Why do we not use a list for implementing the ADT Dynamic Set?

- time for search $\Theta(n)$
- time for insert $\Theta(n)$ (dominated by searching the item)
- ► time for delete Θ(1) if we are given a handle to the object, otw. Θ(n)





How can we improve the search-operation?

Add an express lane:



Let $|L_1|$ denote the number of elements in the "express lane", and $|L_0| = n$ the number of all elements (ignoring dummy elements).

Worst case search time: $|L_1| + \frac{|L_0|}{|L_1|}$ (ignoring additive constants)

Choose $|L_1| = \sqrt{n}$. Then search time $\Theta(\sqrt{n})$.

Add more express lanes. Lane L_i contains roughly every $\frac{L_{i-1}}{L_i}$ -th item from list L_{i-1} .

Search(x) (k + 1 lists L_0, \ldots, L_k)

- ► Find the largest item in list L_k that is smaller than x. At most |L_k| + 2 steps.
- ▶ Find the largest item in list L_{k-1} that is smaller than x. At most $\left\lfloor \frac{|L_{k-1}|}{|L_k|+1} \right\rfloor + 2$ steps.
- ► Find the largest item in list L_{k-2} that is smaller than x. At most [|L_{k-2}|/|L_{k-1}|+1] + 2 steps.
- ▶ ...
- At most $|L_k| + \sum_{i=1}^k \frac{L_{i-1}}{L_i} + 3(k+1)$ steps.



Choose ratios between list-lengths evenly, i.e., $\frac{|L_{i-1}|}{|L_i|} = r$, and, hence, $L_k \approx r^{-k}n$.

Worst case running time is: $\mathcal{O}(r^{-k}n + kr)$. Choose $r = n^{\frac{1}{k+1}}$. Then

$$r^{-k}n + kr = \left(n^{\frac{1}{k+1}}\right)^{-k}n + kn^{\frac{1}{k+1}}$$
$$= n^{1-\frac{k}{k+1}} + kn^{\frac{1}{k+1}}$$
$$= (k+1)n^{\frac{1}{k+1}} .$$

Choosing $k = \Theta(\log n)$ gives a logarithmic running time.



How to do insert and delete?

If we want that in L_i we always skip over roughly the same number of elements in L_{i-1} an insert or delete may require a lot of re-organisation.

Use randomization instead!



Insert:

- A search operation gives you the insert position for element x in every list.
- Flip a coin until it shows head, and record the number $t \in \{1, 2, ...\}$ of trials needed.
- lnsert x into lists L_0, \ldots, L_{t-1} .

Delete:

- > You get all predecessors via backward pointers.
- Delete *x* in all lists it actually appears in.

The time for both operations is dominated by the search time.



Insert (35):





Definition 1 (High Probability)

We say a **randomized** algorithm has running time $O(\log n)$ with high probability if for any constant α the running time is at most $O(\log n)$ with probability at least $1 - \frac{1}{n^{\alpha}}$.

Here the O-notation hides a constant that may depend on α .



High Probability

Suppose there are a polynomially many events $E_1, E_2, ..., E_\ell$, $\ell = n^c$ each holding with high probability (e.g. E_i may be the event that the *i*-th search in a skip list takes time at most $O(\log n)$).

Then the probability that all E_i hold is at least

$$\Pr[E_1 \wedge \cdots \wedge E_{\ell}] = 1 - \Pr[\bar{E}_1 \vee \cdots \vee \bar{E}_{\ell}]$$

$$\geq 1 - n^c \cdot n^{-\alpha}$$

$$= 1 - n^{c-\alpha} .$$

This means $Pr[E_1 \land \cdots \land E_\ell]$ holds with high probability.



Lemma 2

A search (and, hence, also insert and delete) in a skip list with n elements takes time O(logn) with high probability (w. h. p.).





At each point the path goes up with probability 1/2 and left with probability 1/2.

We show that w.h.p:

- A "long" search path must also go very high.
- There are no elements in high lists.

From this it follows that w.h.p. there are no long paths.



$$\left(\frac{n}{k}\right)^k \le \binom{n}{k} \le \left(\frac{en}{k}\right)^k$$

$$\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!} = \frac{n \cdot \ldots \cdot (n-k+1)}{k \cdot \ldots \cdot 1} \ge \left(\frac{n}{k}\right)^k$$

$$\binom{n}{k} = \frac{n \cdot \ldots \cdot (n-k+1)}{k!} \le \frac{n^k}{k!} = \frac{n^k \cdot k^k}{k^k \cdot k!}$$
$$= \left(\frac{n}{k}\right)^k \cdot \frac{k^k}{k!} \le \left(\frac{en}{k}\right)^k$$



Let $E_{z,k}$ denote the event that a search path is of length z (number of edges) but does not visit a list above L_k .

In particular, this means that during the construction in the backward analysis we see at most k heads (i.e., coin flips that tell you to go up) in z trials.



 $\Pr[E_{z,k}] \le \Pr[\text{at most } k \text{ heads in } z \text{ trials}]$

$$\leq \binom{z}{k} 2^{-(z-k)} \leq \left(\frac{ez}{k}\right)^k 2^{-(z-k)} \leq \left(\frac{2ez}{k}\right)^k 2^{-z}$$

choosing $k = \gamma \log n$ with $\gamma \ge 1$ and $z = (\beta + \alpha)\gamma \log n$

$$\leq \left(\frac{2ez}{k}\right)^{k} 2^{-\beta k} \cdot n^{-\gamma \alpha} \leq \left(\frac{2ez}{2^{\beta}k}\right)^{k} \cdot n^{-\alpha}$$
$$\leq \left(\frac{2e(\beta + \alpha)}{2^{\beta}}\right)^{k} n^{-\alpha}$$

now choosing $\beta = 6\alpha$ gives

$$\leq \left(\frac{42\alpha}{64^{\alpha}}\right)^k n^{-\alpha} \leq n^{-\alpha}$$

for $\alpha \geq 1$.



So far we fixed $k = \gamma \log n$, $\gamma \ge 1$, and $z = 7\alpha \gamma \log n$, $\alpha \ge 1$.

This means that a search path of length $\Omega(\log n)$ visits a list on a level $\Omega(\log n)$, w.h.p.

Let A_{k+1} denote the event that the list L_{k+1} is non-empty. Then

$$\Pr[A_{k+1}] \le n2^{-(k+1)} \le n^{-(\gamma-1)}$$

For the search to take at least $z = 7\alpha \gamma \log n$ steps either the event $E_{z,k}$ or the even A_{k+1} must hold. Hence,

 $\begin{aligned} &\Pr[\text{search requires } z \text{ steps}] \leq \Pr[E_{z,k}] + \Pr[A_{k+1}] \\ &\leq n^{-\alpha} + n^{-(\gamma-1)} \end{aligned}$

This means, the search requires at most *z* steps, w. h. p.

Skip Lists

Bibliography

[GT98] Michael T. Goodrich, Roberto Tamassia Data Structures and Algorithms in JAVA, John Wiley, 1998

Skip lists are covered in Chapter 7.5 of [GT98].

