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### Several possibilities:

- ▶ Choose path with maximum bottleneck capacity.
- ▶ Choose path with sufficiently large bottleneck capacity.
- ▶ Choose the shortest augmenting path.

# Capacity Scaling

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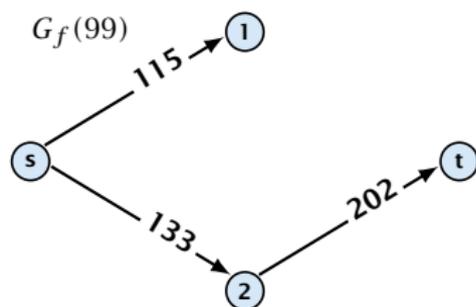
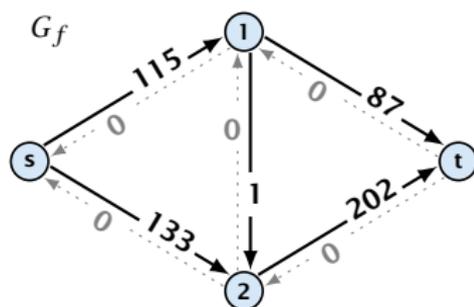
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## Algorithm 2 maxflow( $G, s, t, c$ )

```
1: foreach  $e \in E$  do  $f_e \leftarrow 0$ ;  
2:  $\Delta \leftarrow 2^{\lceil \log_2 C \rceil}$   
3: while  $\Delta \geq 1$  do  
4:    $G_f(\Delta) \leftarrow \Delta$ -residual graph  
5:   while there is augmenting path  $P$  in  $G_f(\Delta)$  do  
6:      $f \leftarrow \text{augment}(f, c, P)$   
7:      $\text{update}(G_f(\Delta))$   
8:    $\Delta \leftarrow \Delta/2$   
9: return  $f$ 
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- ▶ therefore after the last phase there are no augmenting paths anymore
- ▶ this means we have a maximum flow.

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- ▶ In  $G_f$  this cut can have capacity at most  $m\Delta$ .
- ▶ This gives me an upper bound on the flow that I can still add.

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## Theorem 4

*We need  $\mathcal{O}(m \log C)$  augmentations. The algorithm can be implemented in time  $\mathcal{O}(m^2 \log C)$ .*