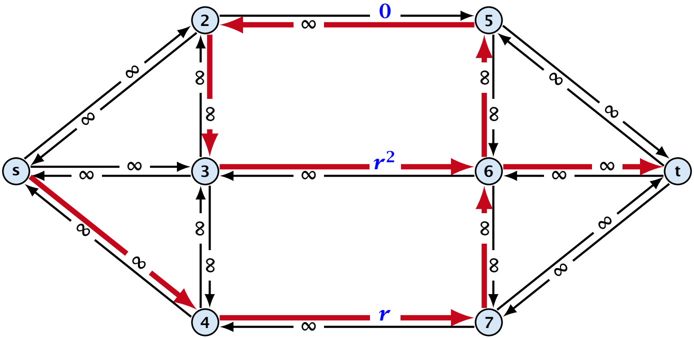

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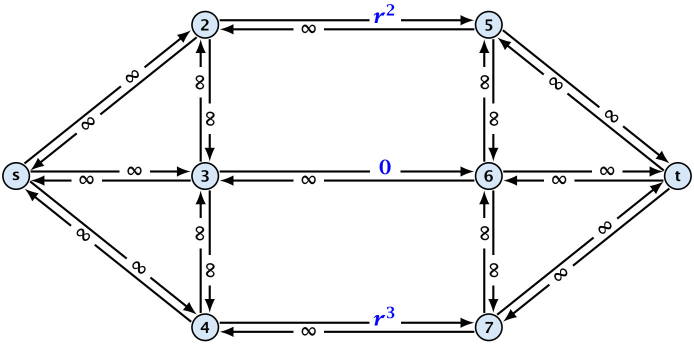

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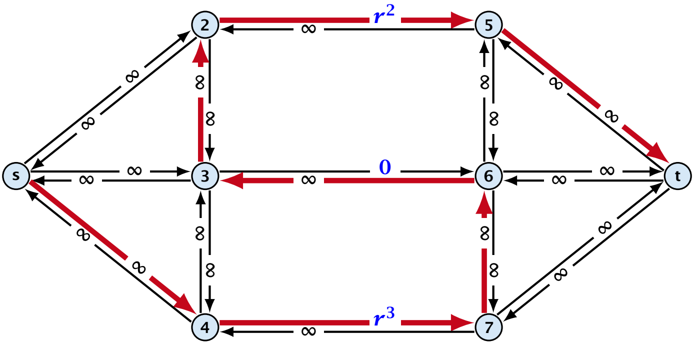

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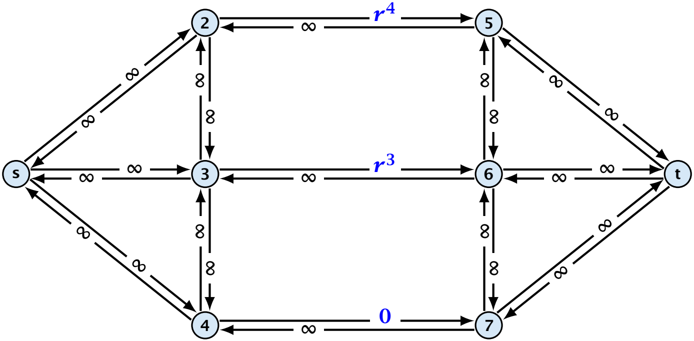

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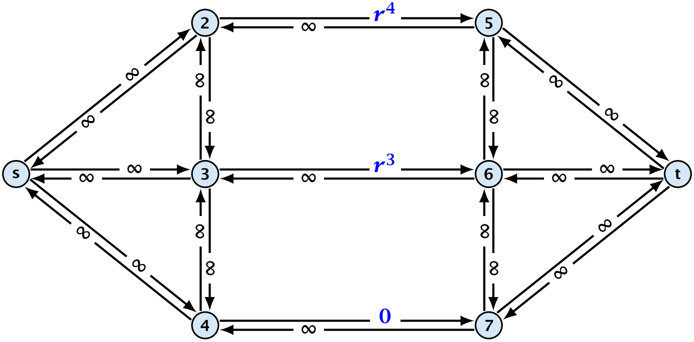

Running time may be infinite!!!

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Several possibilities:

- Choose path with maximum bottleneck capacity.
- Choose path with sufficiently large bottleneck capacity.
- Choose the shortest augmenting path.