

7 Dictionary

Dictionary:

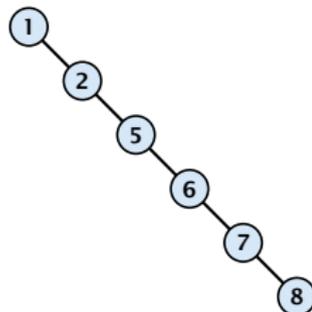
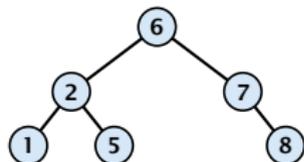
- ▶ **S . insert(x)**: Insert an element x .
- ▶ **S . delete(x)**: Delete the element pointed to by x .
- ▶ **S . search(k)**: Return a pointer to an element e with $\text{key}[e] = k$ in S if it exists; otherwise return **null**.

7.1 Binary Search Trees

An (**internal**) **binary search tree** stores the elements in a binary tree. Each tree-node corresponds to an element. All elements in the left sub-tree of a node v have a smaller key-value than $\text{key}[v]$ and elements in the right sub-tree have a larger-key value. We assume that all key-values are different.

(**External** Search Trees store objects only at leaf-vertices)

Examples:

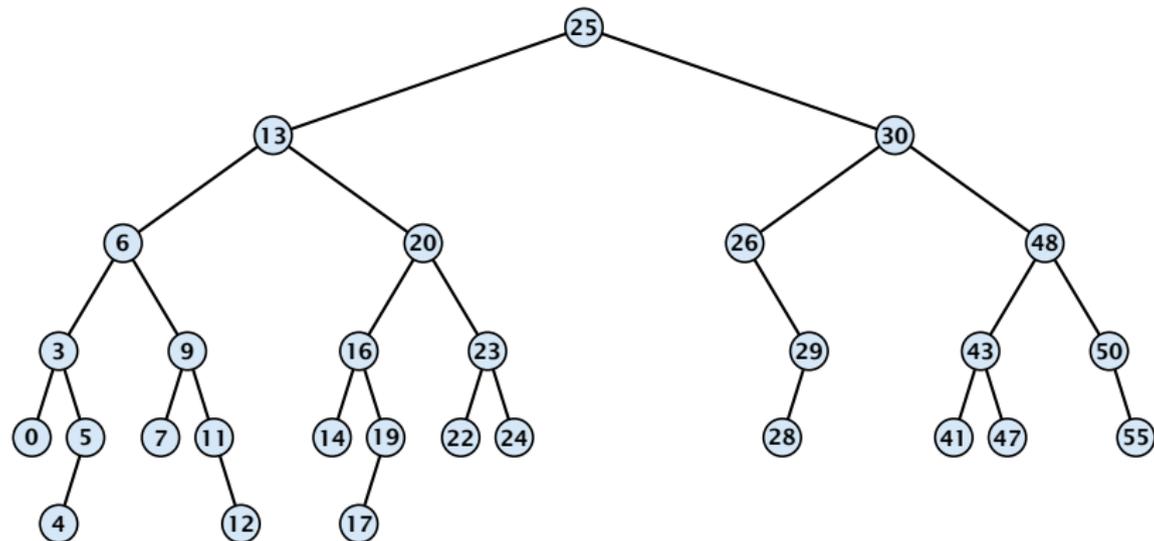


7.1 Binary Search Trees

We consider the following operations on binary search trees. Note that this is a super-set of the dictionary-operations.

- ▶ $T.\text{insert}(x)$
- ▶ $T.\text{delete}(x)$
- ▶ $T.\text{search}(k)$
- ▶ $T.\text{successor}(x)$
- ▶ $T.\text{predecessor}(x)$
- ▶ $T.\text{minimum}()$
- ▶ $T.\text{maximum}()$

Binary Search Trees: Searching

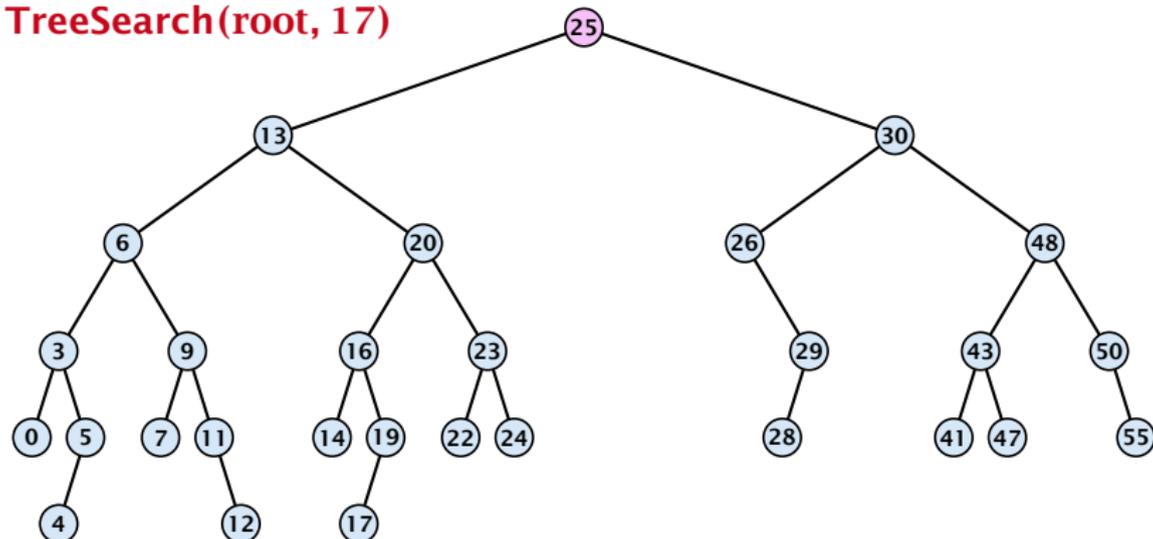


Algorithm 5 TreeSearch(x, k)

- 1: **if** $x = \text{null}$ **or** $k = \text{key}[x]$ **return** x
- 2: **if** $k < \text{key}[x]$ **return** TreeSearch(left[x], k)
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Binary Search Trees: Searching

TreeSearch(root, 17)

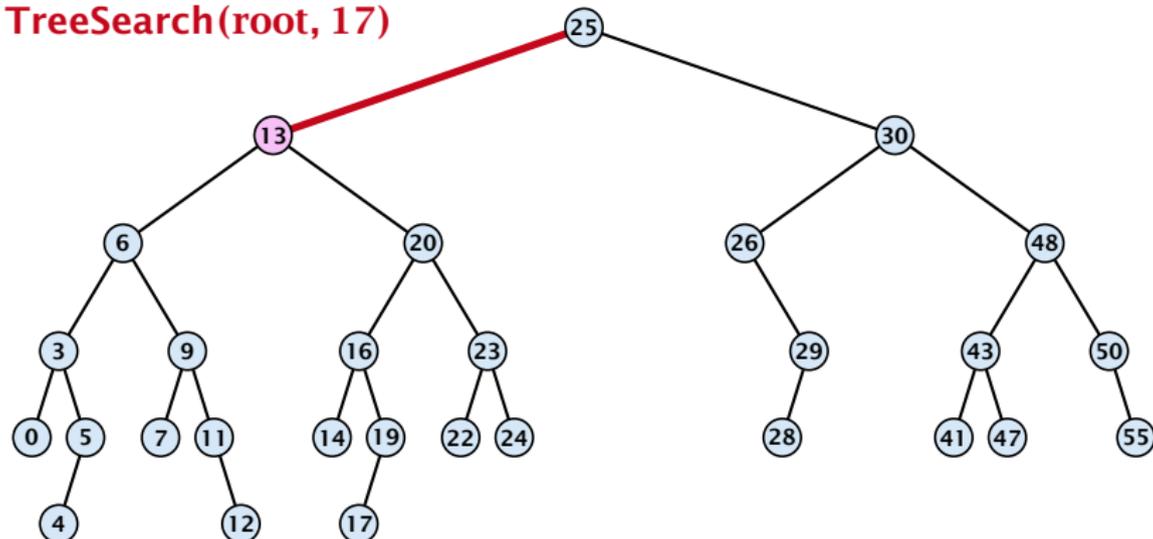


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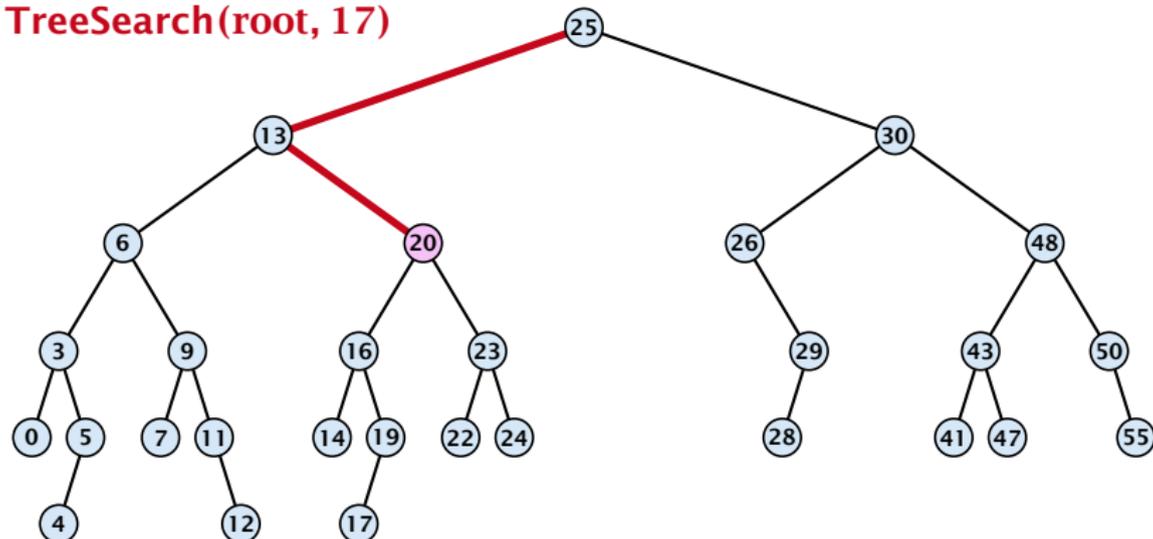


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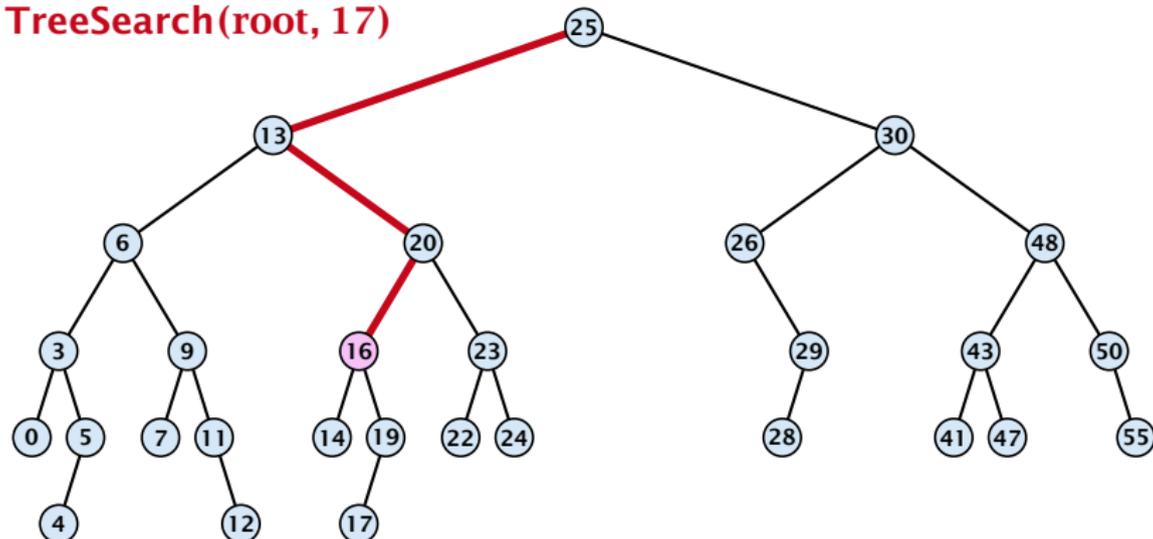


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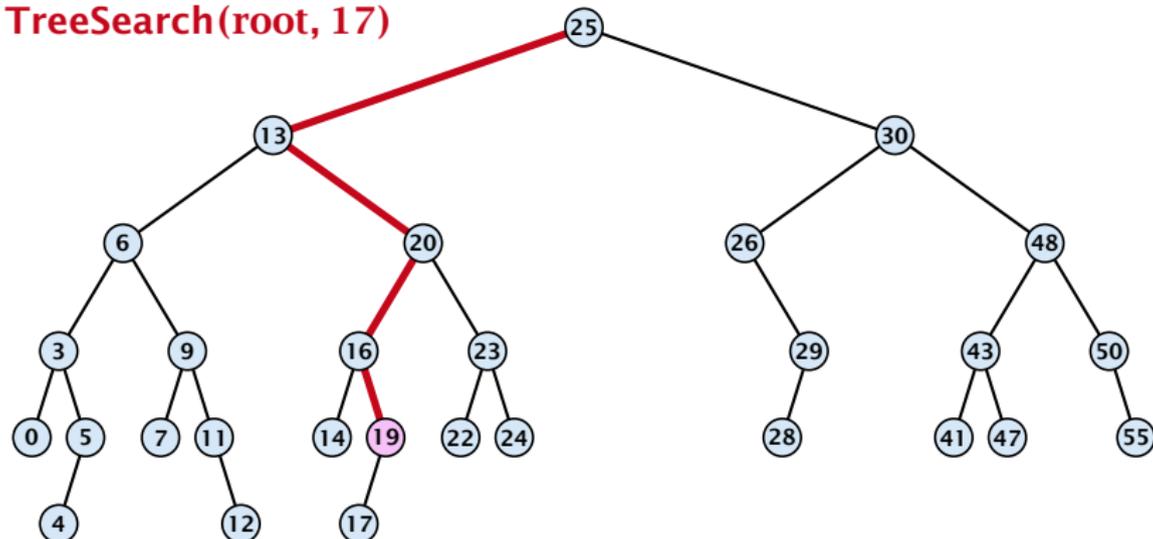


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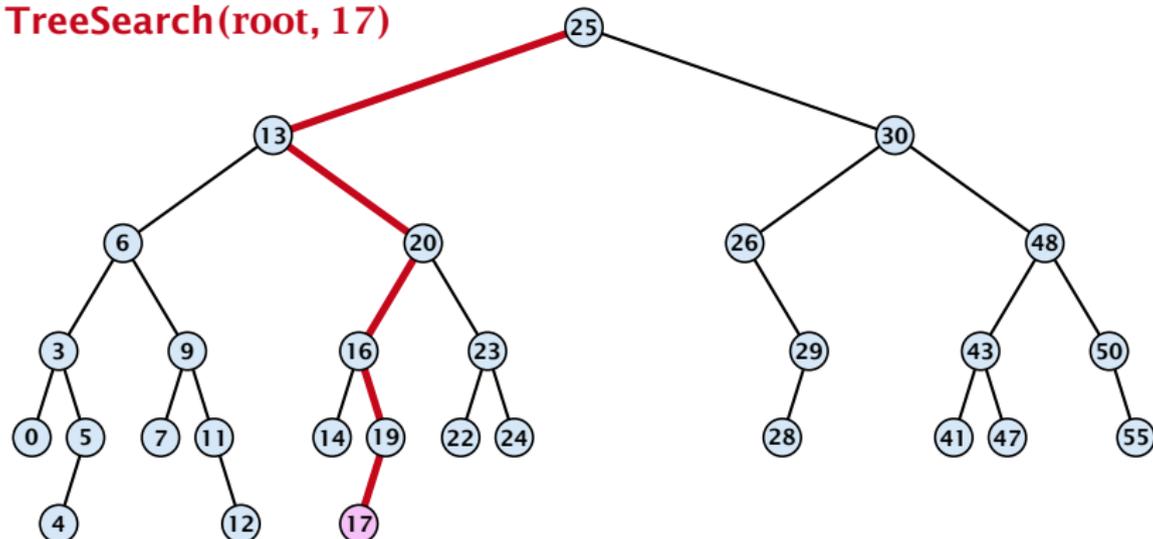


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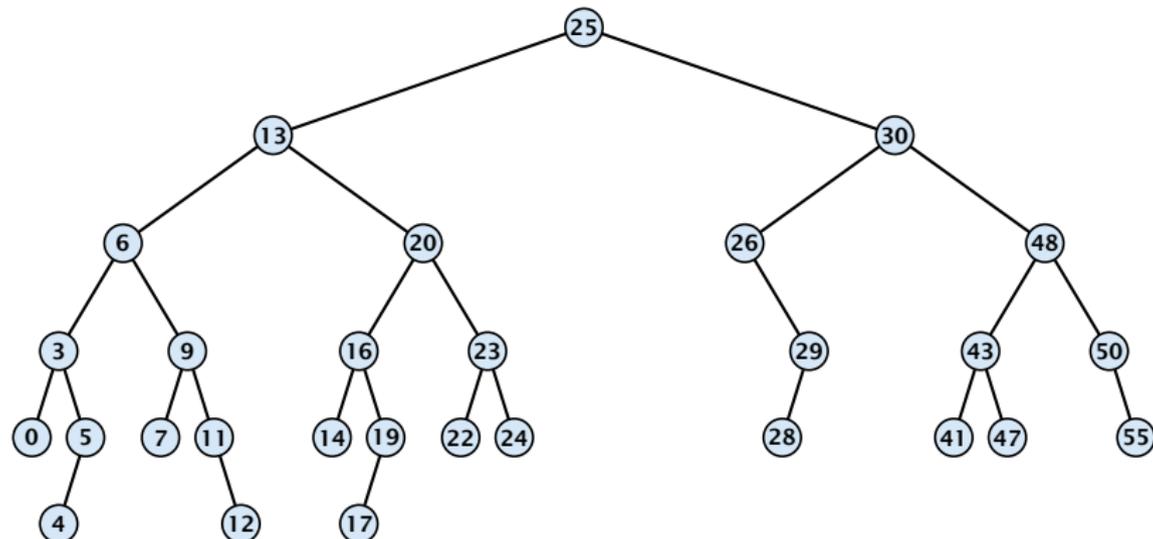
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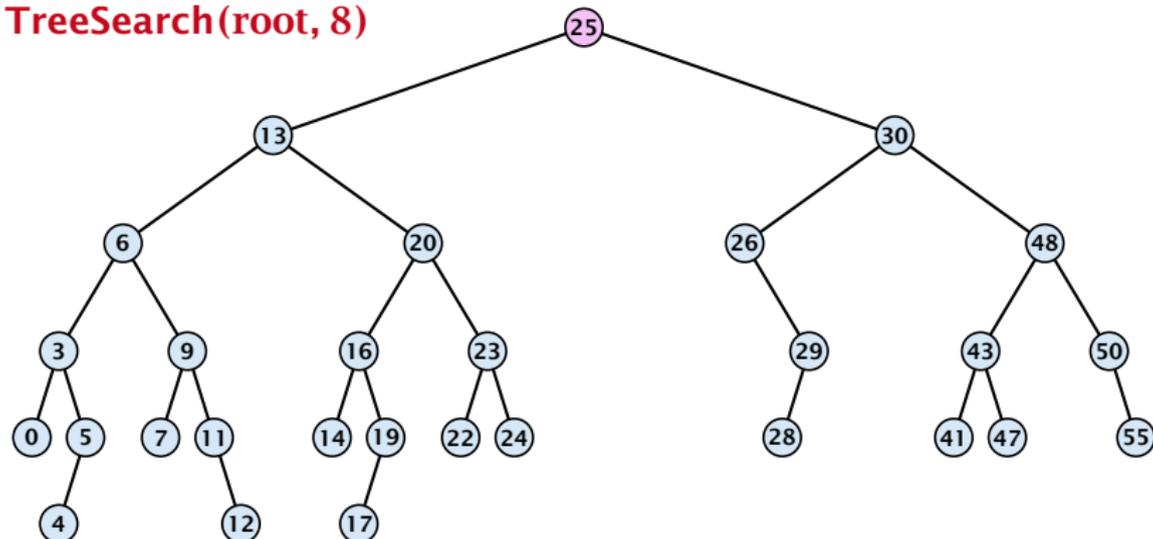


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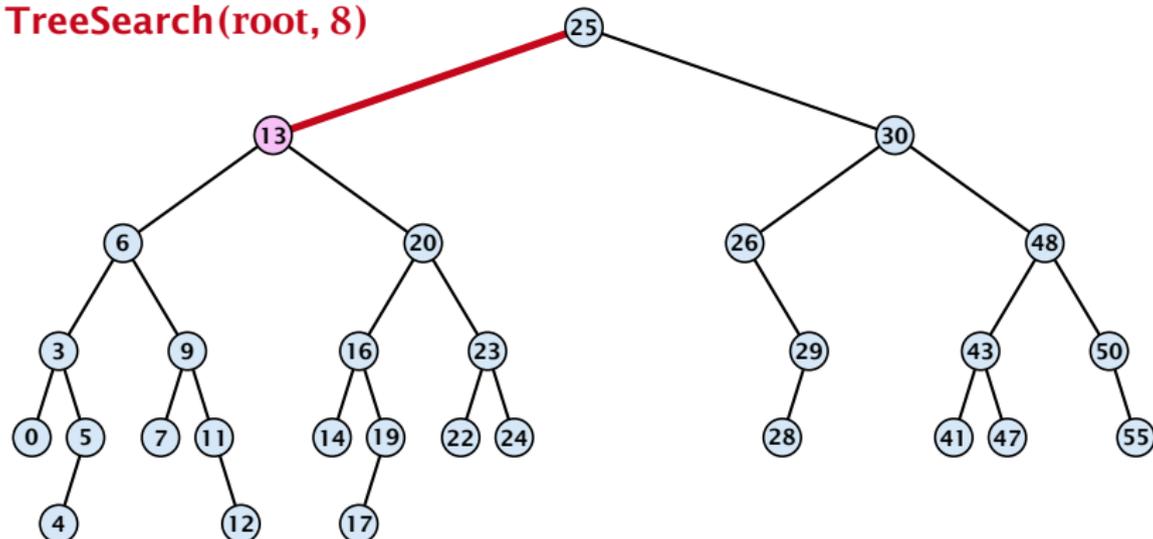


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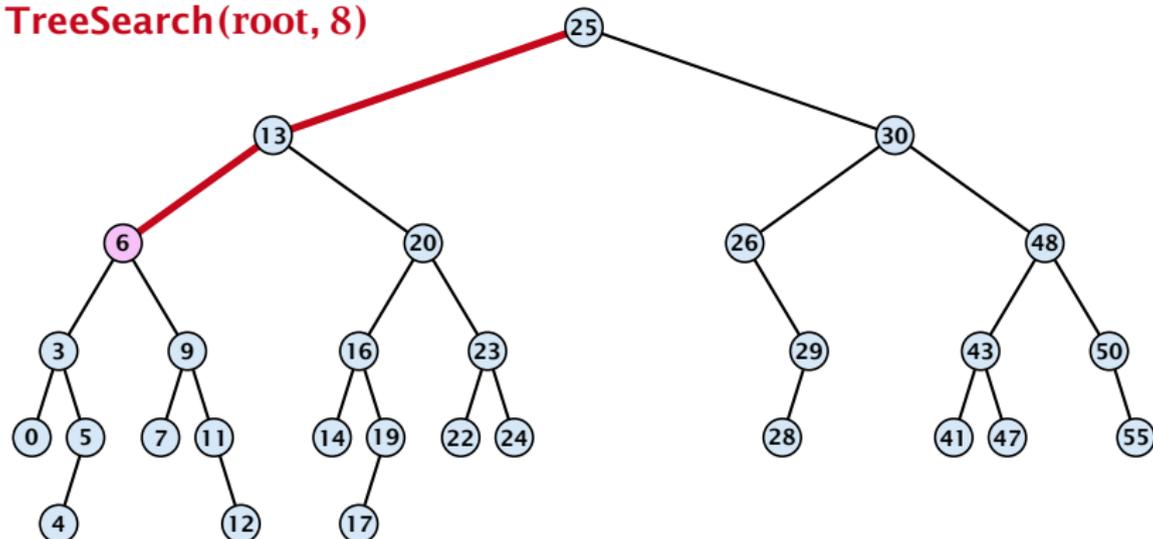


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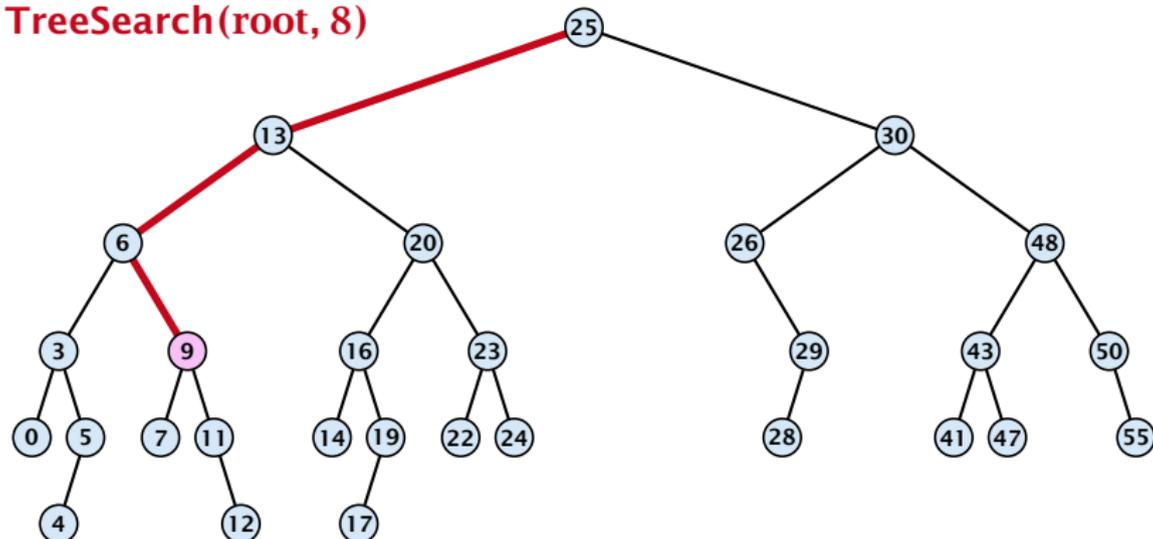


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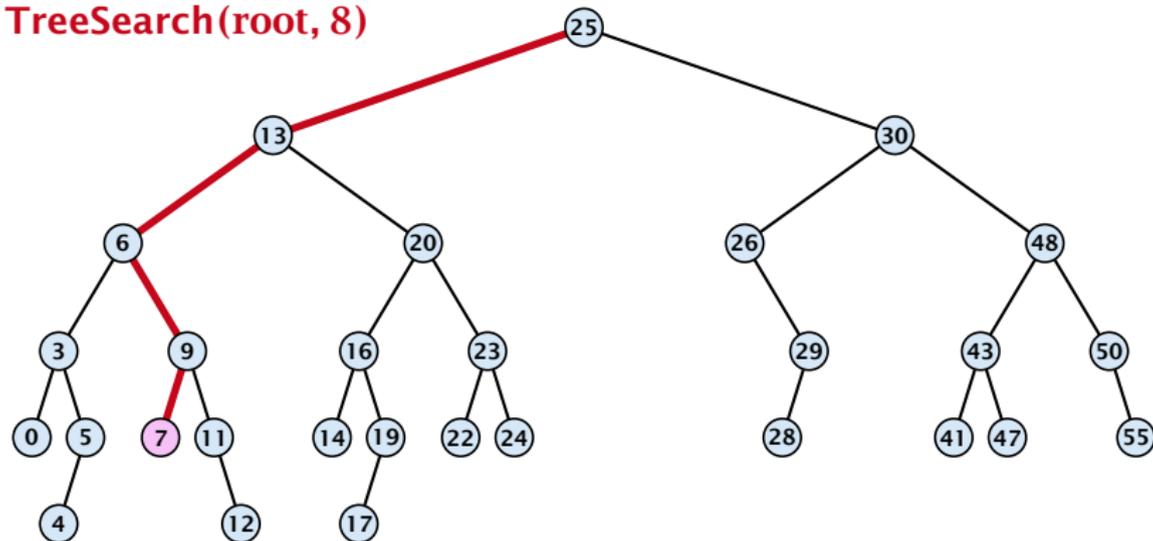


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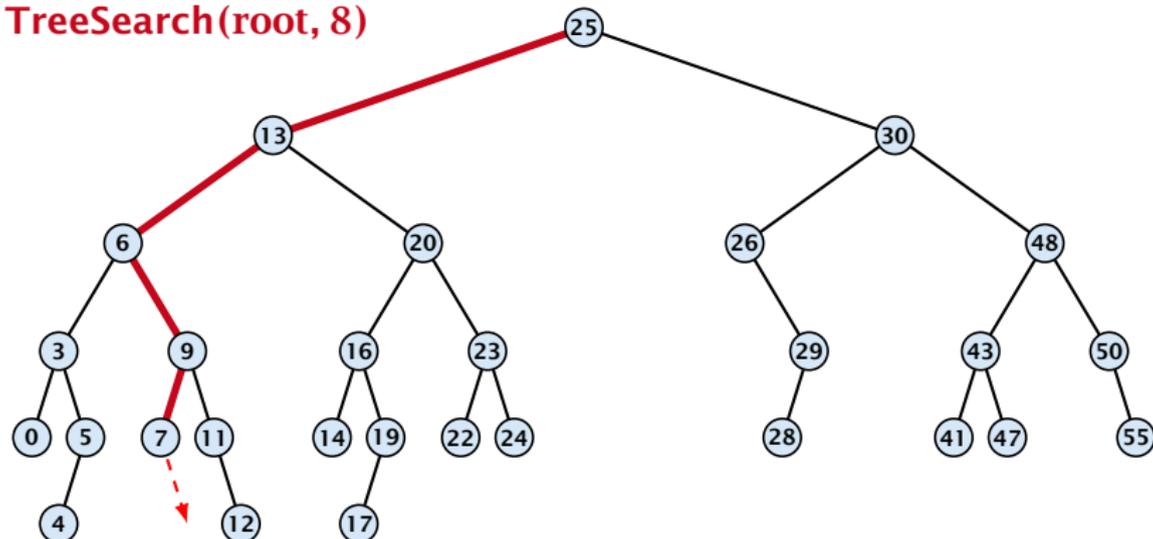


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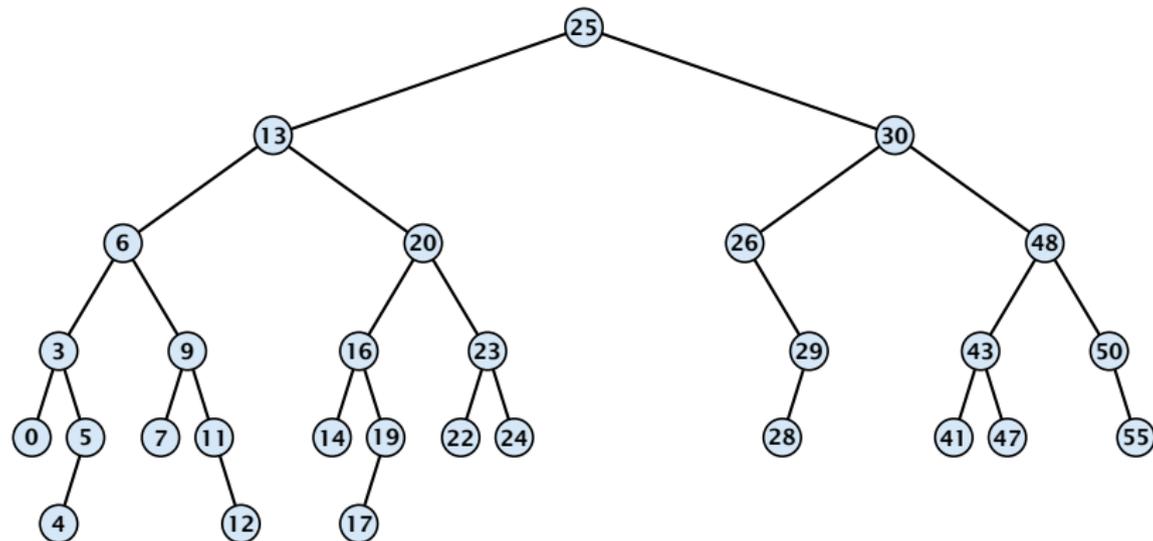
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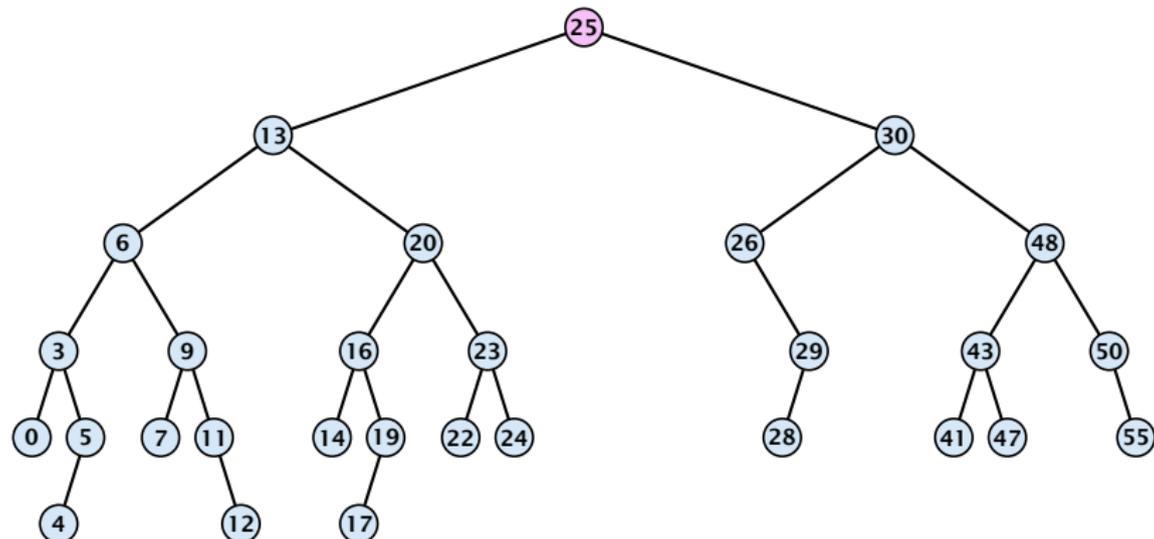
Binary Search Trees: Minimum



Algorithm 6 TreeMin(x)

- 1: **if** $x = \text{null}$ **or** $\text{left}[x] = \text{null}$ **return** x
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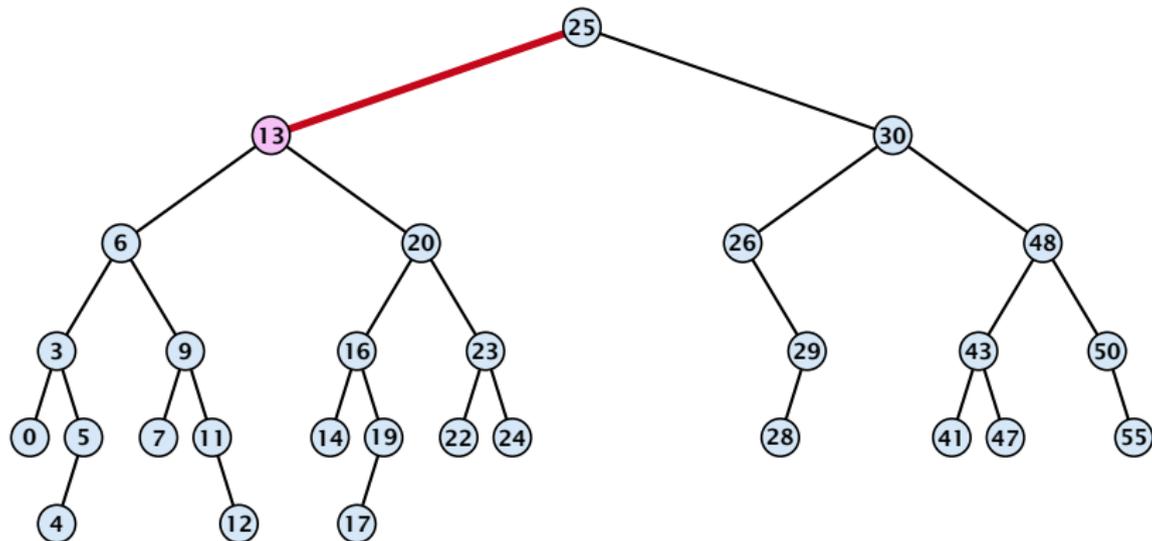
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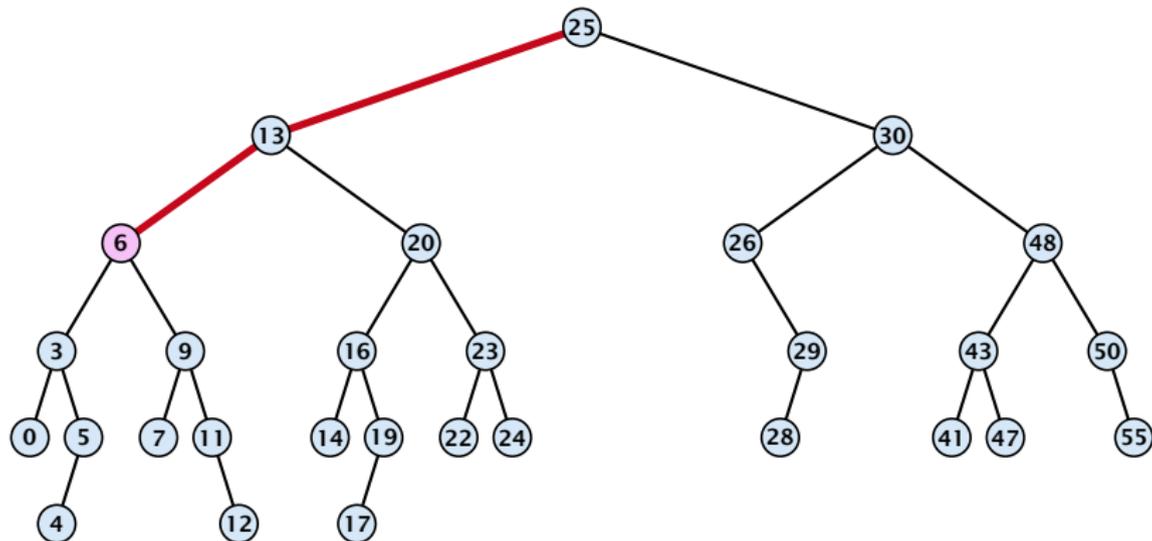
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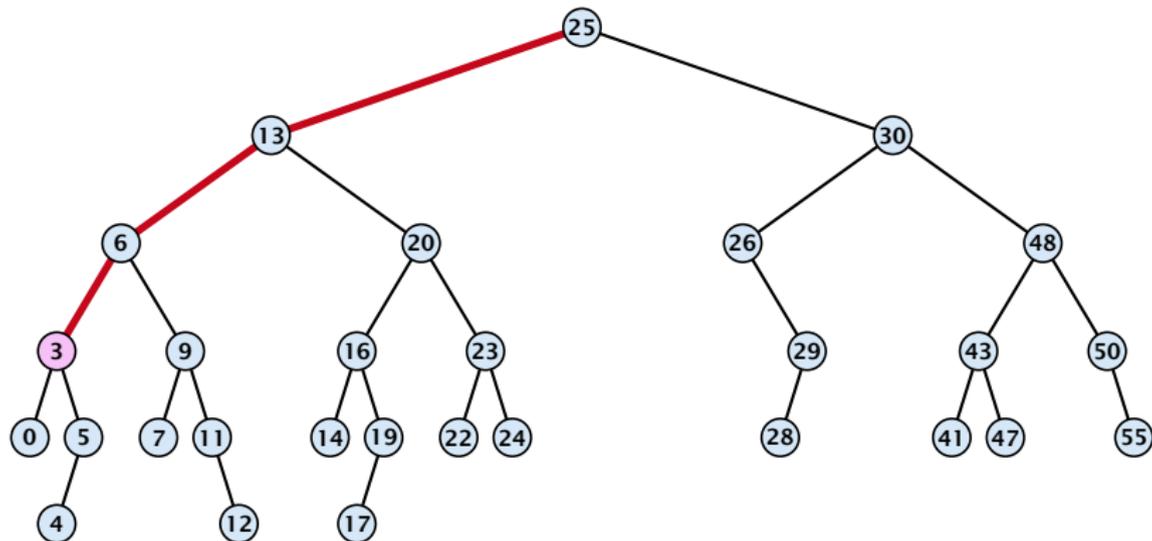
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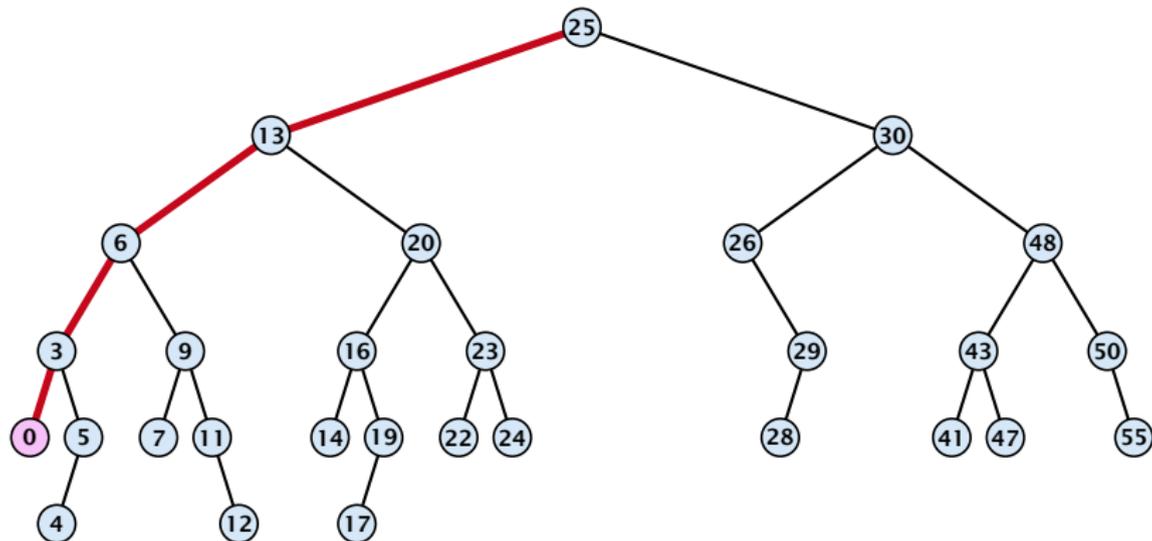
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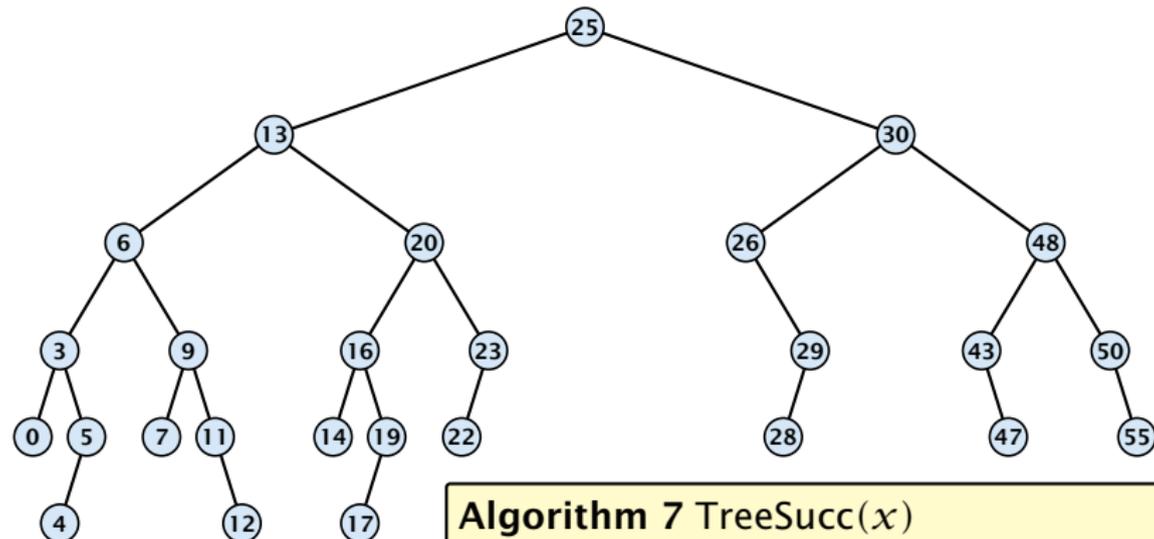
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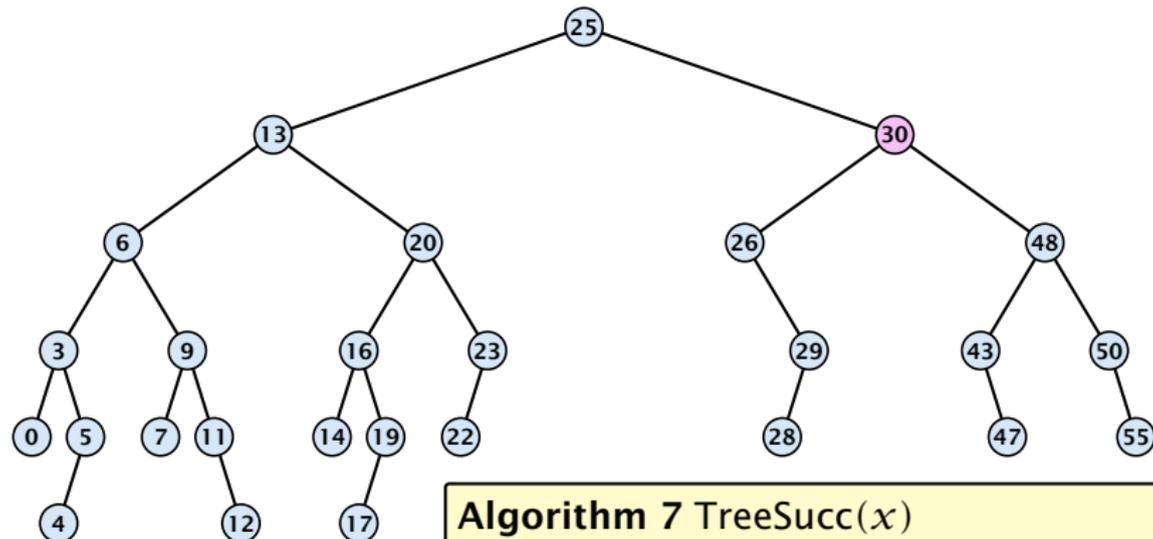
Binary Search Trees: Successor



Algorithm 7 TreeSucc(x)

- 1: **if** $\text{right}[x] \neq \text{null}$ **return** $\text{TreeMin}(\text{right}[x])$
- 2: $y \leftarrow \text{parent}[x]$
- 3: **while** $y \neq \text{null}$ **and** $x = \text{right}[y]$ **do**
- 4: $x \leftarrow y; y \leftarrow \text{parent}[x]$
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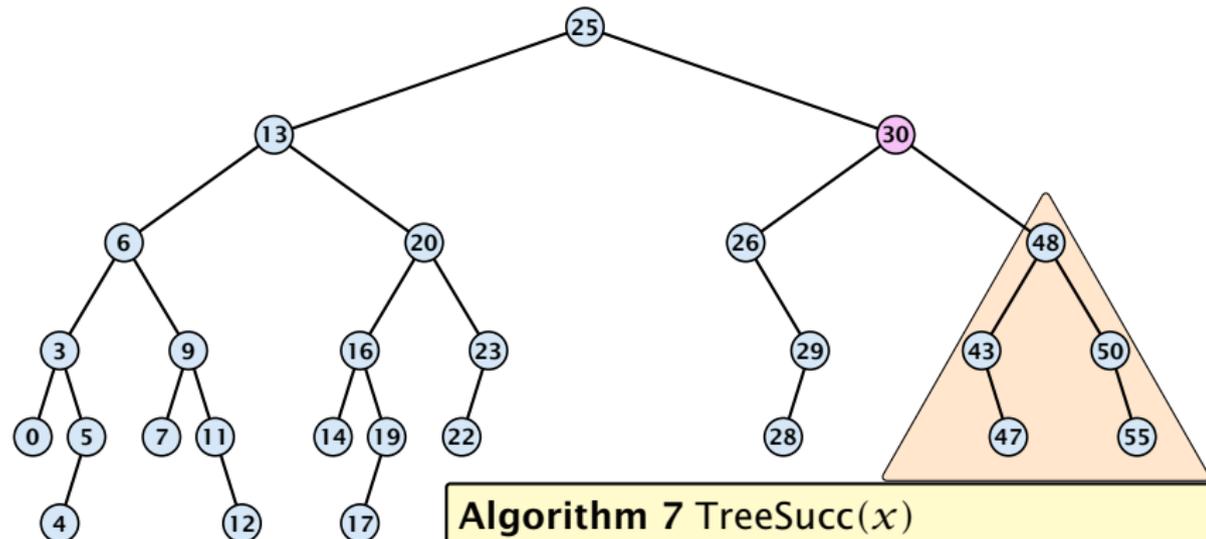
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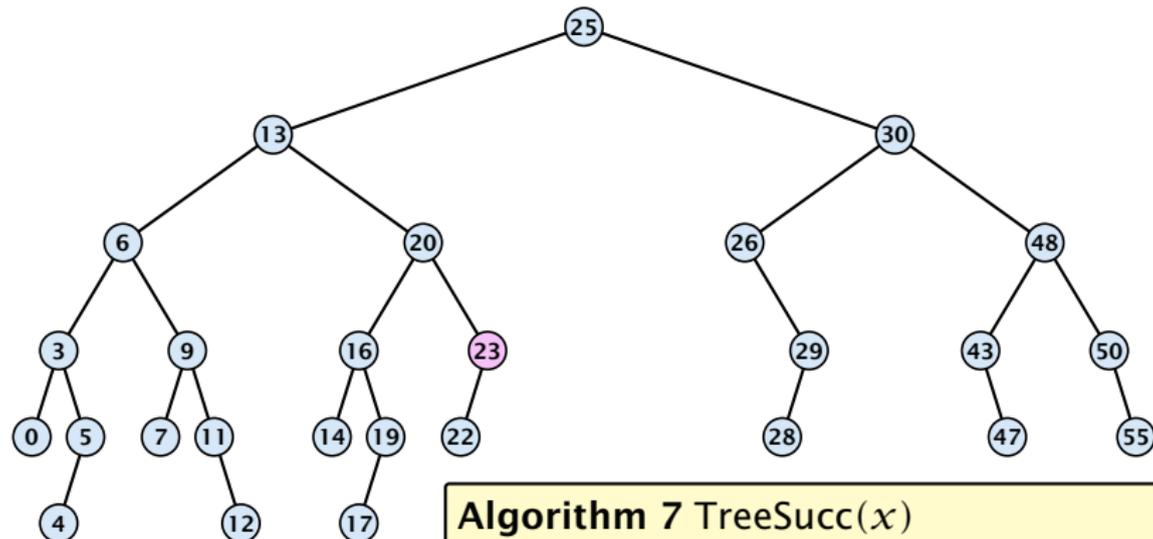
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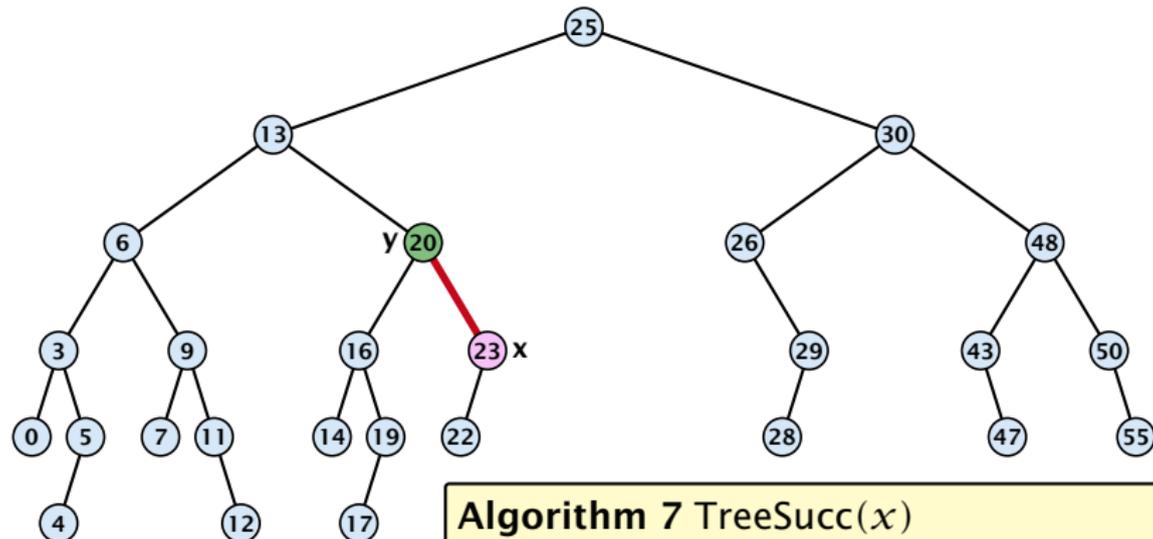
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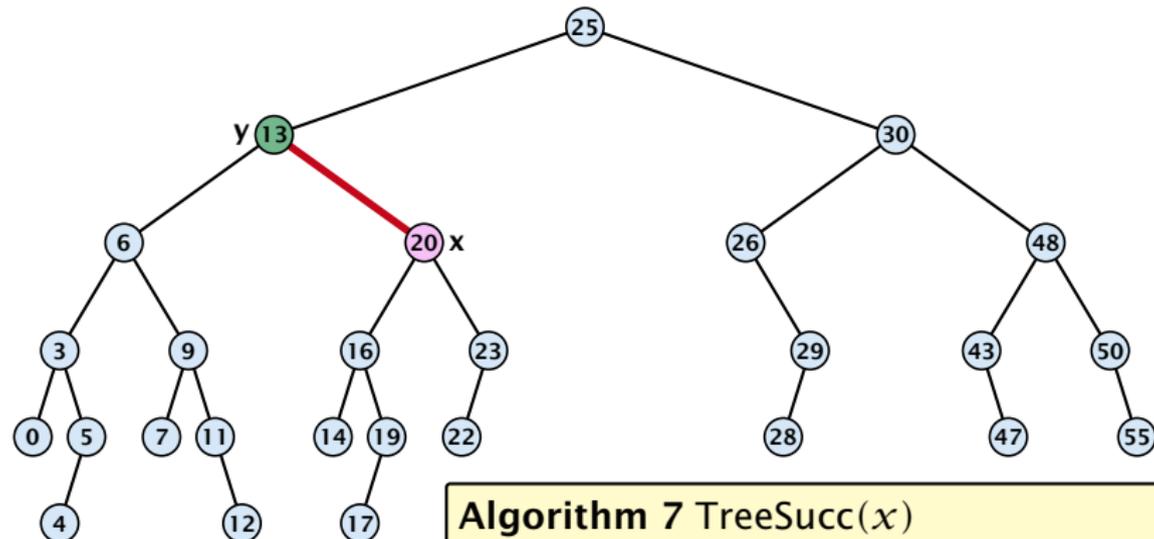
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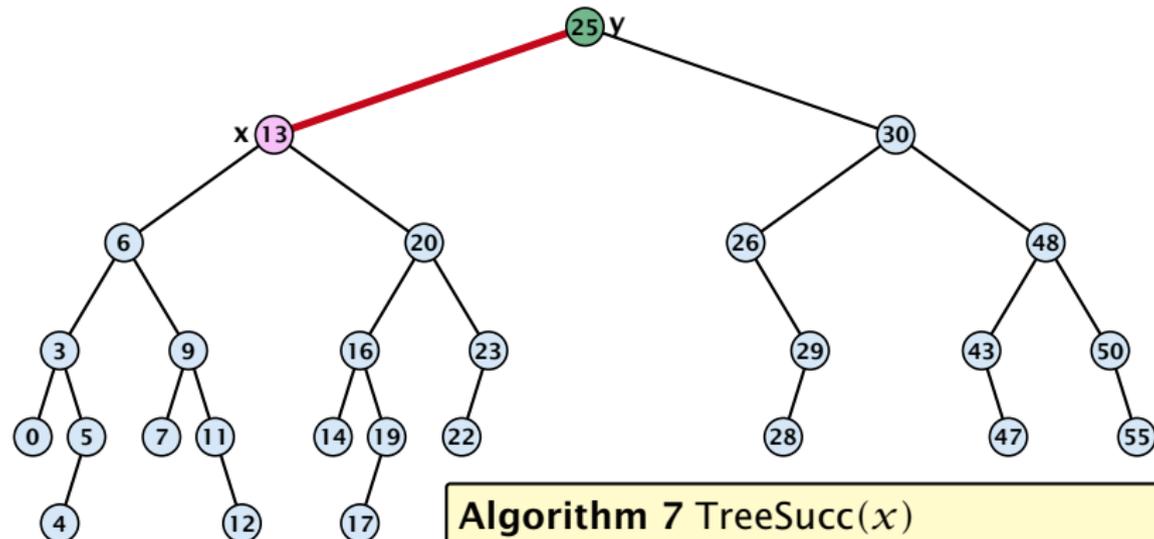
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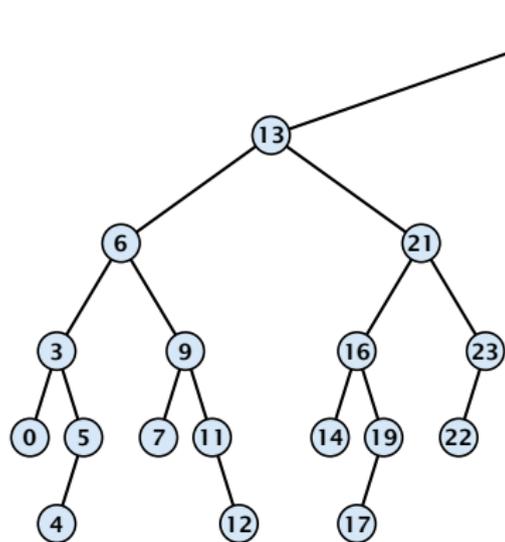
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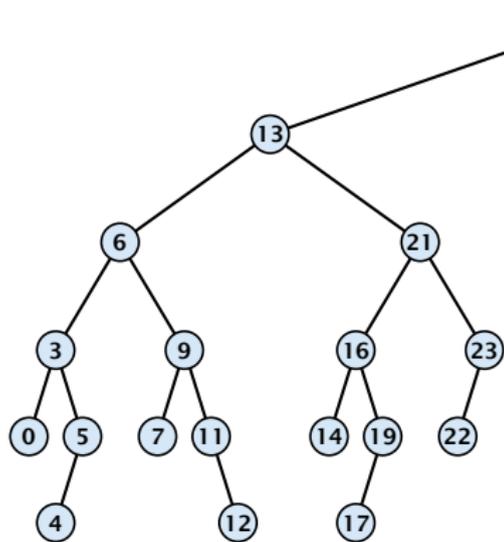


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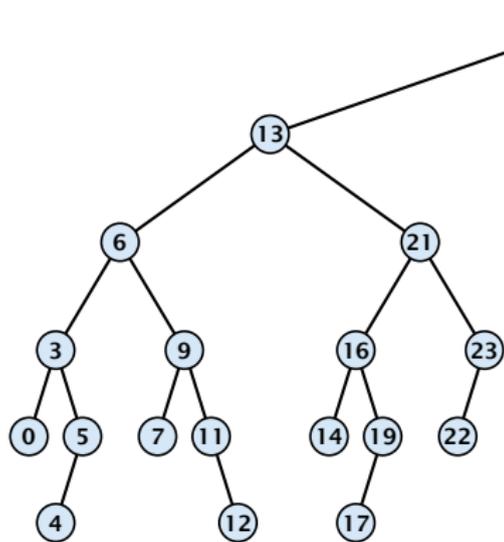


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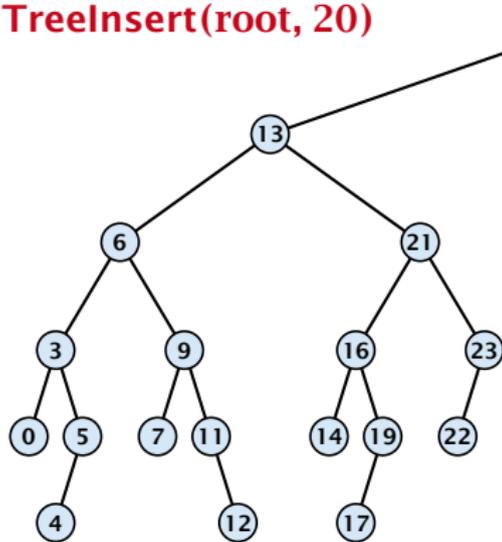
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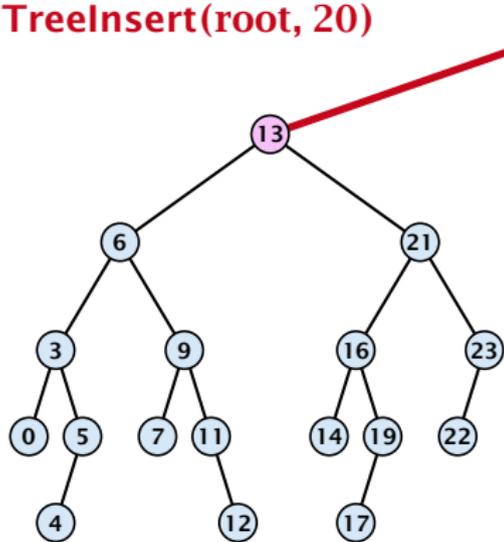
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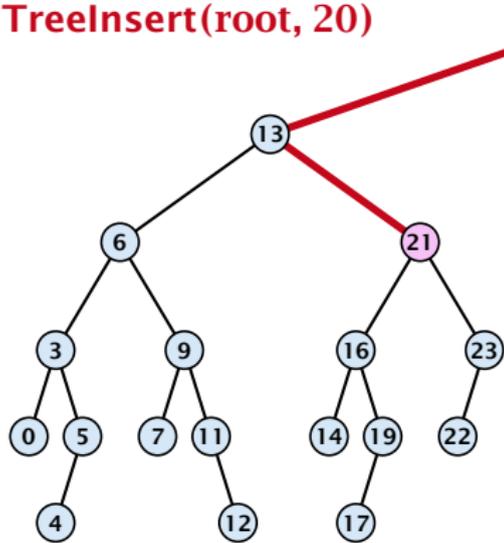
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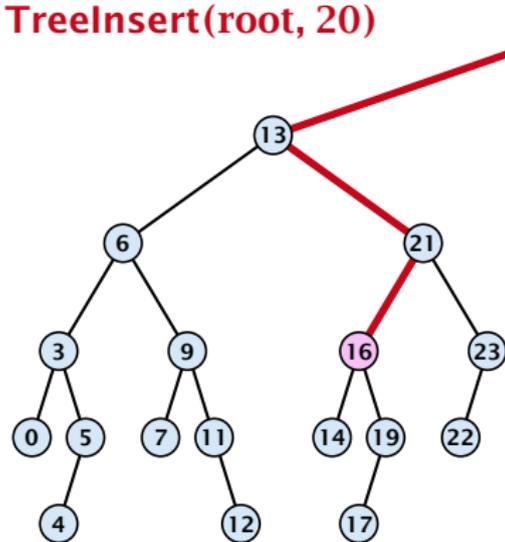
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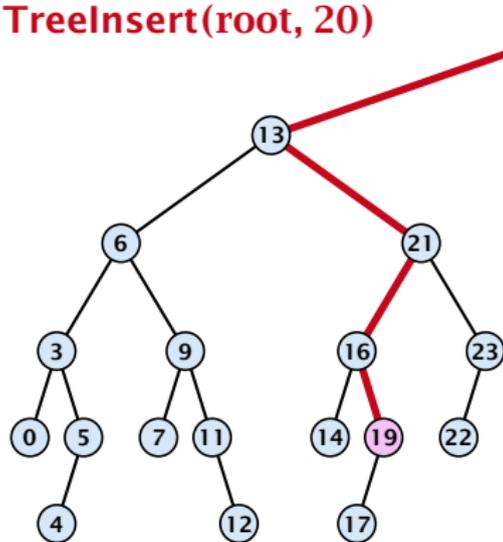
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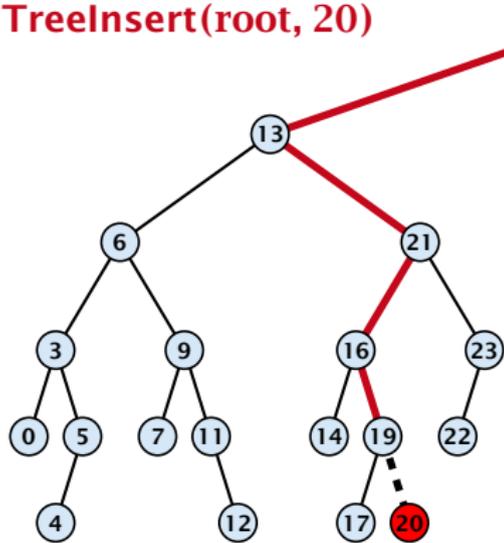
Algorithm 8 TreeInsert(x, z)

- 1: **if** $x = \text{null}$ **then**
- 2: $\text{root}[T] \leftarrow z$; $\text{parent}[z] \leftarrow \text{null}$;
- 3: **return**;
- 4: **if** $\text{key}[x] > \text{key}[z]$ **then**
- 5: **if** $\text{left}[x] = \text{null}$ **then**
- 6: $\text{left}[x] \leftarrow z$; $\text{parent}[z] \leftarrow x$;
- 7: **else** TreeInsert($\text{left}[x], z$);
- 8: **else**
- 9: **if** $\text{right}[x] = \text{null}$ **then**
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- 11: **else** TreeInsert($\text{right}[x], z$);

Binary Search Trees: Insert

Insert element **not** in the tree.

TreeInsert(root, 20)

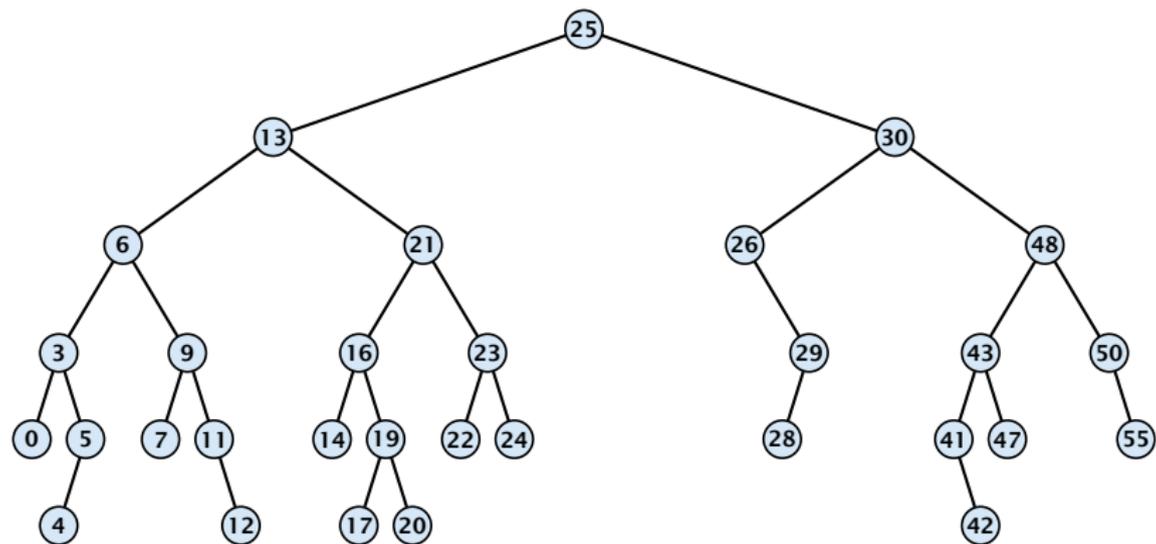


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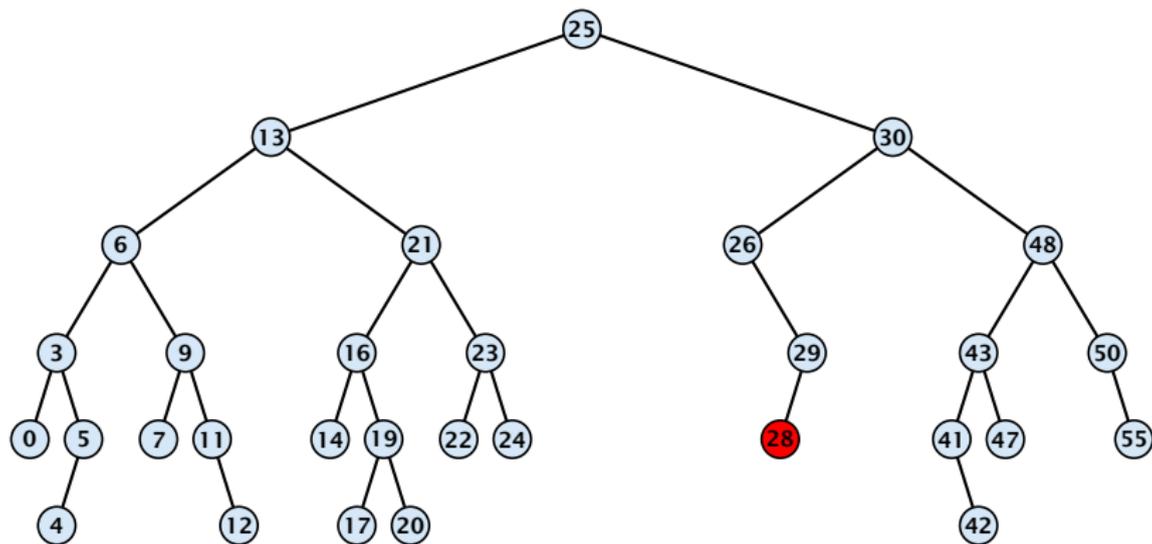
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Binary Search Trees: Delete



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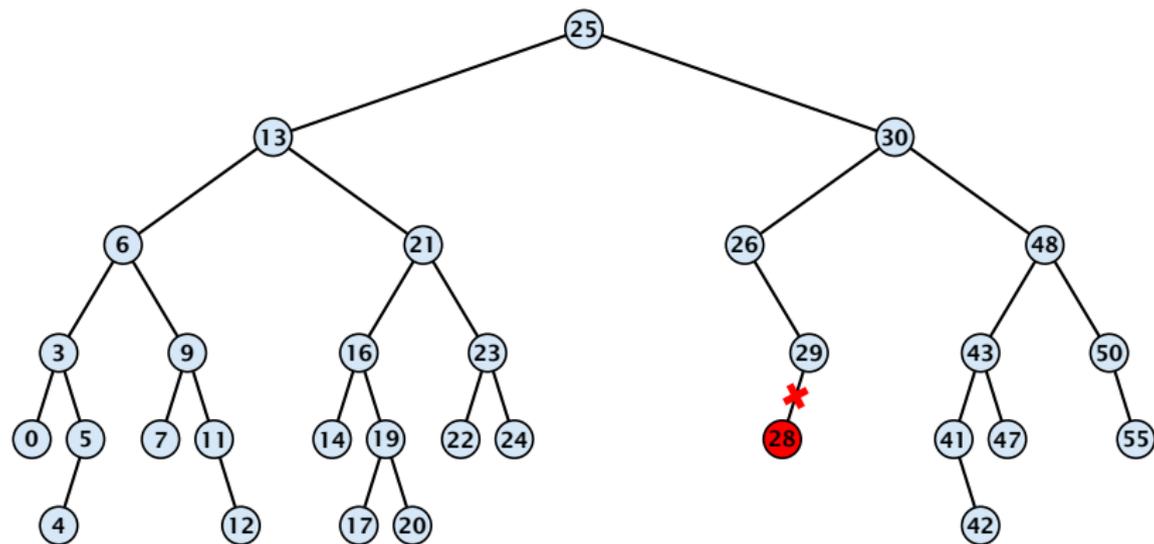


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Element does not have any children

- ▶ Simply go to the parent and set the corresponding pointer to **null**.

Binary Search Trees: Delete

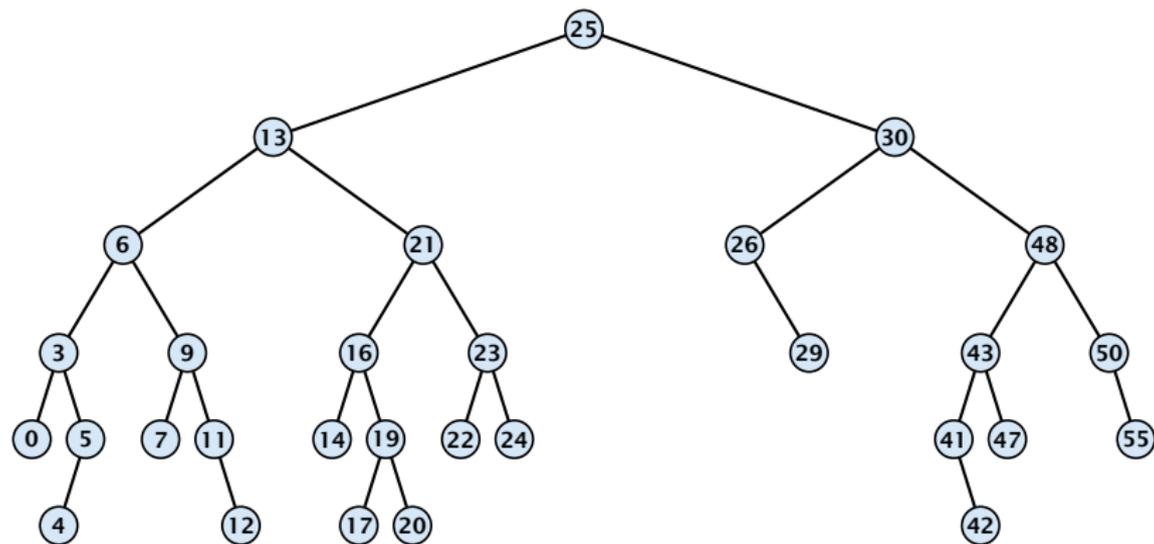


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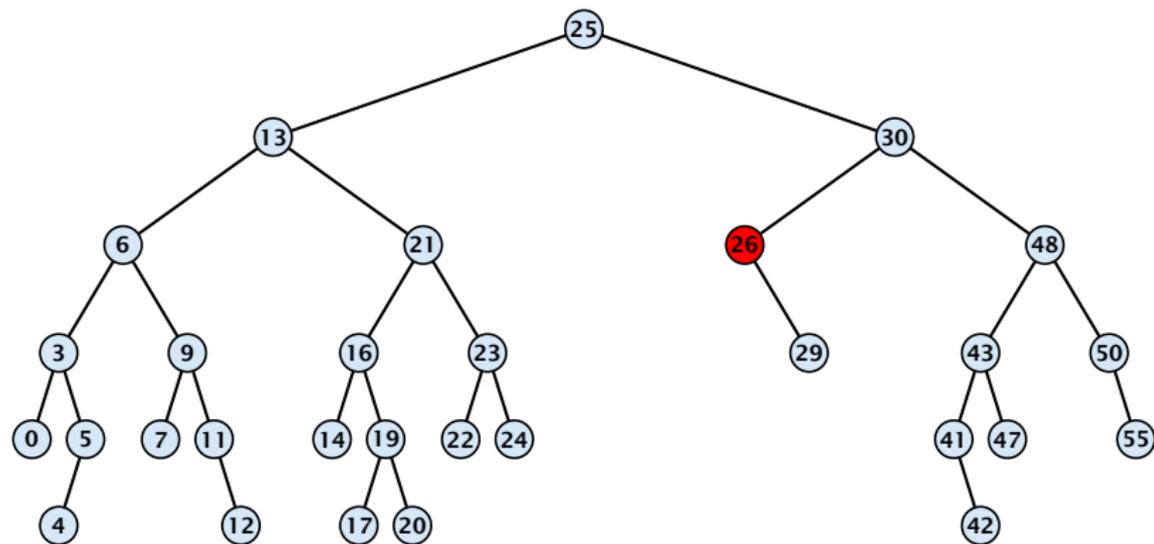


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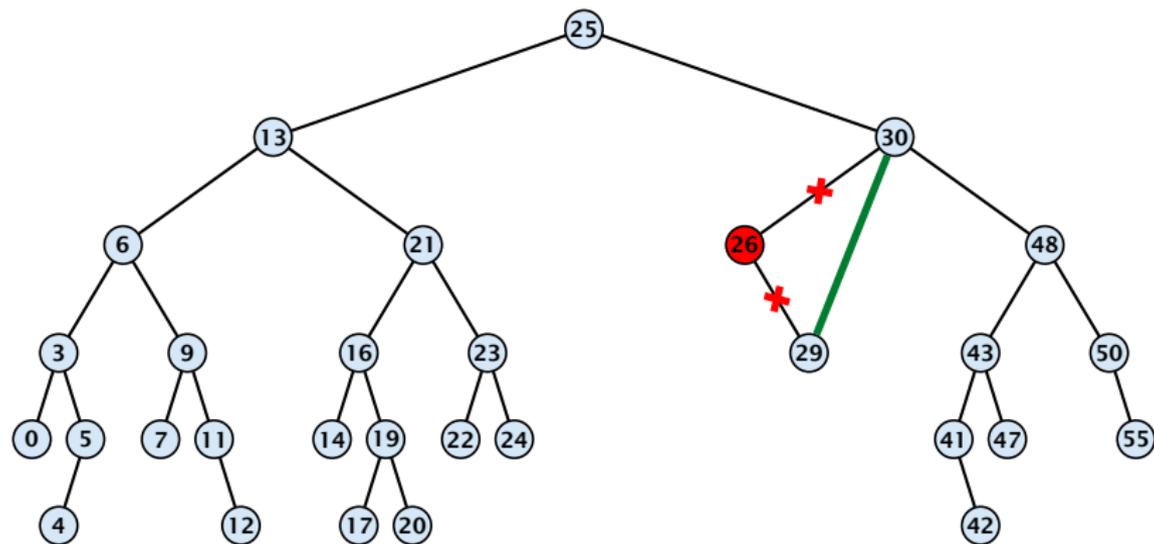


Case 2:

Element has exactly one child

- ▶ Splice the element out of the tree by connecting its parent to its successor.

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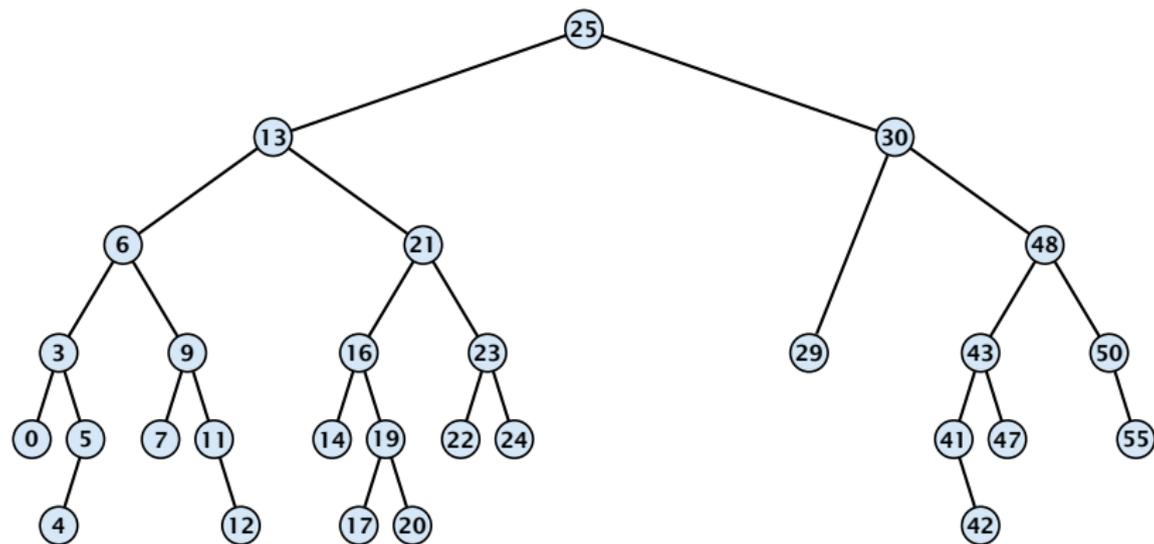


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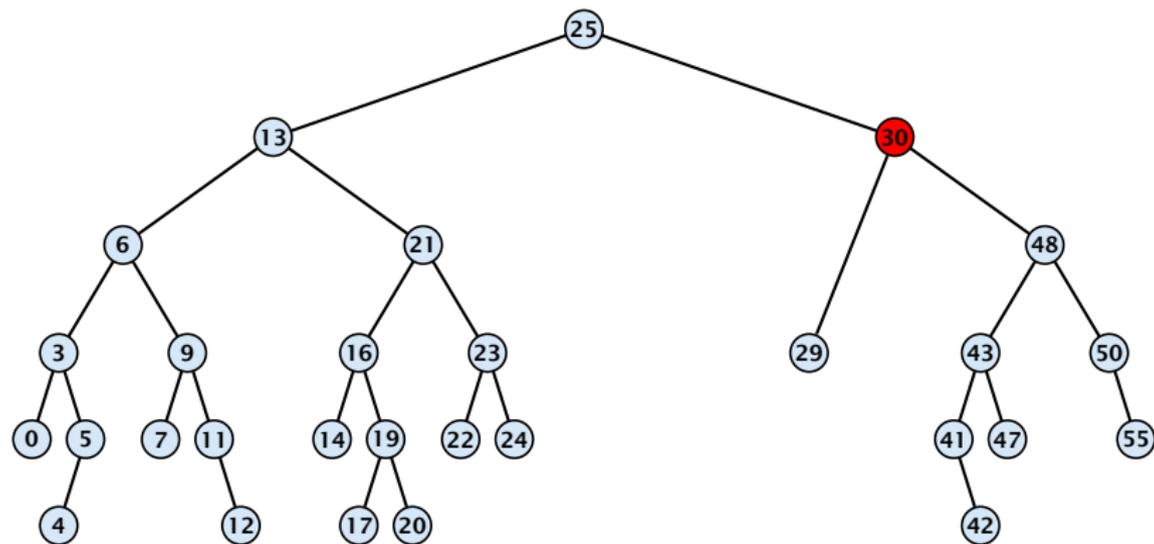


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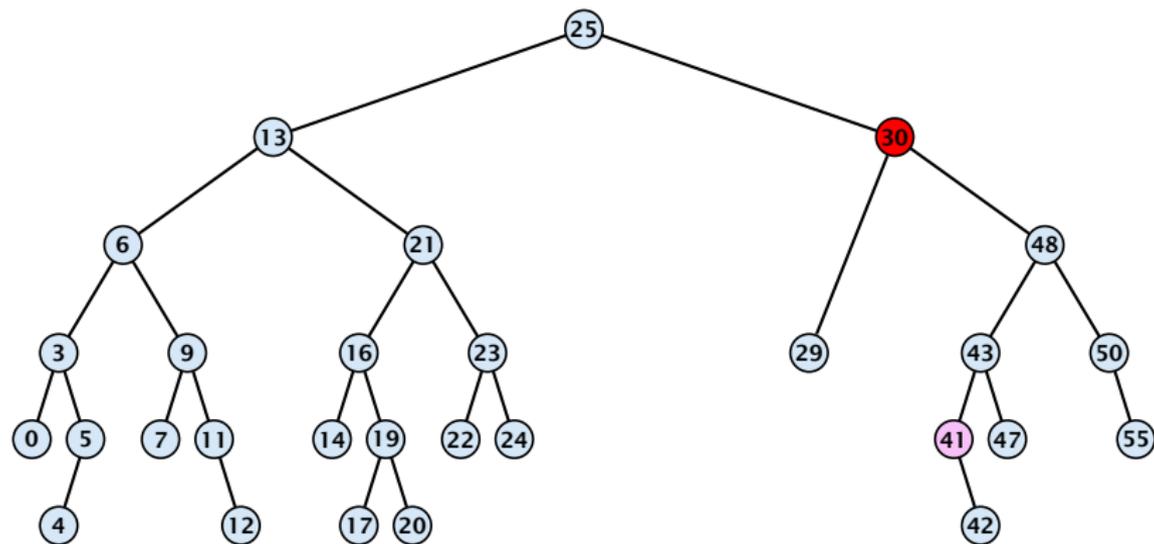


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Element has two children

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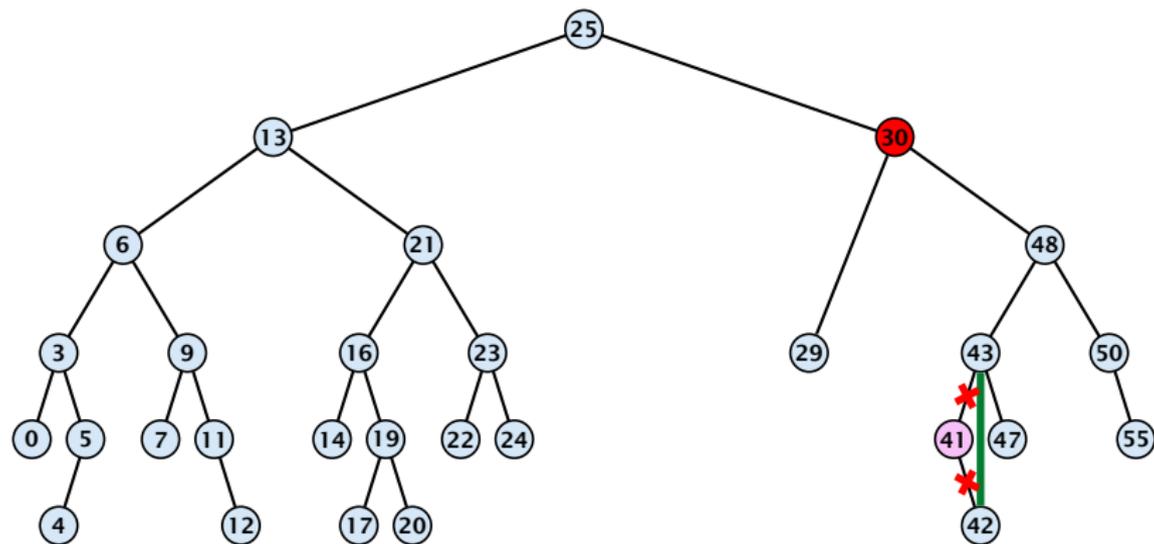


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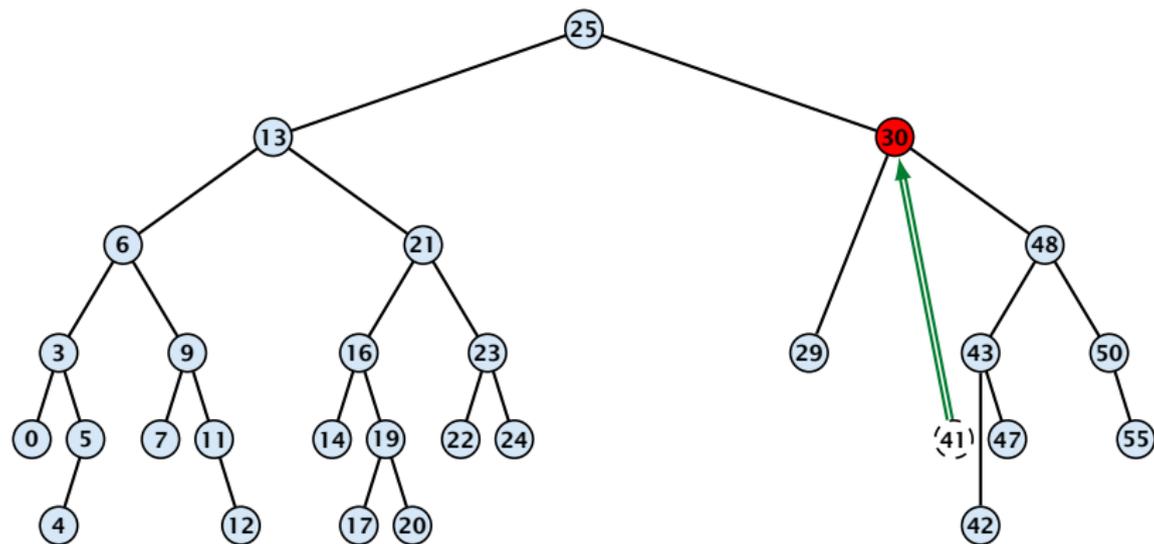


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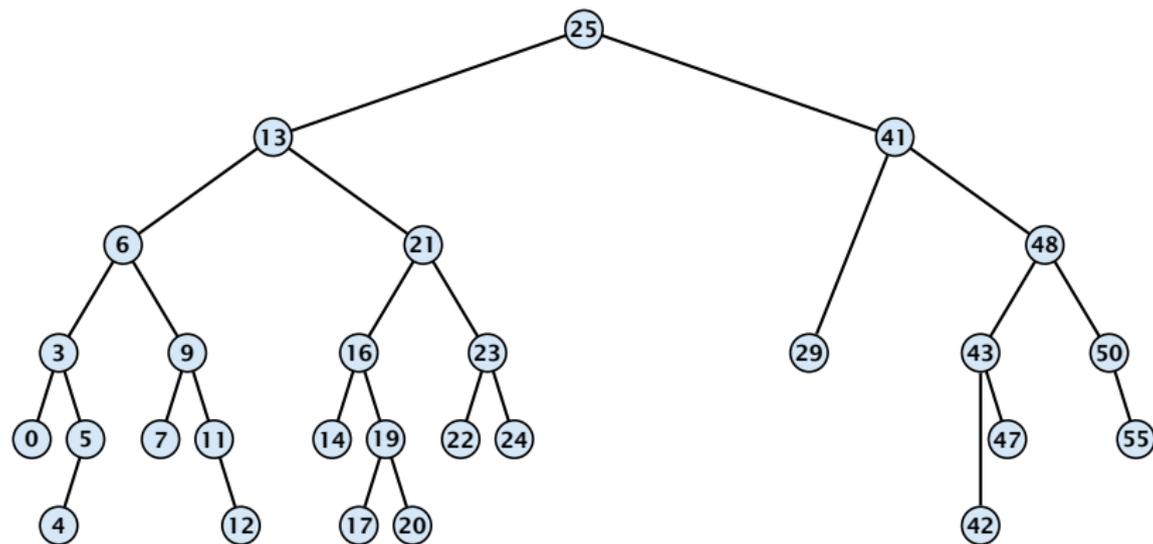


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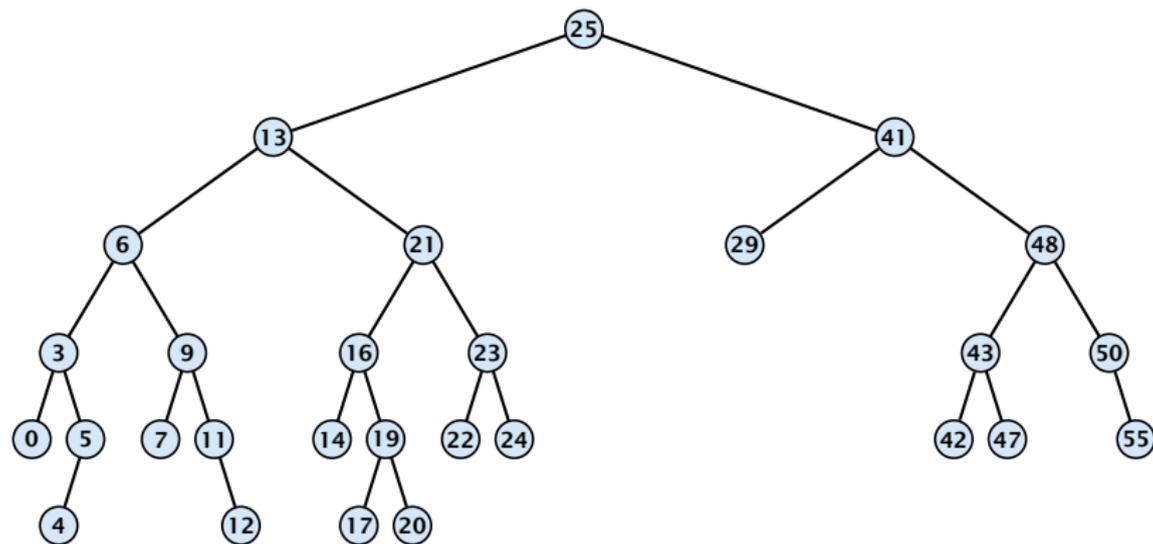


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Binary Search Trees: Delete

Algorithm 9 TreeDelete(z)

```
1: if left[ $z$ ] = null or right[ $z$ ] = null
2:   then  $y \leftarrow z$  else  $y \leftarrow \text{TreeSucc}(z)$ ;   select  $y$  to splice out
3:   if left[ $y$ ]  $\neq$  null
4:     then  $x \leftarrow \text{left}[y]$  else  $x \leftarrow \text{right}[y]$ ;  $x$  is child of  $y$  (or null)
5:   if  $x \neq \text{null}$  then parent[ $x$ ]  $\leftarrow$  parent[ $y$ ];   parent[ $x$ ] is correct
6:   if parent[ $y$ ] = null then
7:     root[ $T$ ]  $\leftarrow x$ 
8:   else
9:     if  $y = \text{left}[\text{parent}[y]]$  then
10:      left[parent[ $y$ ]]  $\leftarrow x$ 
11:     else
12:      right[parent[ $y$ ]]  $\leftarrow x$ 
13:   if  $y \neq z$  then copy  $y$ -data to  $z$ 
```

} fix pointer to x

Balanced Binary Search Trees

All operations on a binary search tree can be performed in time $\mathcal{O}(h)$, where h denotes the height of the tree.

However the height of the tree may become as large as $\Theta(n)$.

Balanced Binary Search Trees

With each insert- and delete-operation perform local adjustments to guarantee a height of $\mathcal{O}(\log n)$.

AVL-trees, Red-black trees, Scapegoat trees, 2-3 trees, B-trees, AA trees, Treaps

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7.2 Red Black Trees

Definition 1

A red black tree is a balanced binary search tree in which each internal node has two children. Each internal node has a color, such that

1. The root is black.
2. All leaf nodes are black.
3. For each node, all paths to descendant leaves contain the same number of black nodes.
4. If a node is red then both its children are black.

The null-pointers in a binary search tree are replaced by pointers to special null-vertices, that do not carry any object-data

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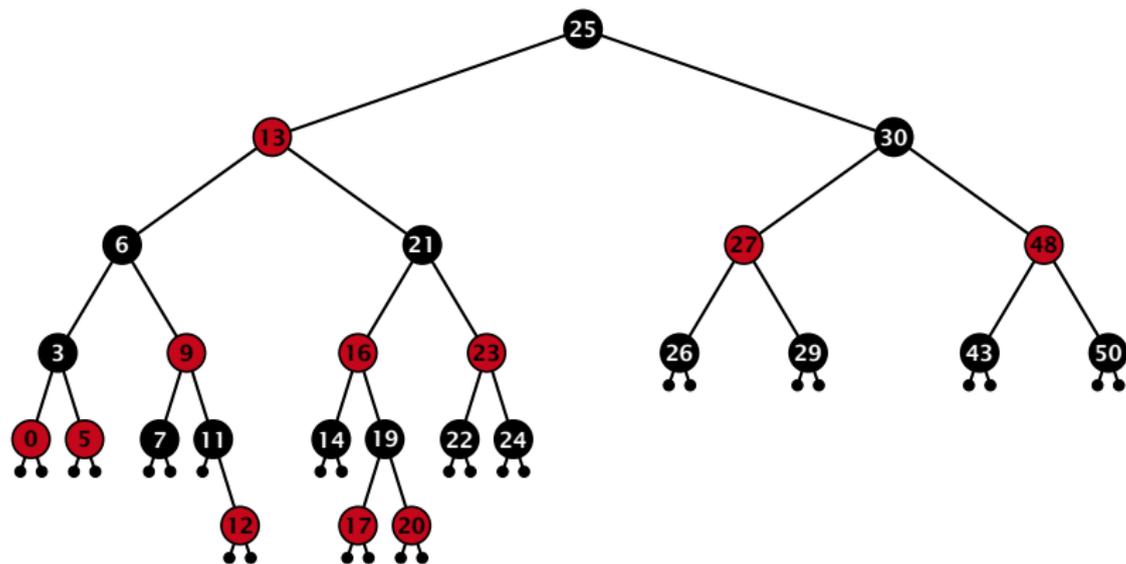
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Red Black Trees: Example



7.2 Red Black Trees

Lemma 2

A red-black tree with n internal nodes has height at most $\mathcal{O}(\log n)$.

Definition 3

The black height $\text{bh}(v)$ of a node v in a red black tree is the number of black nodes on a path from v to a leaf vertex (not counting v).

We first show:

Lemma 4

A sub-tree of black height $\text{bh}(v)$ in a red black tree contains at least $2^{\text{bh}(v)} - 1$ internal vertices.

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Proof of Lemma 4.

Induction on the height of v .

base case ($\text{height}(v) = 0$)

if $\text{height}(v)$ (maximum distance from v and a node in the subtree rooted at v) is 0 then v is a leaf.

The black height of v is 0.

The subtree rooted at v contains only v and is therefore

balanced.

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if $\text{height}(v)$ (maximum distance from v to a node in the subtree rooted at v) is 0 then $\text{black}(v) = 1$.

The black height of v is 1.

The subtree rooted at v contains only v and v is black.

□

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- ▶ The black height of v is 0.
- ▶ The sub-tree rooted at v contains $0 = 2^{\text{bh}(v)} - 1$ inner vertices.

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7.2 Red Black Trees

Proof (cont.)

induction step

- Suppose T is a node with height h .
 - If T has no children with strictly smaller height, then T is a leaf node and the claim is true.
 - These children (if any) either have height $h-1$ or $h-2$.
 - By induction hypothesis both sub-trees contain at least $\frac{1}{2} \cdot 2^{h-1}$ internal vertices.
 - Their total is at least 2^{h-1} .



7.2 Red Black Trees

Proof (cont.)

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- ▶ Suppose v is a node with $\text{height}(v) > 0$.
- ▶ v has two children with strictly smaller height.
- ▶ These children (c_1, c_2) either have $\text{bh}(c_i) = \text{bh}(v)$ or $\text{bh}(c_i) = \text{bh}(v) - 1$.
- ▶ By induction hypothesis both sub-trees contain at least $2^{\text{bh}(v)-1} - 1$ internal vertices.
- ▶ Then T_v contains at least $2(2^{\text{bh}(v)-1} - 1) + 1 \geq 2^{\text{bh}(v)} - 1$ vertices.



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Proof of Lemma 2.

Let h denote the height of the red-black tree, and let P denote a path from the root to the furthest leaf.

At least half of the nodes on P must be black, since a red node must be followed by a black node.

Hence, the black height of the root is at least $h/2$.

The tree contains at least $2^{h/2} - 1$ internal vertices. Hence,
 $2^{h/2} - 1 \leq n$.

Hence, $h \leq 2 \log(n + 1) = \mathcal{O}(\log n)$. □

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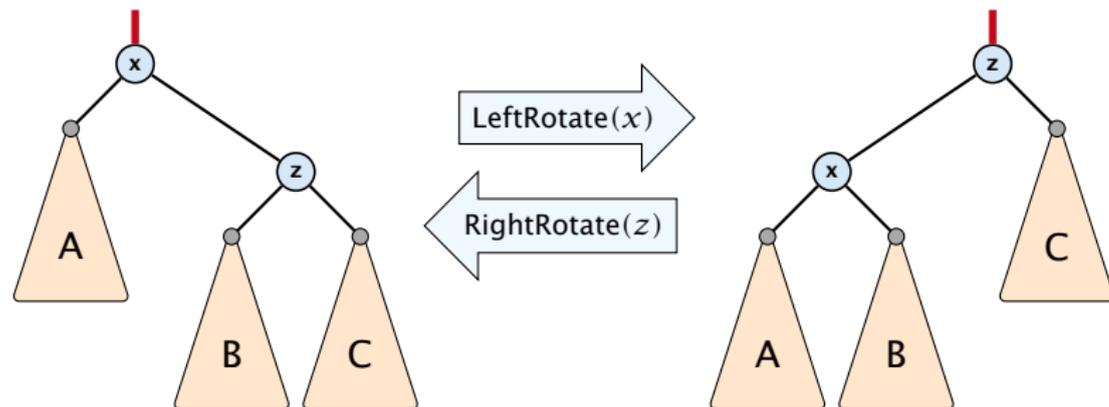
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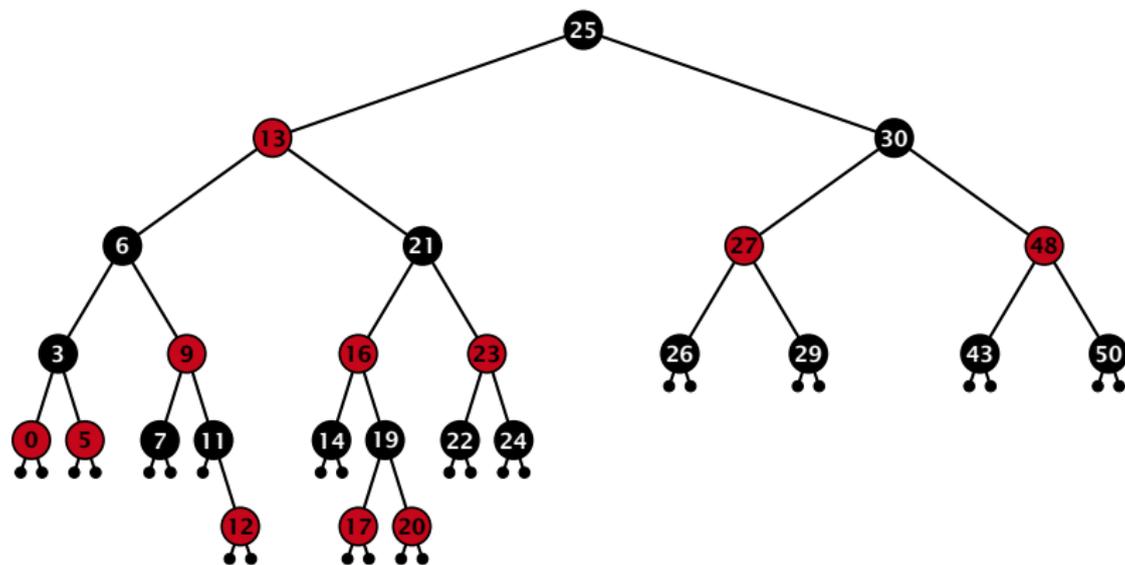
We need to adapt the insert and delete operations so that the red black properties are maintained.

Rotations

The properties will be maintained through rotations:



Red Black Trees: Insert

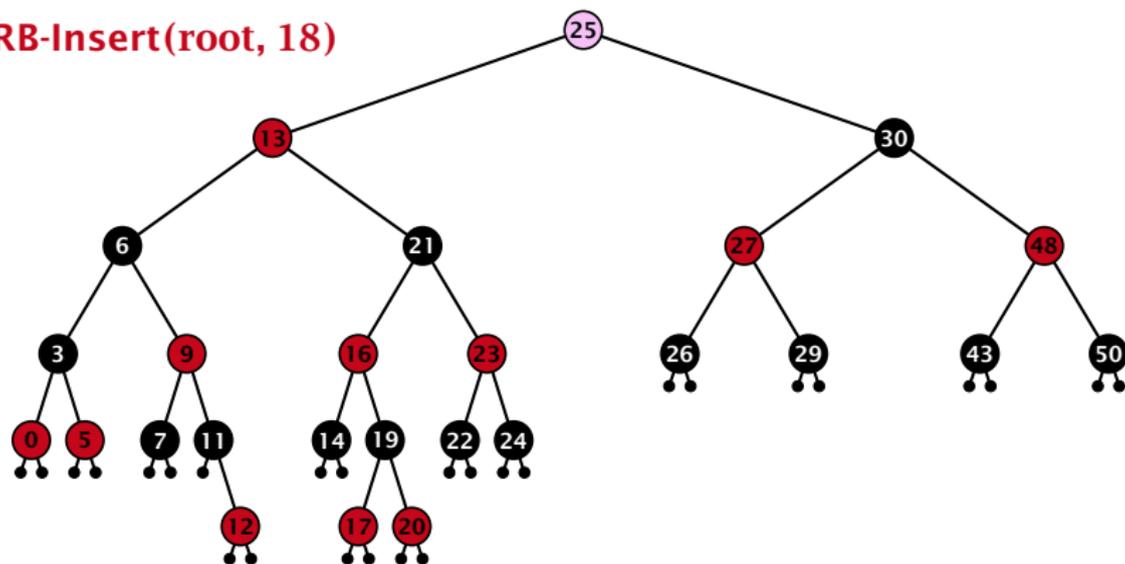


Insert:

- ▶ first make a normal insert into a binary search tree
- ▶ then fix red-black properties

Red Black Trees: Insert

RB-Insert(root, 18)

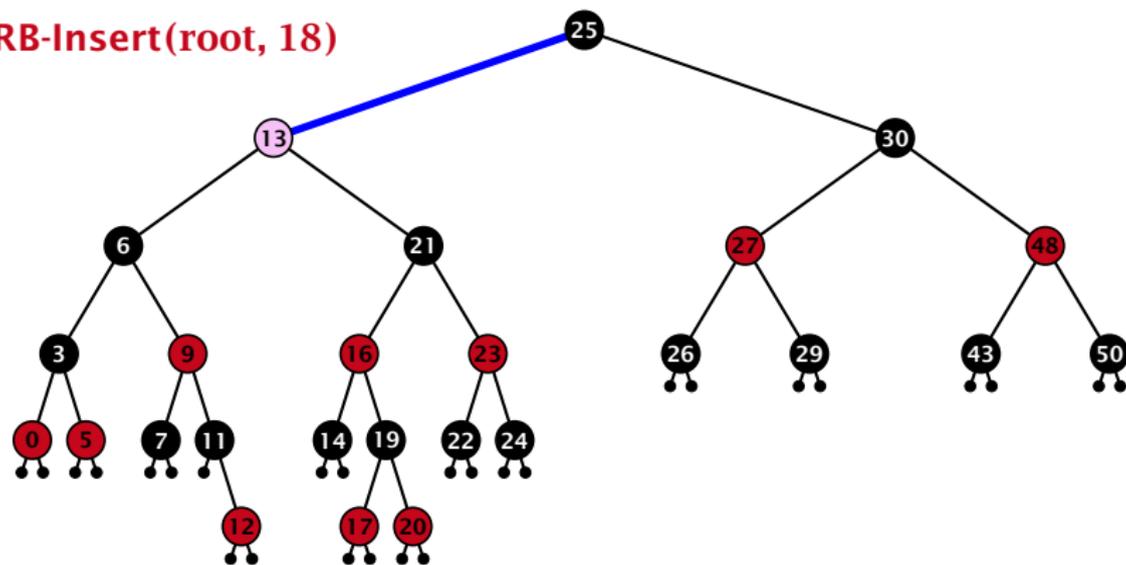


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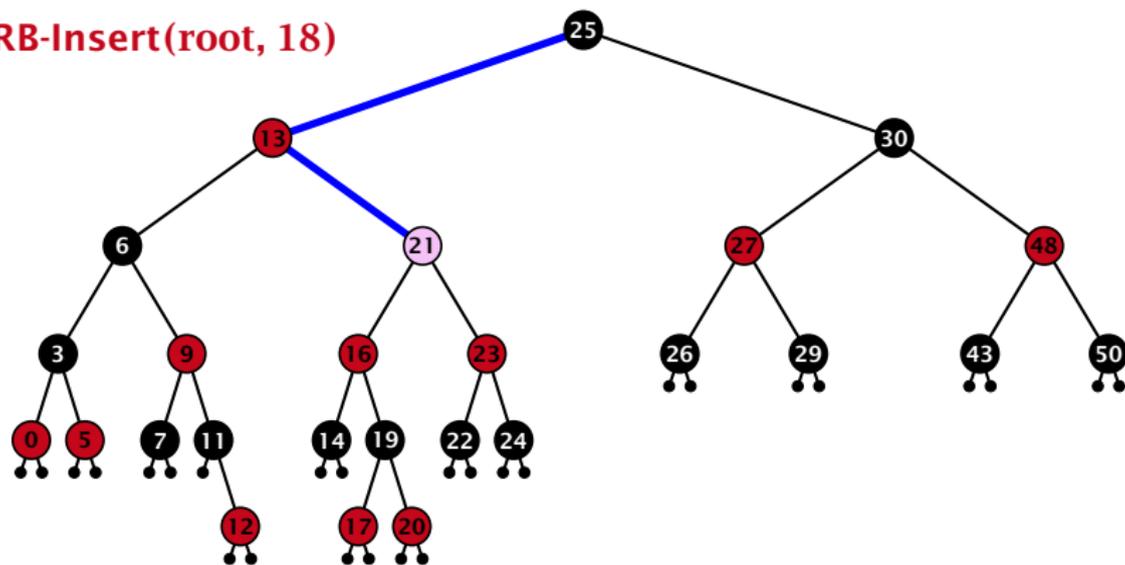


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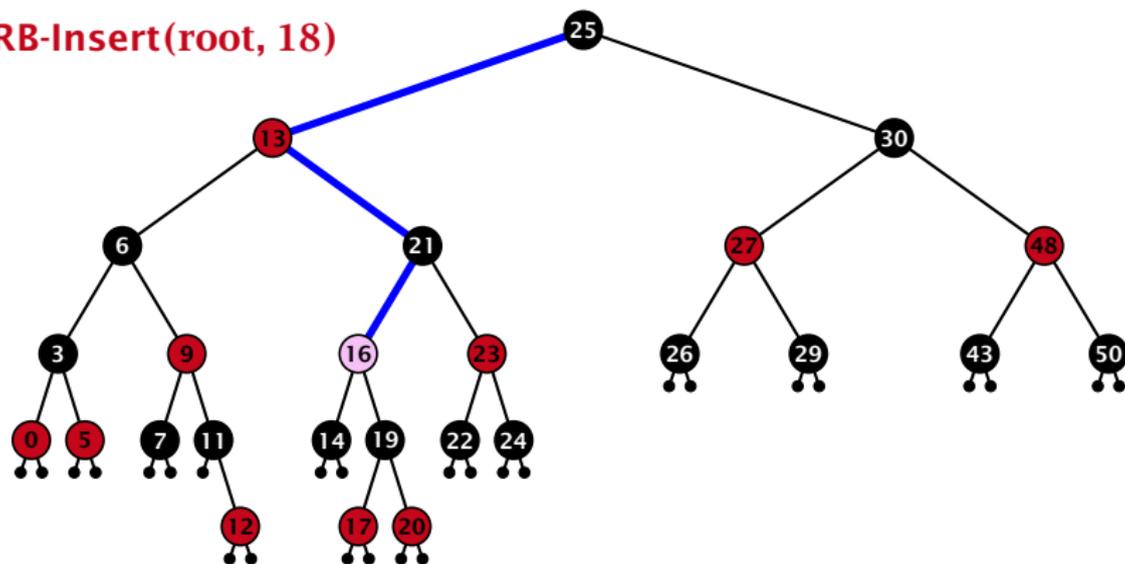


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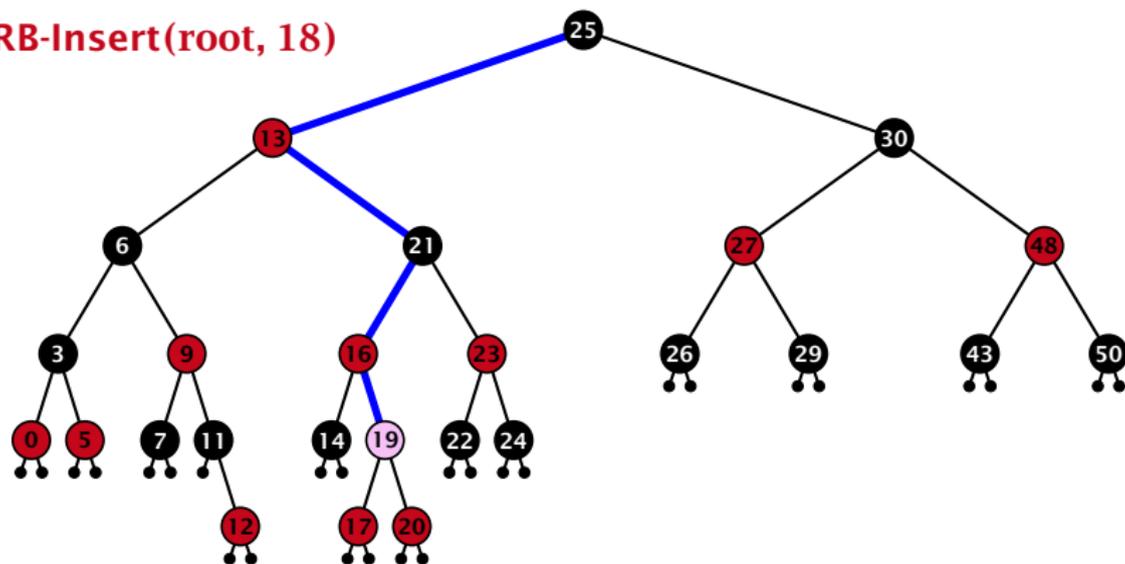


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RB-Insert(root, 18)

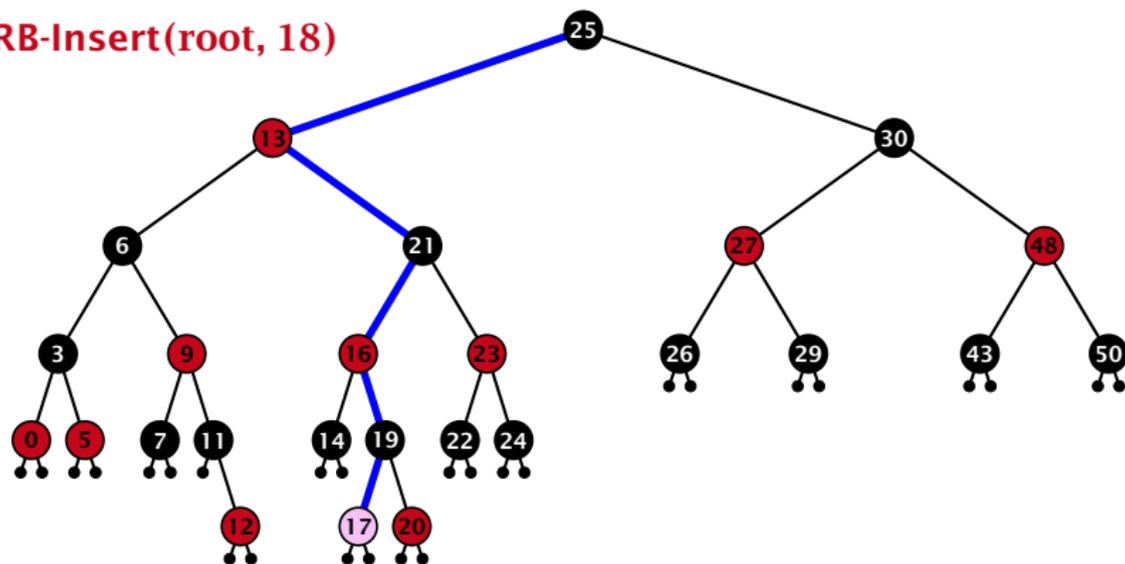


Insert:

- ▶ first make a normal insert into a binary search tree
- ▶ then fix red-black properties

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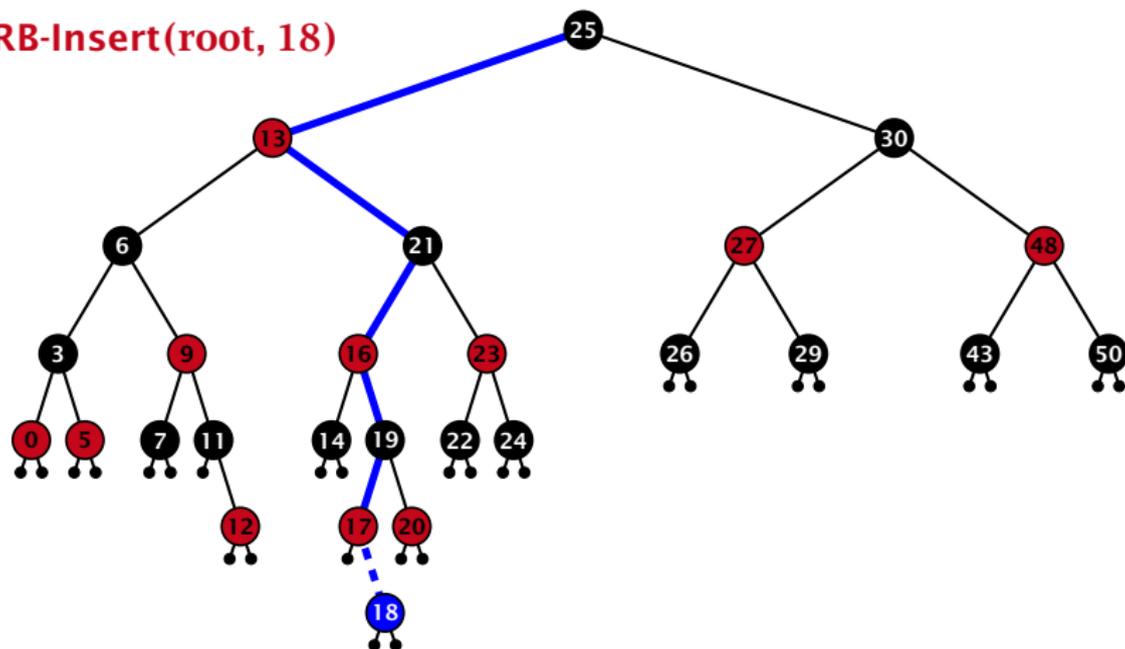


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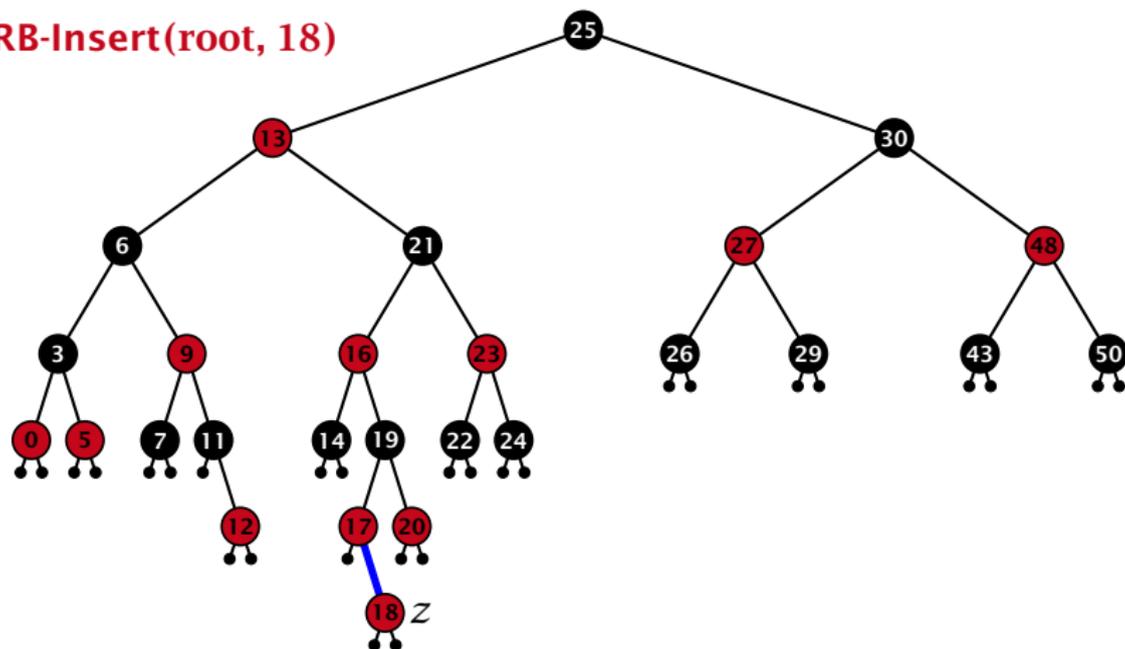


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RB-Insert(root, 18)



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Red Black Trees: Insert

Invariant of the fix-up algorithm:

- ▶ z is a red node
- ▶ the black-height property is fulfilled at every node
- ▶ the only violation of red-black properties occurs at z and $\text{parent}[z]$
 - either both of them are red (most important case)
 - or the parent does not exist (violation since root must be black)

If z has a parent but no grand-parent we could simply color the parent/root black; however this case never happens.

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Red Black Trees: Insert

Algorithm 10 InsertFix(z)

```
1: while parent[ $z$ ]  $\neq$  null and col[parent[ $z$ ]] = red do
2:   if parent[ $z$ ] = left[gp[ $z$ ]] then
3:      $uncle \leftarrow$  right[grandparent[ $z$ ]]
4:     if col[ $uncle$ ] = red then
5:       col[p[ $z$ ]]  $\leftarrow$  black; col[ $u$ ]  $\leftarrow$  black;
6:       col[gp[ $z$ ]]  $\leftarrow$  red;  $z \leftarrow$  grandparent[ $z$ ];
7:     else
8:       if  $z$  = right[parent[ $z$ ]] then
9:          $z \leftarrow$  p[ $z$ ]; LeftRotate( $z$ );
10:      col[p[ $z$ ]]  $\leftarrow$  black; col[gp[ $z$ ]]  $\leftarrow$  red;
11:      RightRotate(gp[ $z$ ]);
12:     else same as then-clause but right and left exchanged
13: col(root[ $T$ ])  $\leftarrow$  black;
```

Red Black Trees: Insert

Algorithm 10 InsertFix(z)

```
1: while parent[ $z$ ]  $\neq$  null and col[parent[ $z$ ]] = red do
2:   if parent[ $z$ ] = left[gp[ $z$ ]] then  $z$  in left subtree of grandparent
3:      $uncle \leftarrow$  right[grandparent[ $z$ ]]
4:     if col[ $uncle$ ] = red then
5:       col[p[ $z$ ]]  $\leftarrow$  black; col[ $u$ ]  $\leftarrow$  black;
6:       col[gp[ $z$ ]]  $\leftarrow$  red;  $z \leftarrow$  grandparent[ $z$ ];
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2:   if parent[ $z$ ] = left[gp[ $z$ ]] then
3:      $uncle \leftarrow$  right[grandparent[ $z$ ]]
4:     if col[ $uncle$ ] = red then Case 1: uncle red
5:       col[p[ $z$ ]]  $\leftarrow$  black; col[ $u$ ]  $\leftarrow$  black;
6:       col[gp[ $z$ ]]  $\leftarrow$  red;  $z \leftarrow$  grandparent[ $z$ ];
7:     else
8:       if  $z$  = right[parent[ $z$ ]] then
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10:      col[p[ $z$ ]]  $\leftarrow$  black; col[gp[ $z$ ]]  $\leftarrow$  red;
11:      RightRotate(gp[ $z$ ]);
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```

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6:       col[gp[ $z$ ]]  $\leftarrow$  red;  $z \leftarrow$  grandparent[ $z$ ];
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10:    col[p[ $z$ ]]  $\leftarrow$  black; col[gp[ $z$ ]]  $\leftarrow$  red;
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Red Black Trees: Insert

Algorithm 10 InsertFix(z)

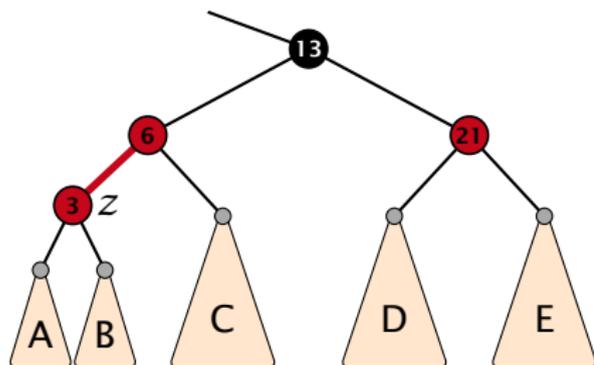
```
1: while parent[ $z$ ]  $\neq$  null and col[parent[ $z$ ]] = red do
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3:     uncle  $\leftarrow$  right[grandparent[ $z$ ]]
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8:       if  $z$  = right[parent[ $z$ ]] then 2a:  $z$  right child
9:          $z \leftarrow$  p[ $z$ ]; LeftRotate( $z$ );
10:        col[p[ $z$ ]]  $\leftarrow$  black; col[gp[ $z$ ]]  $\leftarrow$  red;
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Red Black Trees: Insert

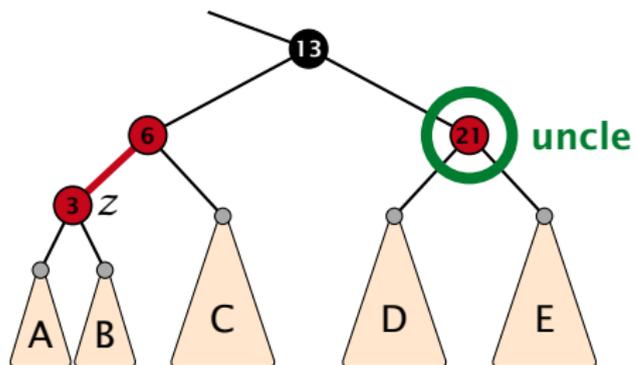
Algorithm 10 InsertFix(z)

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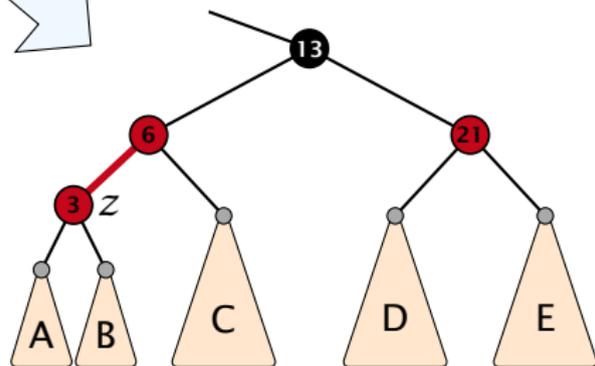
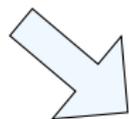
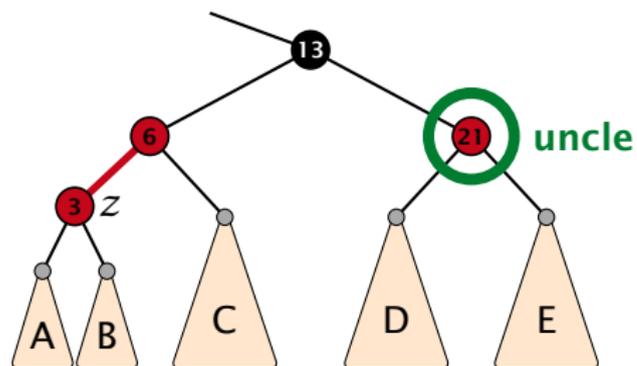
Case 1: Red Uncle



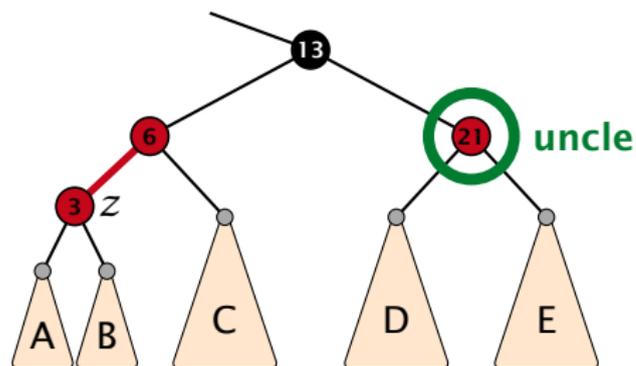
Case 1: Red Uncle



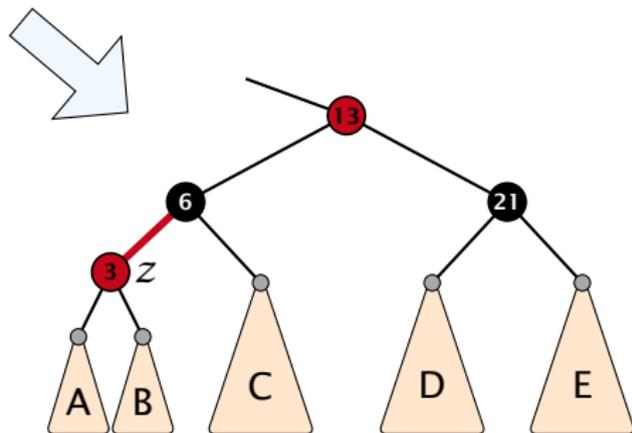
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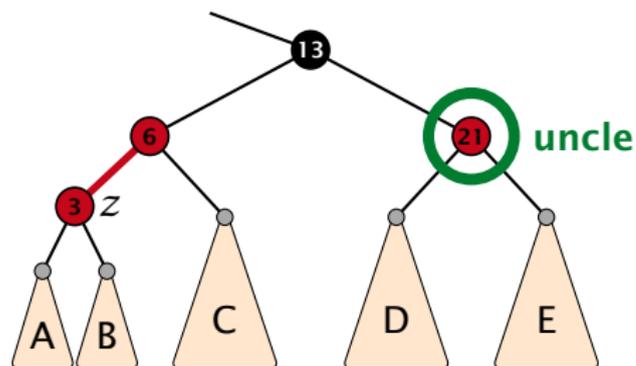
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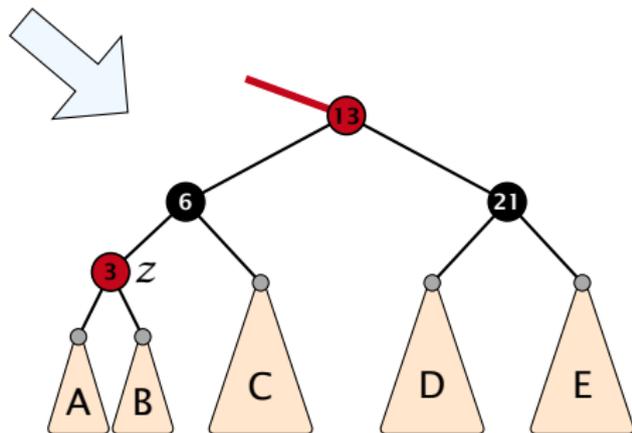
1. recolour



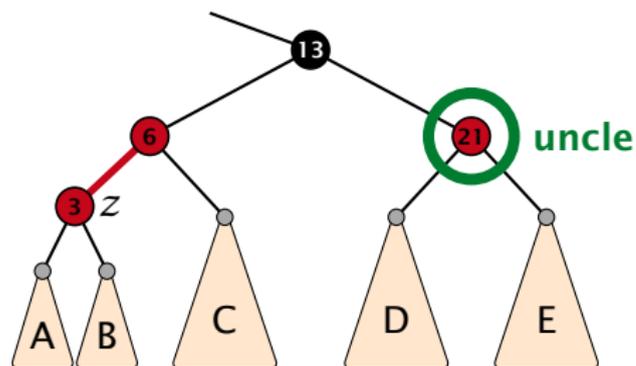
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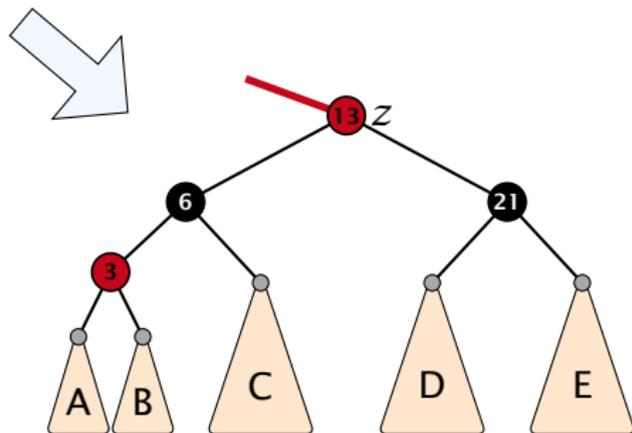
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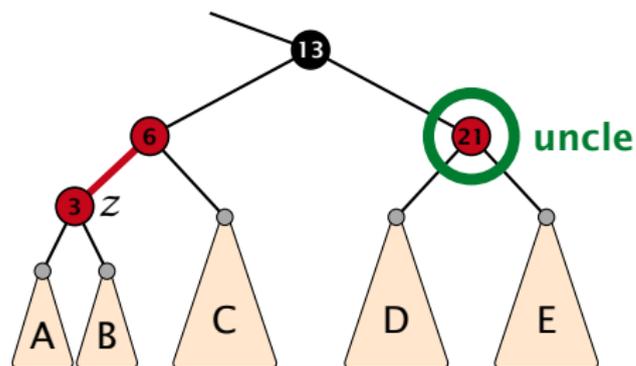
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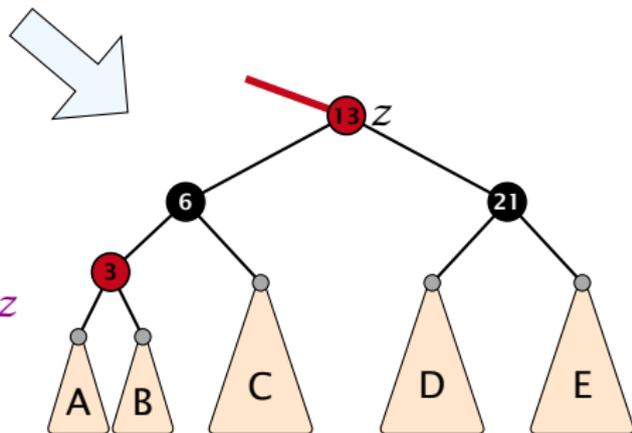
1. recolour
2. move z to grand-parent



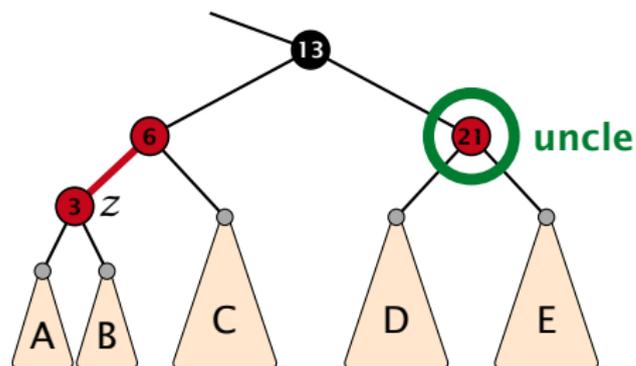
Case 1: Red Uncle



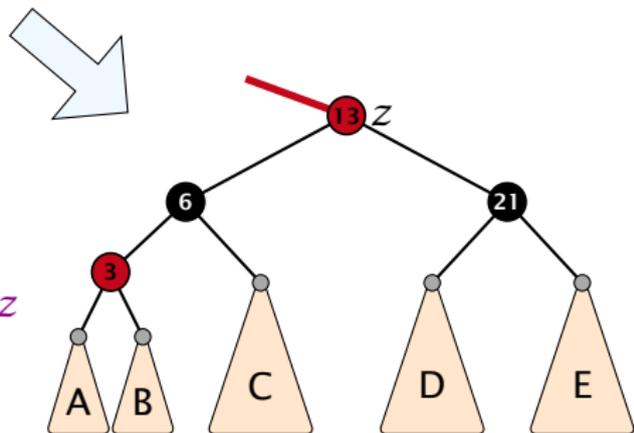
1. recolour
2. move z to grand-parent
3. invariant is fulfilled for new z



Case 1: Red Uncle

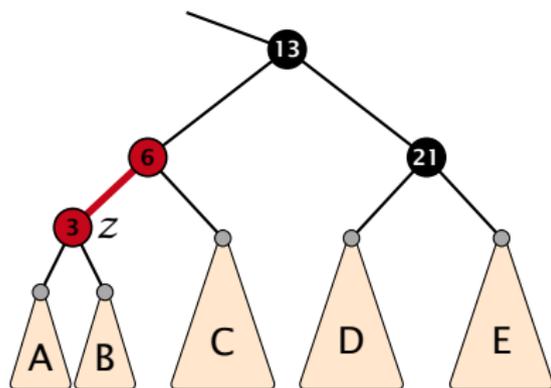


1. recolour
2. move z to grand-parent
3. invariant is fulfilled for new z
4. you made progress



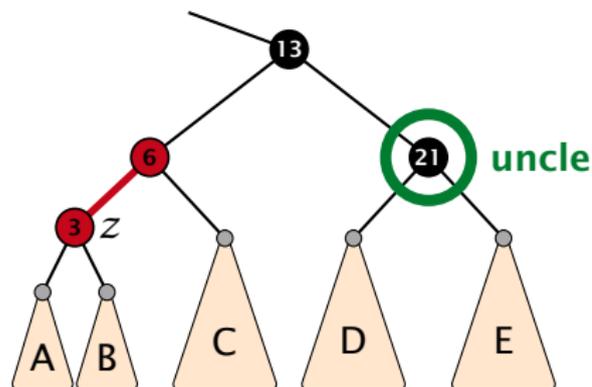
Case 2b: Black uncle and z is left child

1. rotate around grandparent
2. re-colour to ensure that black height property holds
3. you have a red black tree



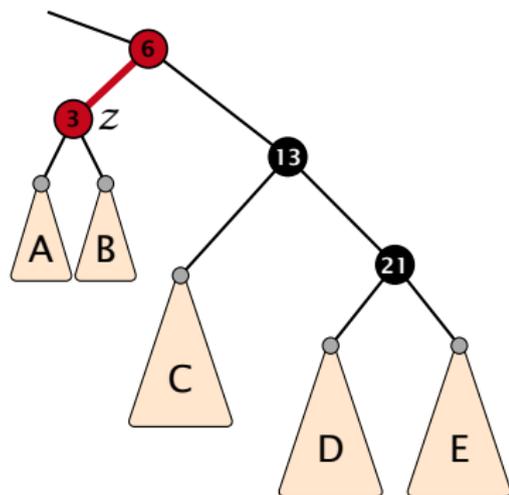
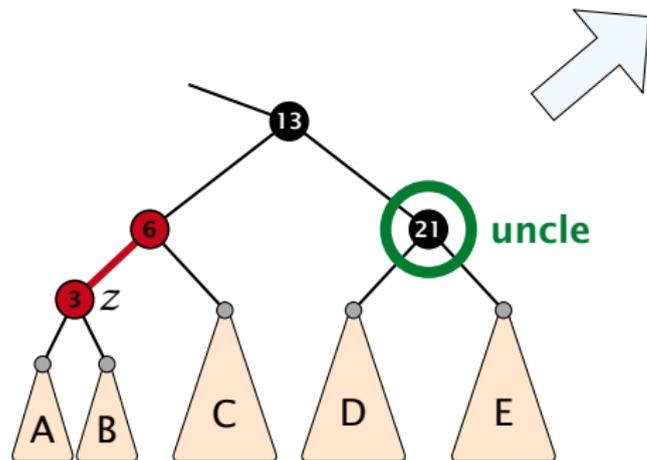
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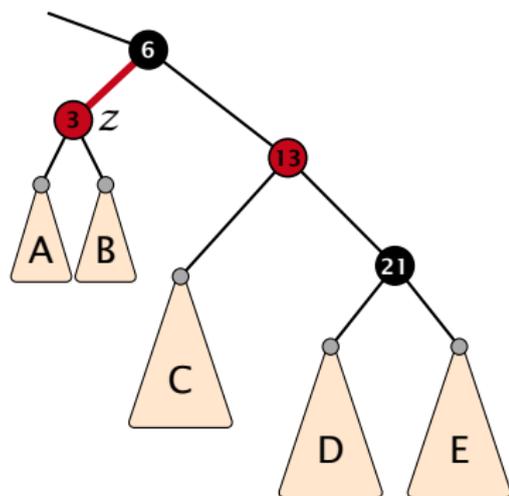
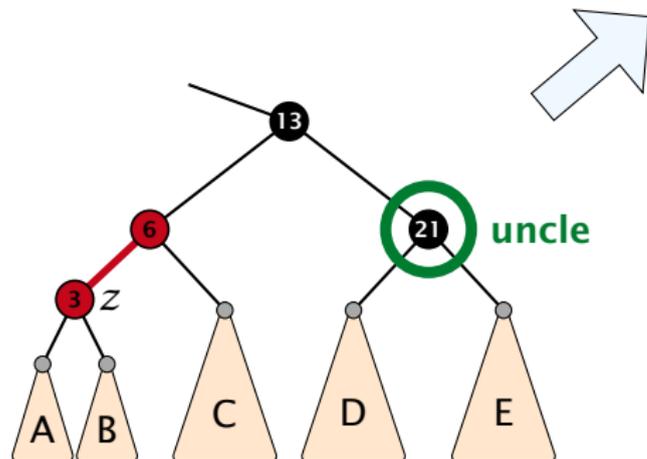
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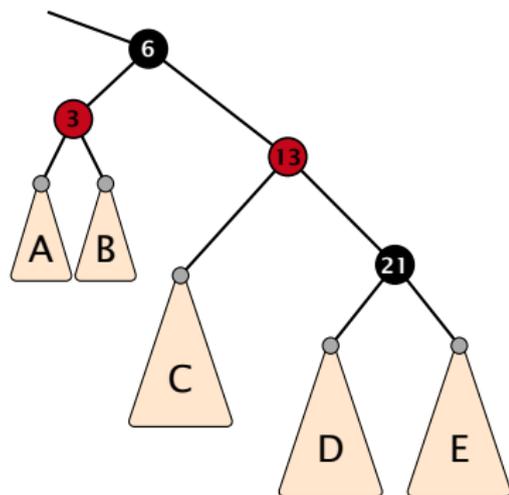
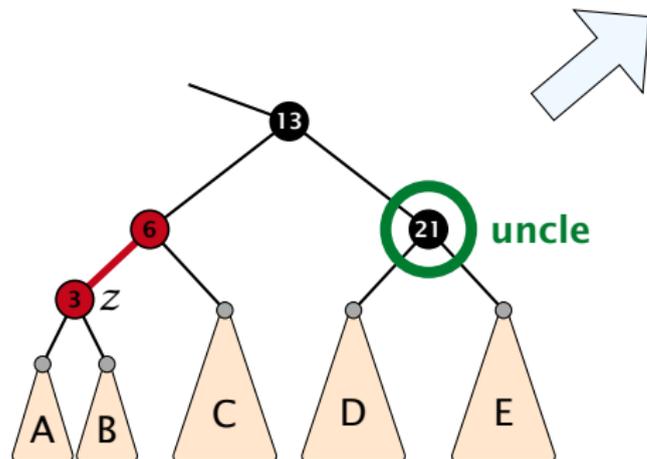
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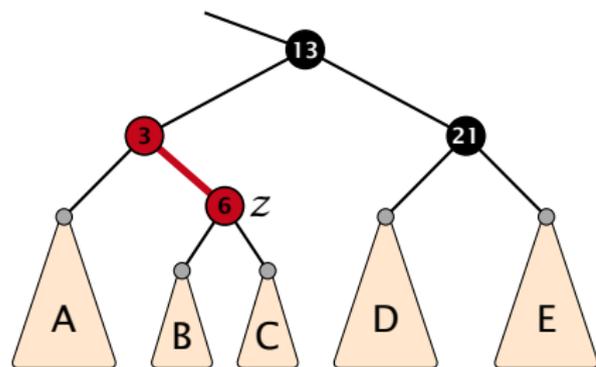
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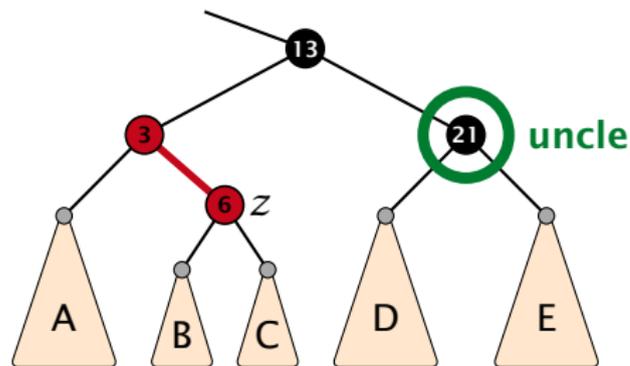
Case 2a: Black uncle and z is right child

1. rotate around parent
2. move z downwards
3. you have Case 2b.



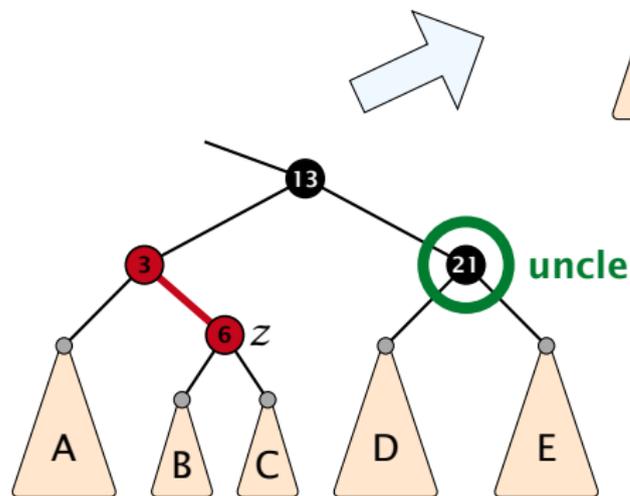
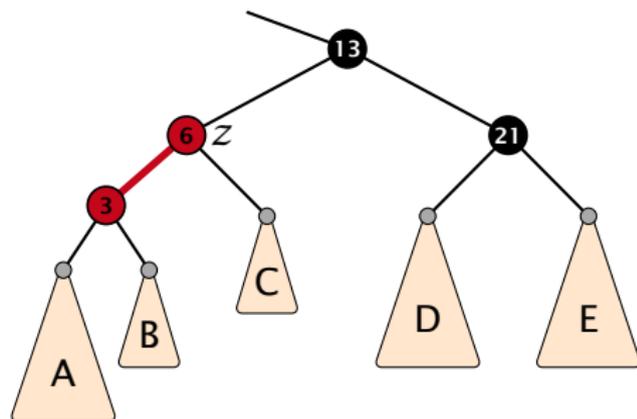
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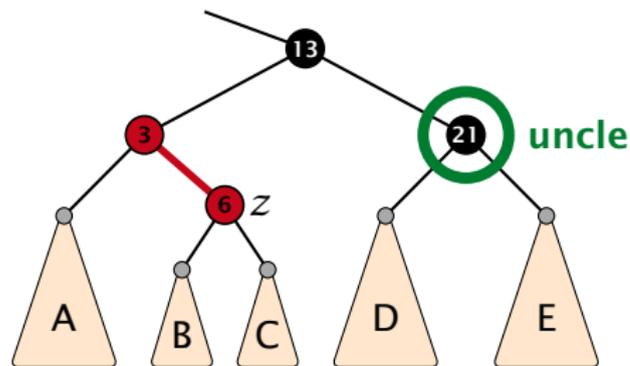
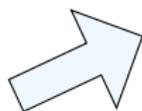
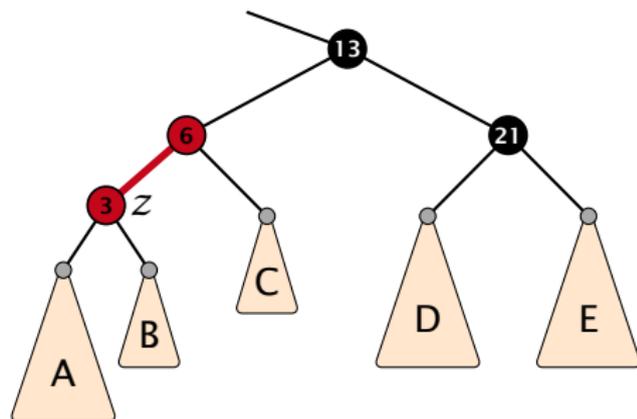
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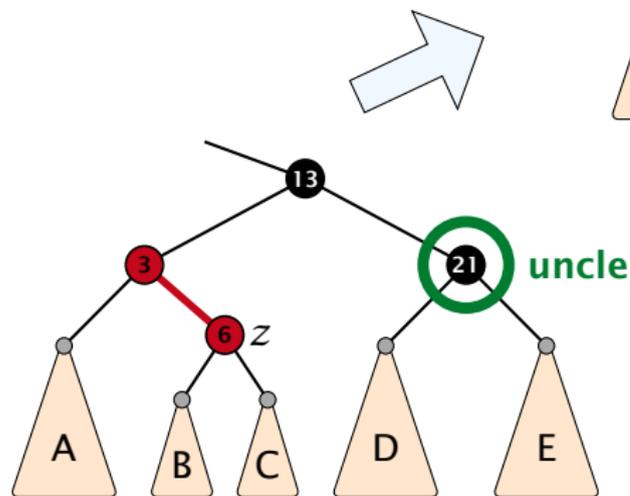
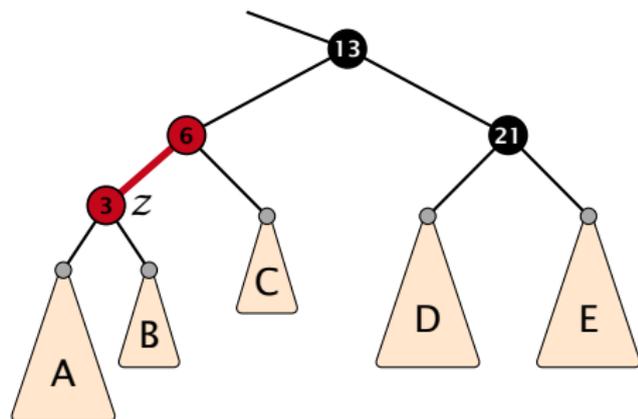
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Red Black Trees: Insert

Running time:

- ▶ Only Case 1 may repeat; but only $h/2$ many steps, where h is the height of the tree.
- ▶ Case 2a → Case 2b → red-black tree
- ▶ Case 2b → red-black tree

Performing Case 1 at most $\mathcal{O}(\log n)$ times and every other case at most once, we get a red-black tree. Hence $\mathcal{O}(\log n)$ re-colorings and at most 2 rotations.

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Red Black Trees: Delete

First do a standard delete.

If the spliced out node x was red everything is fine.

If it was black there may be the following problems.

1. Parent and child of x were red; two adjacent red vertices.

2. If you delete the root, the root may now be red.

3. Every path from an ancestor of x to a descendant leaf of x changes the number of black nodes. Black height property might be violated.

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If it was black there may be the following problems.

• Parent and child of x were red; two adjacent red nodes.

• x was the root; the root may now be red.

• x was left child of a red node; left-right imbalance.

• x was right child of a red node; right-left imbalance.

• x was black; the number of black nodes (Black Height) property may not be violated.

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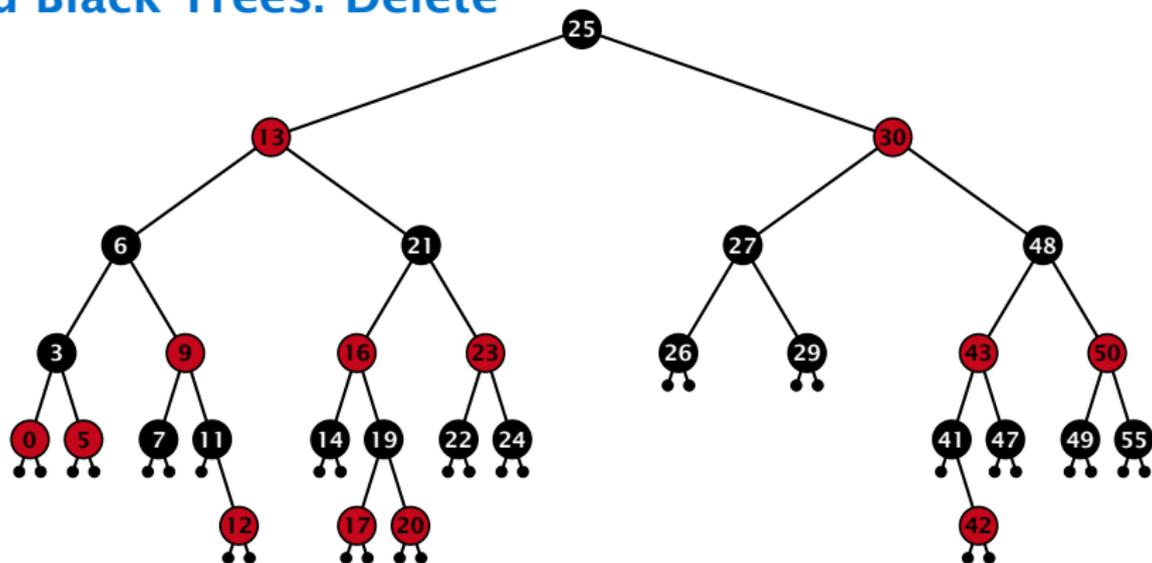
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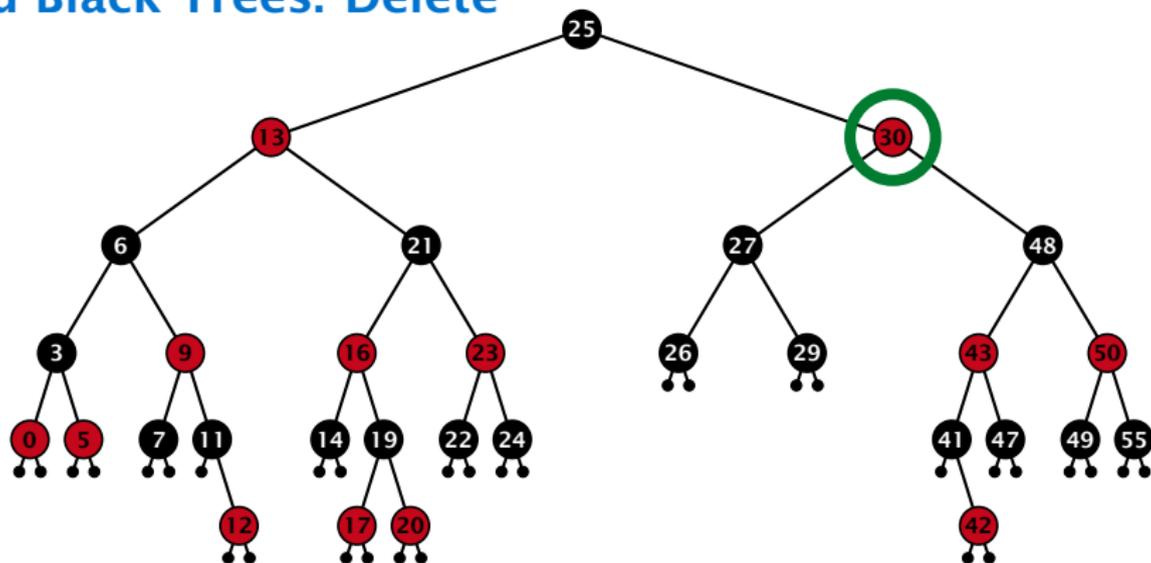
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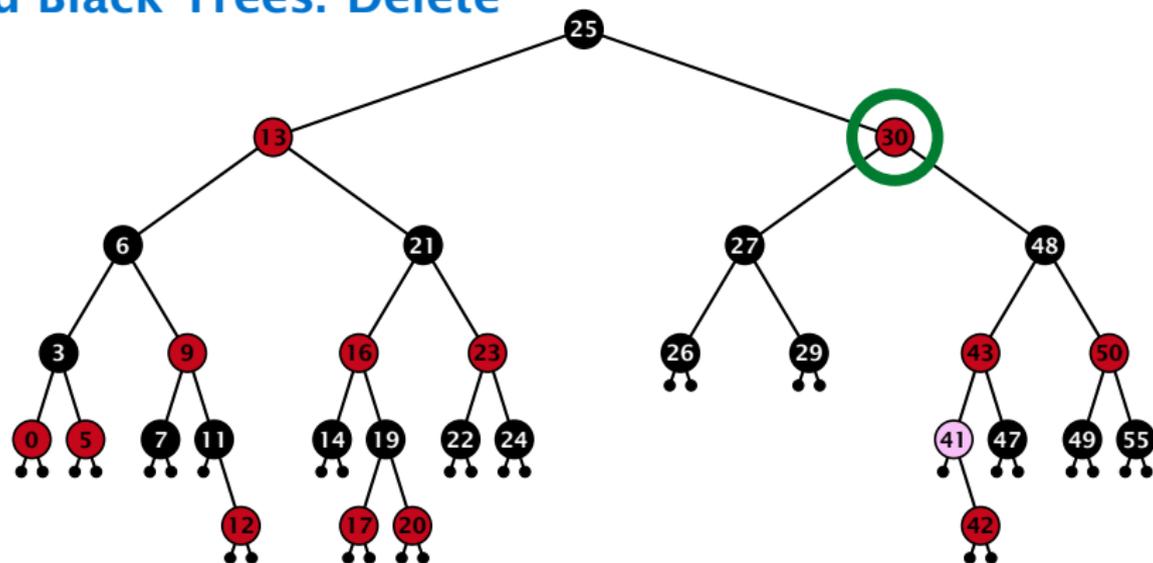


Case 3:

Element has two children

- ▶ do normal delete
- ▶ when replacing content by content of successor, don't change color of node

Red Black Trees: Delete

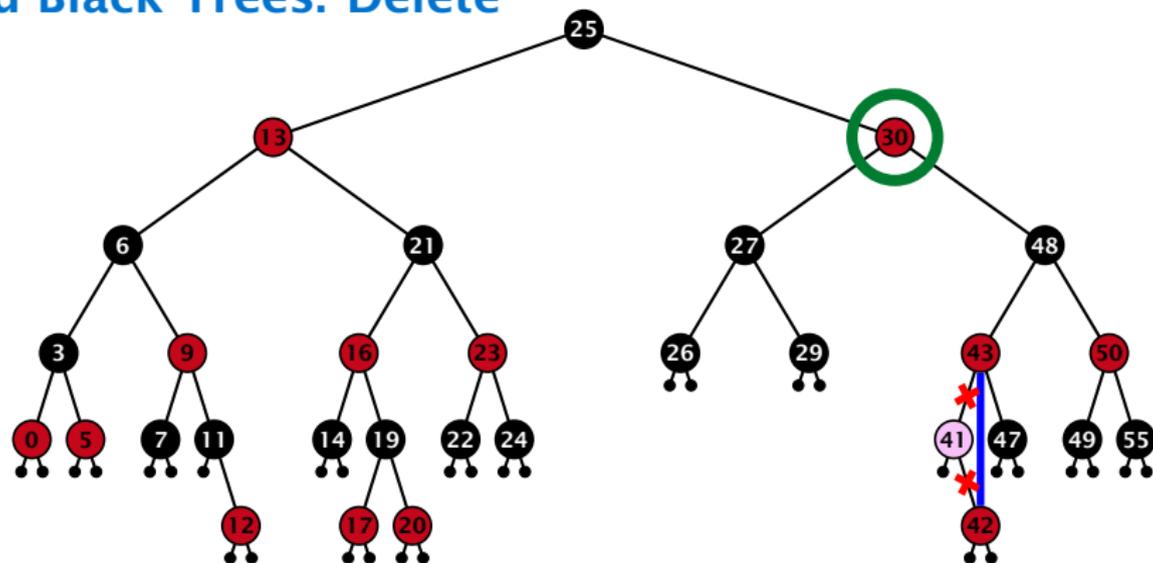


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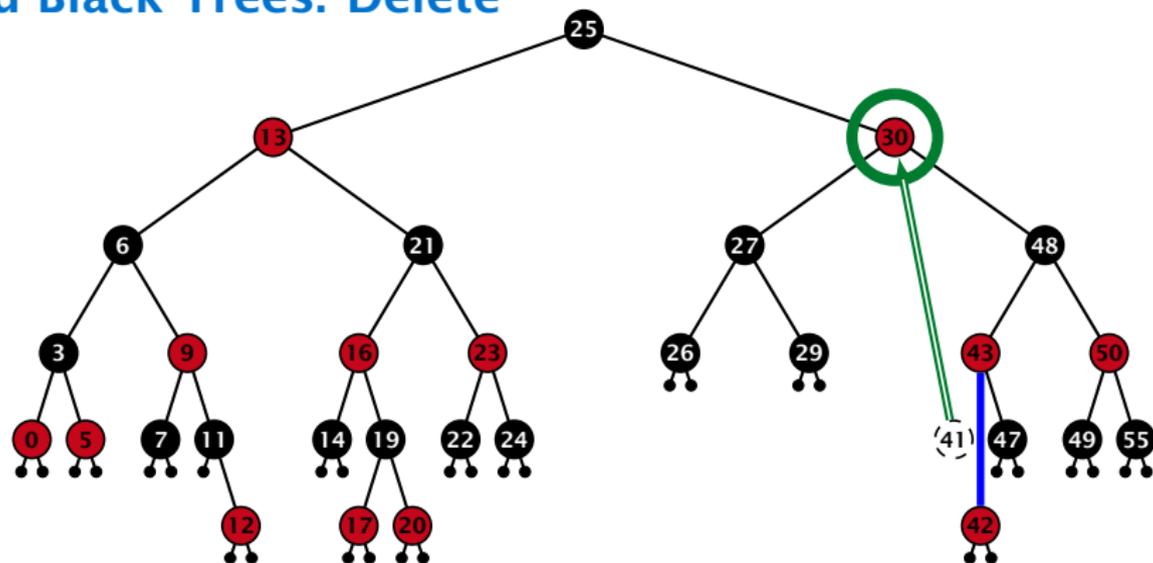


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Element has two children

- ▶ do normal delete
- ▶ when replacing content by content of successor, don't change color of node

Red Black Trees: Delete

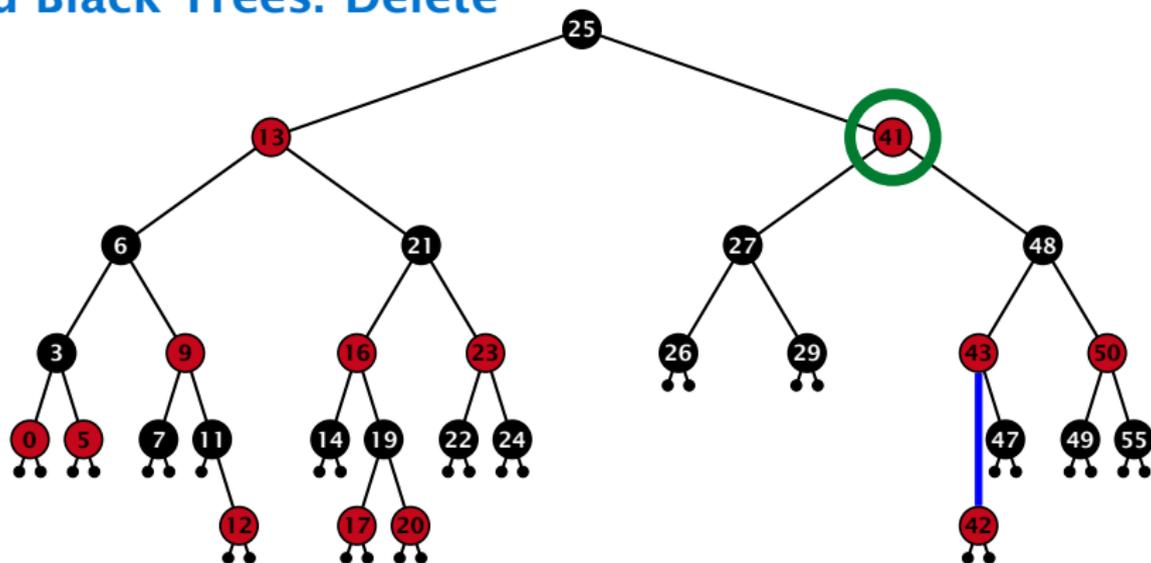


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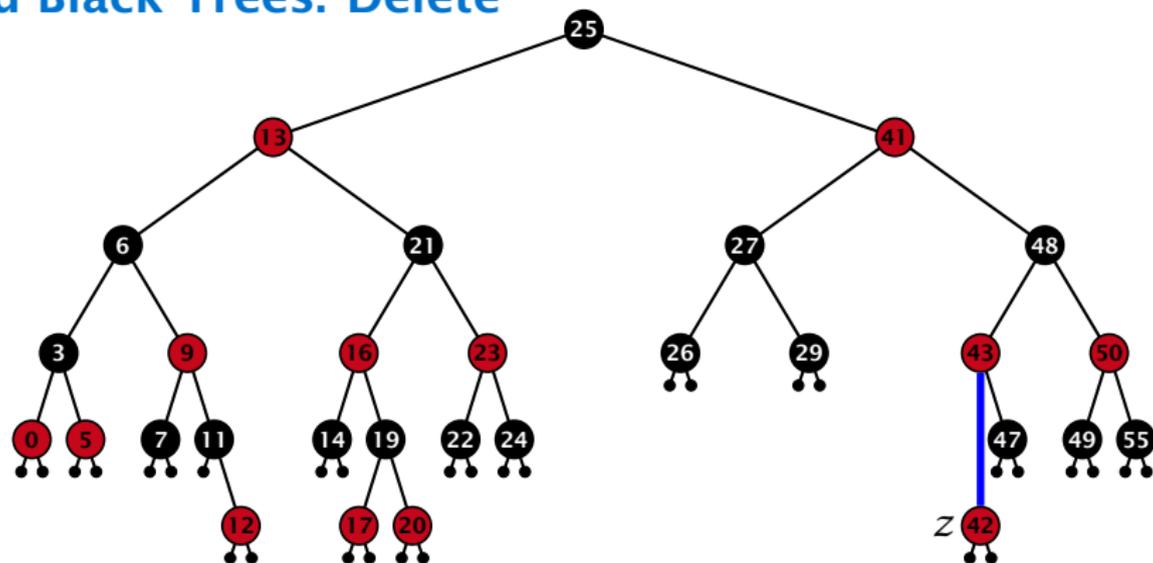


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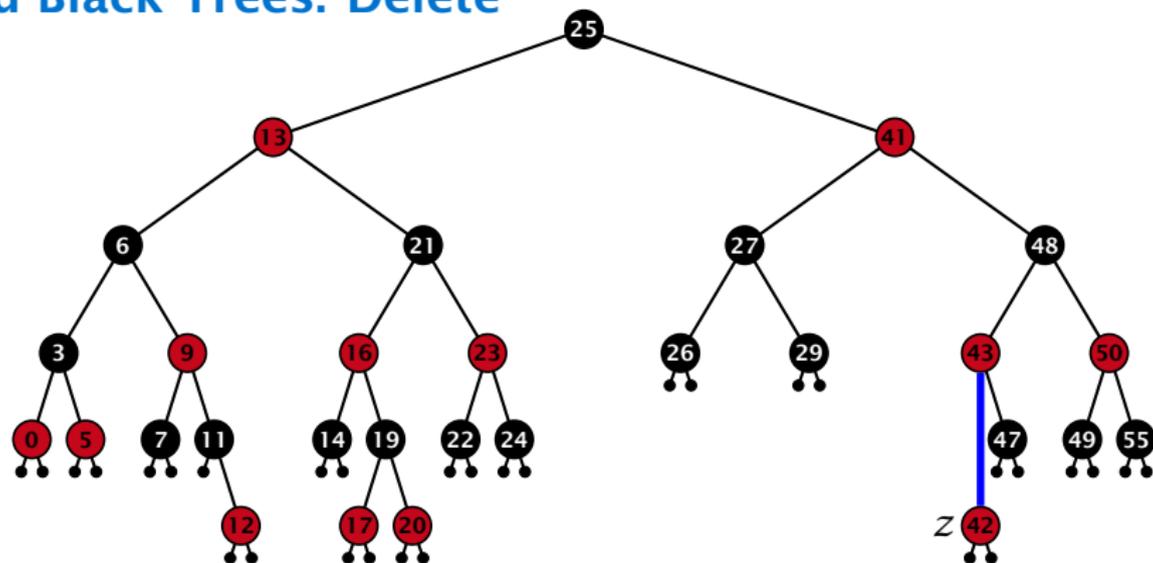
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Delete:

- ▶ deleting black node messes up black-height property
- ▶ if z is red, we can simply color it black and everything is fine
- ▶ the problem is if z is black (e.g. a dummy-leaf); we call a fix-up procedure to fix the problem.

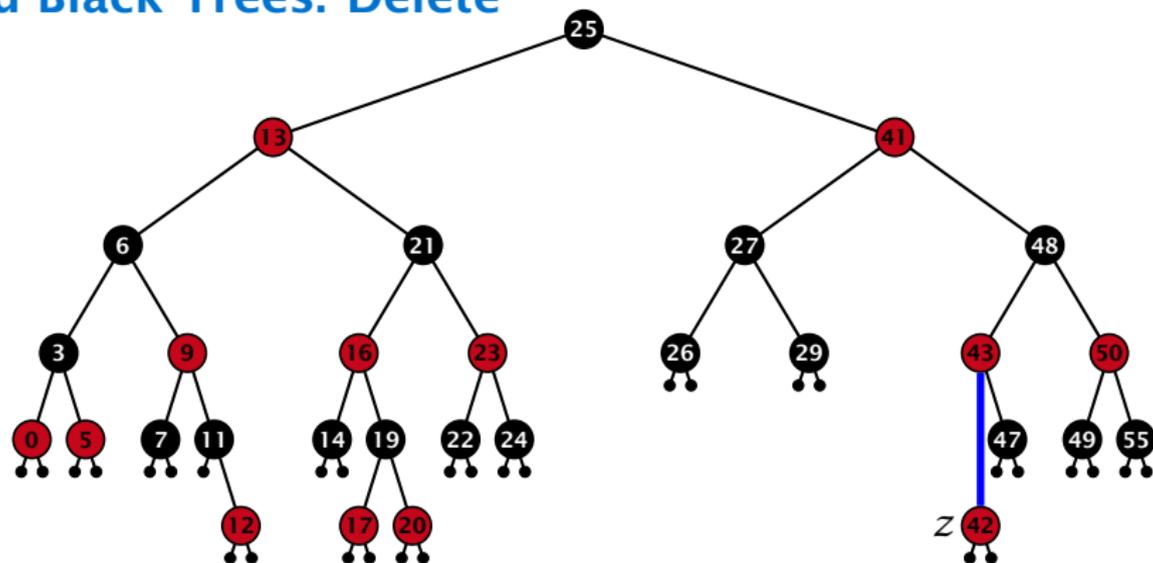
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Invariant of the fix-up algorithm

- ▶ the node z is black
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Goal: make rotations in such a way that you at some point can remove the fake black unit from the edge.

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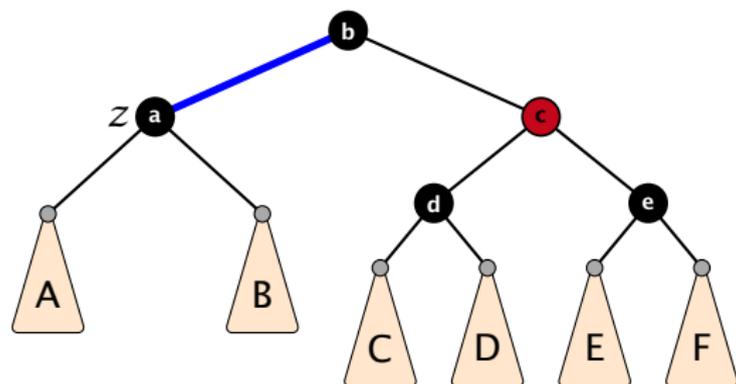
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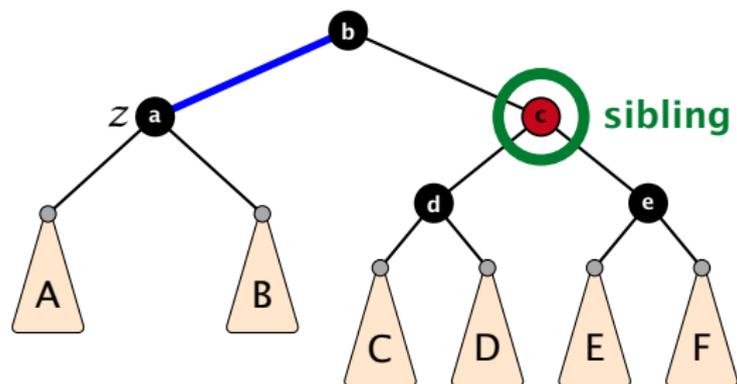
Case 1: Sibling of z is red



1. left-rotate around parent of z
2. recolor nodes b and c
3. the new sibling is black
(and parent of z is red)
4. Case 2 (special),
or Case 3, or Case 4



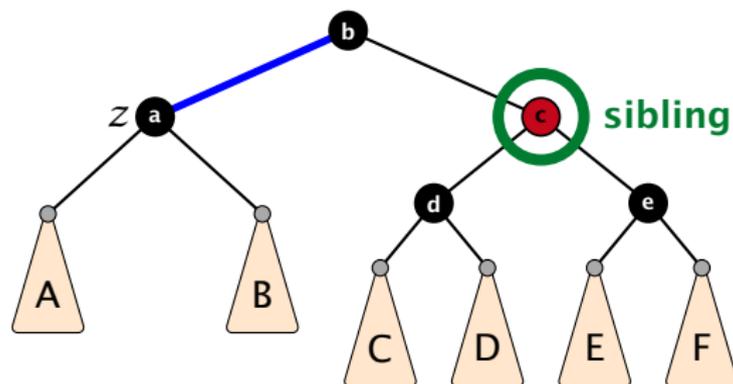
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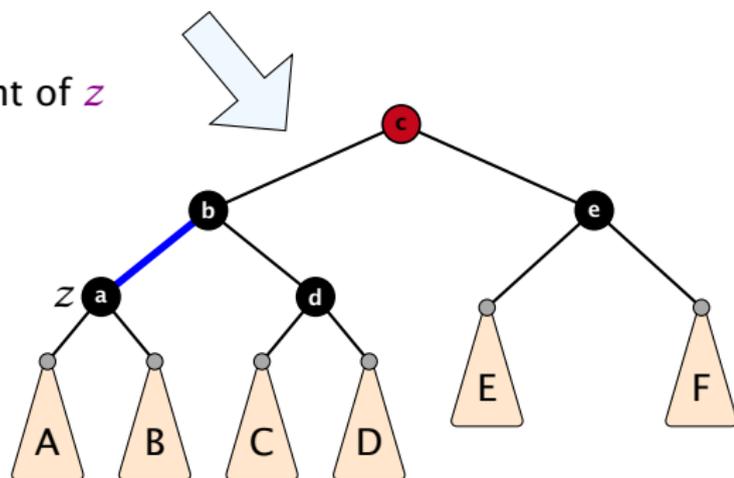


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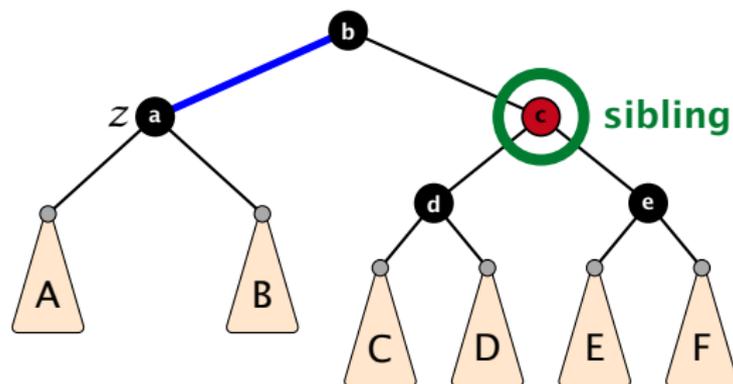
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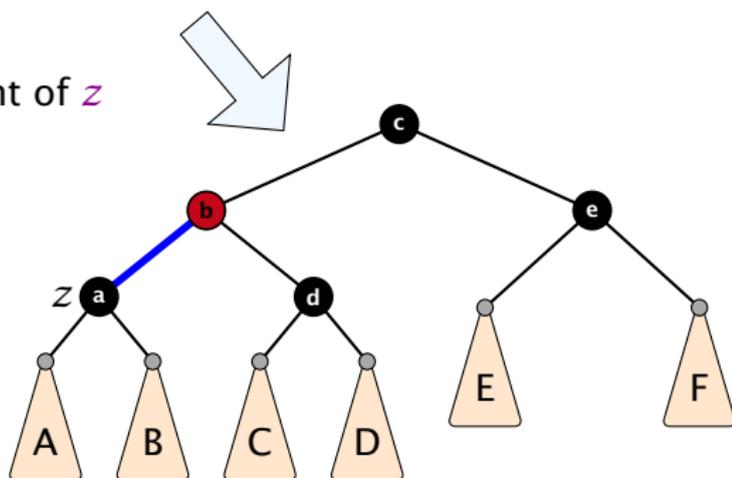
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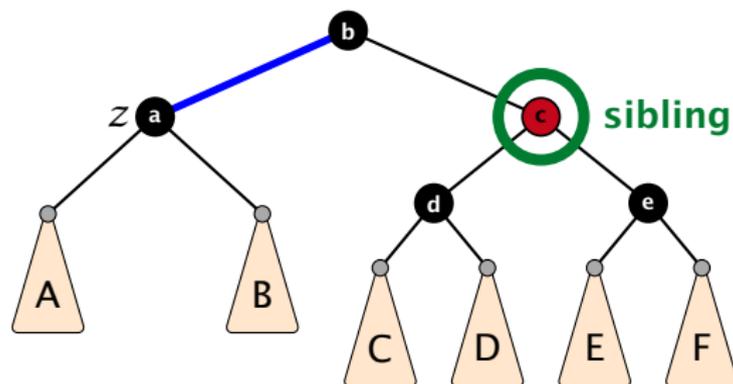
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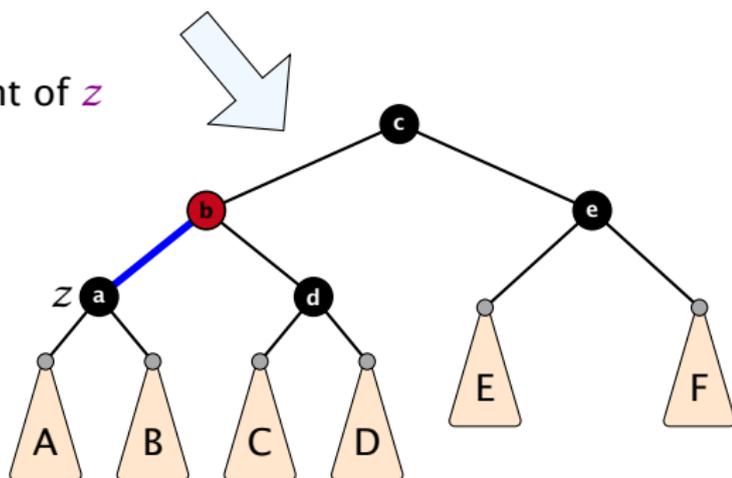
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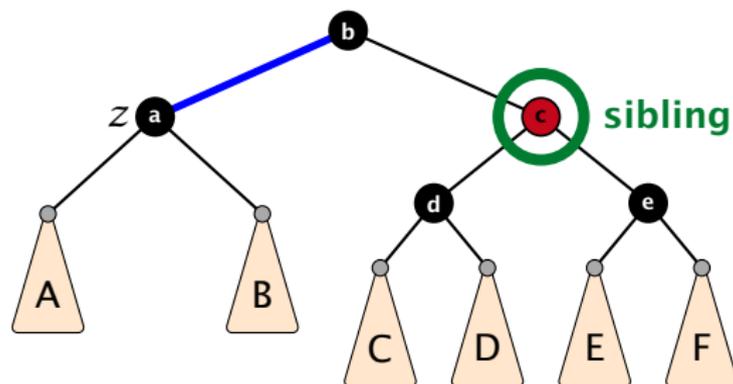
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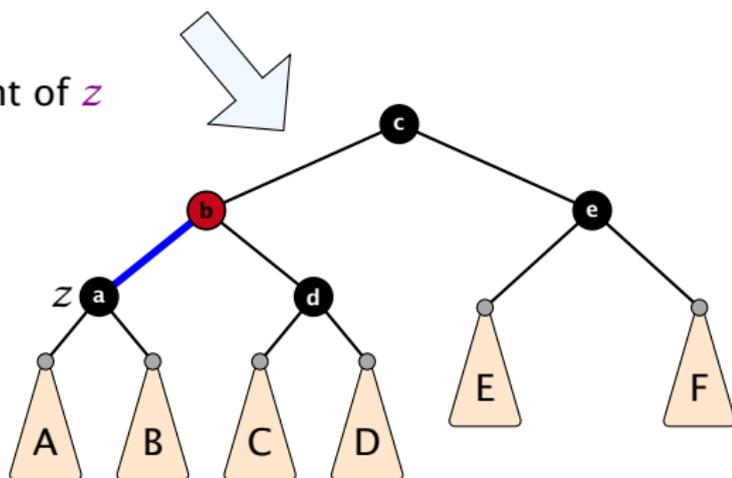
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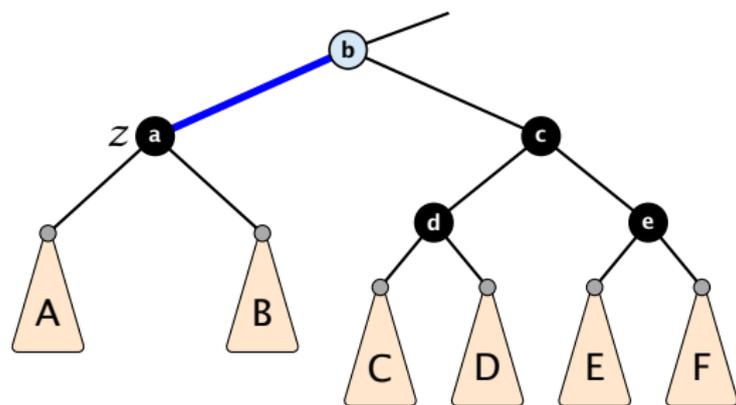
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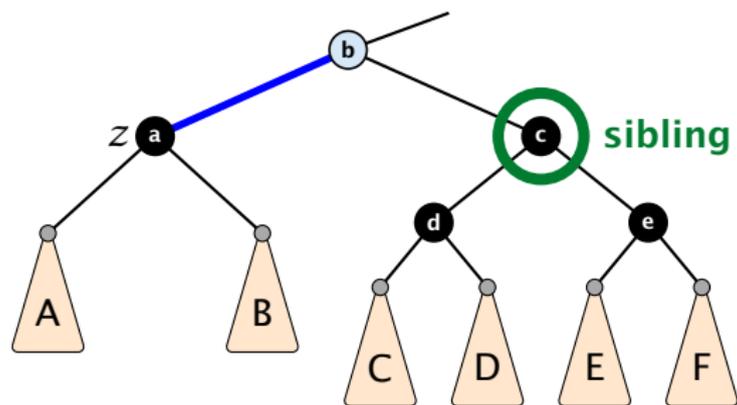
Case 2: Sibling is black with two black children



1. re-color node c
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4. we made progress
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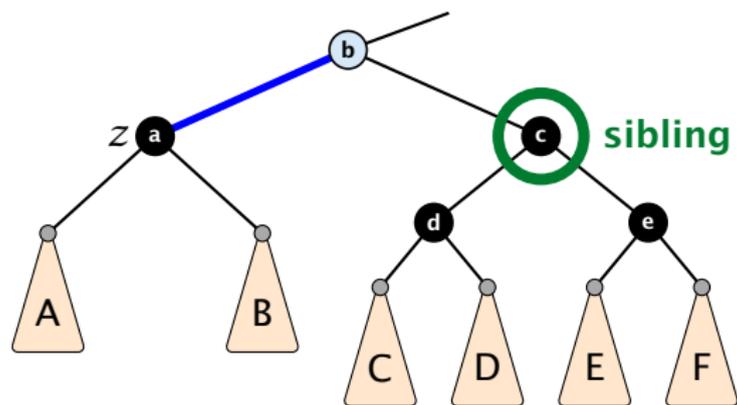
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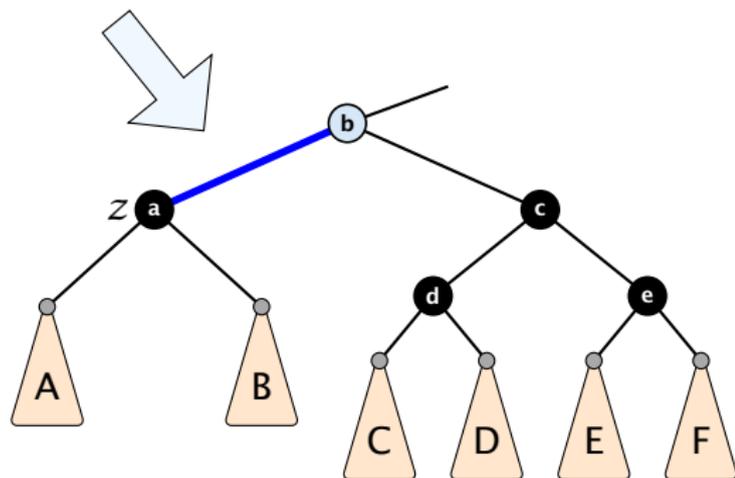
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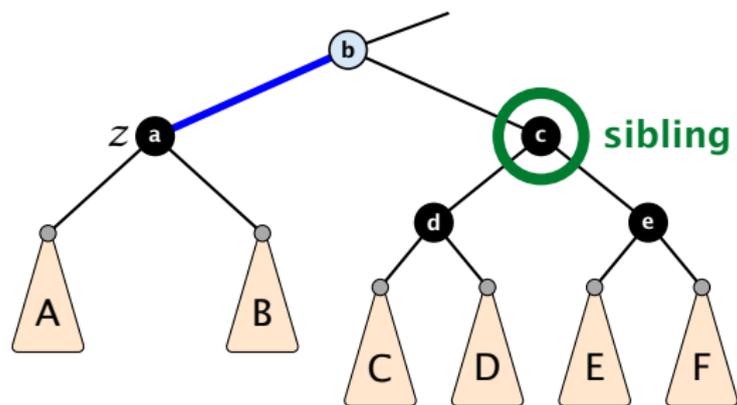
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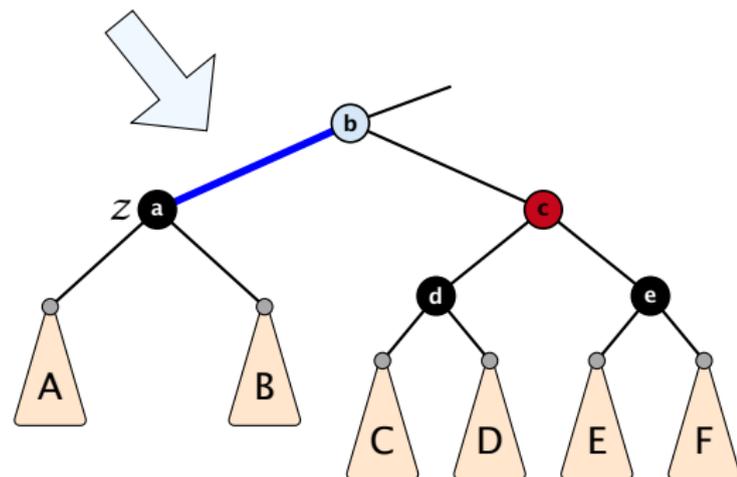
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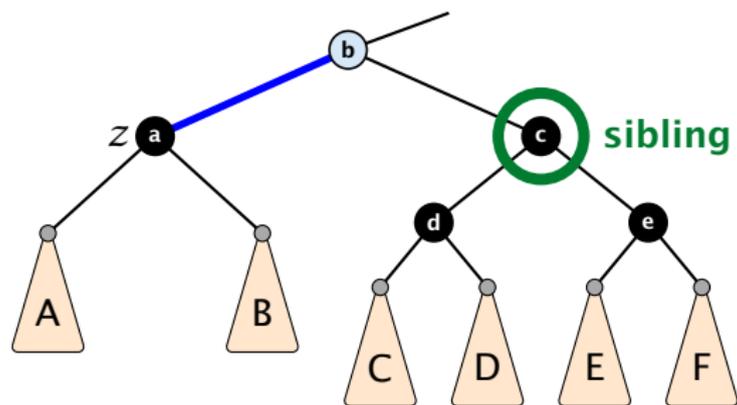
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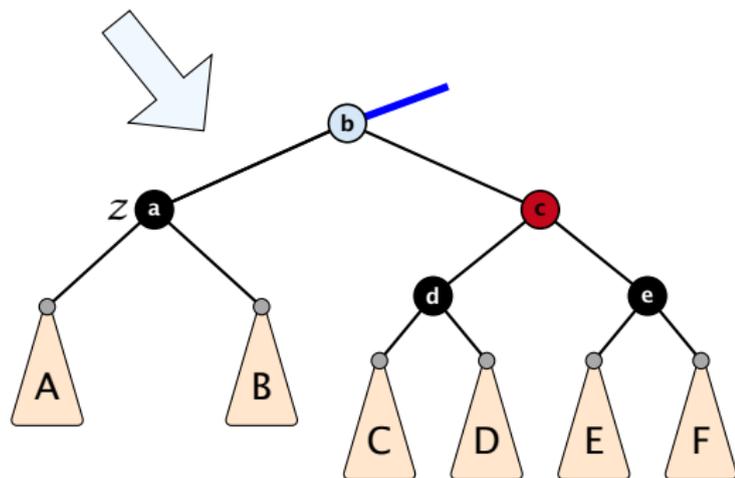
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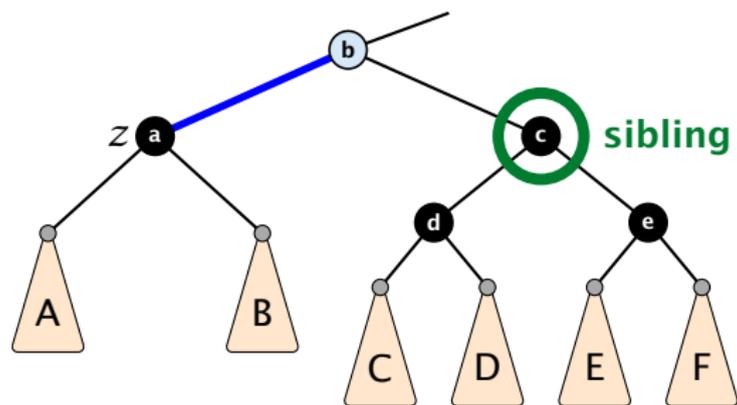
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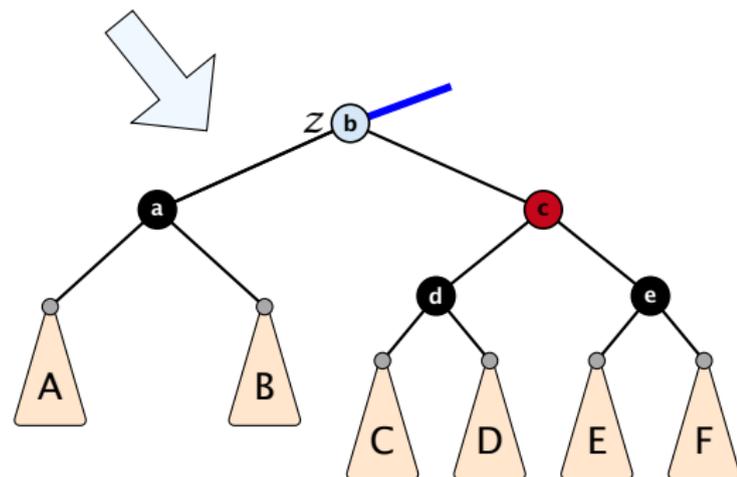
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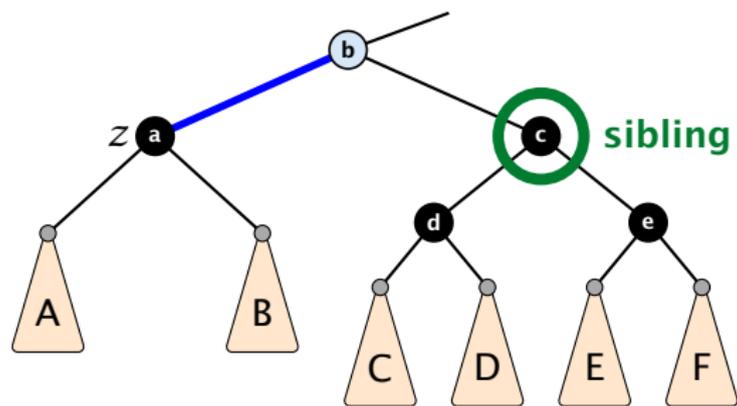
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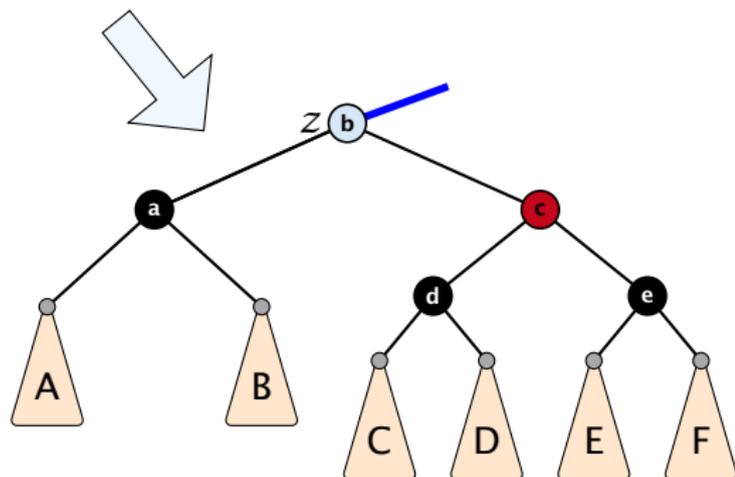
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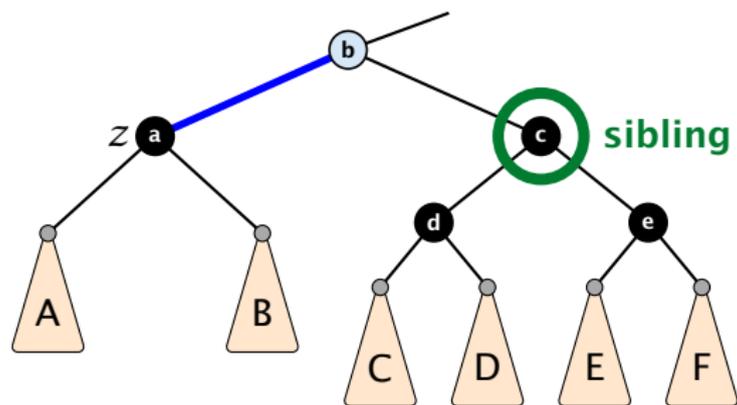
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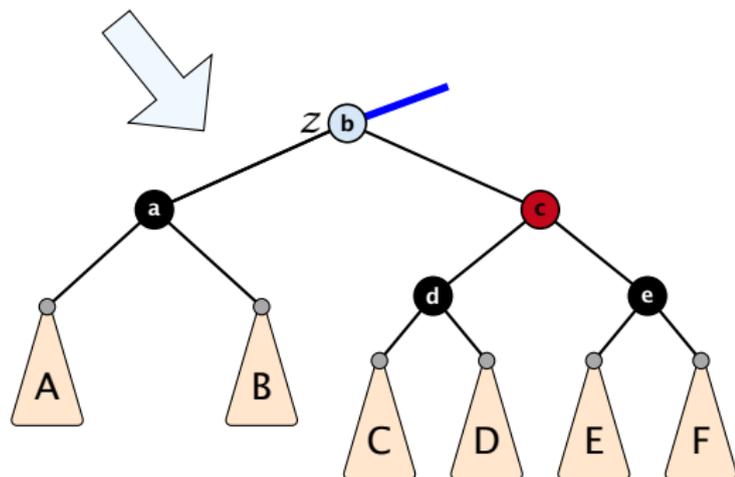
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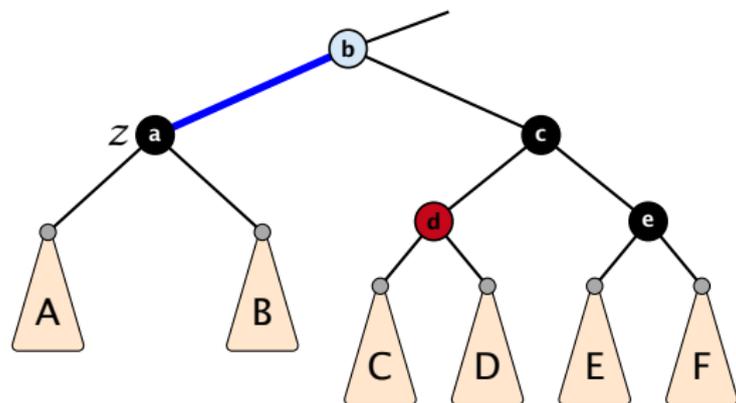


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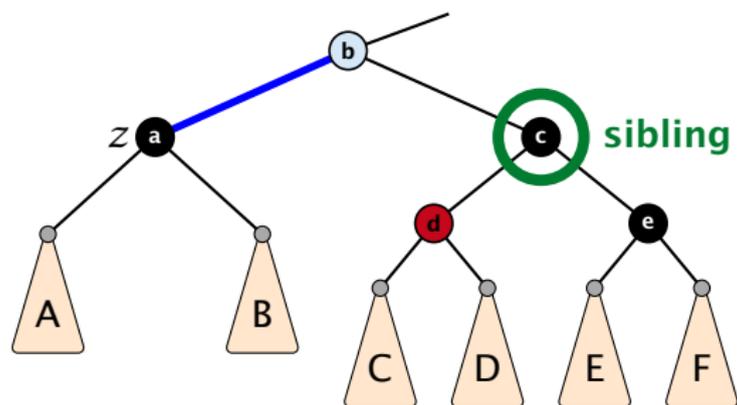
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1. do a right-rotation at sibling
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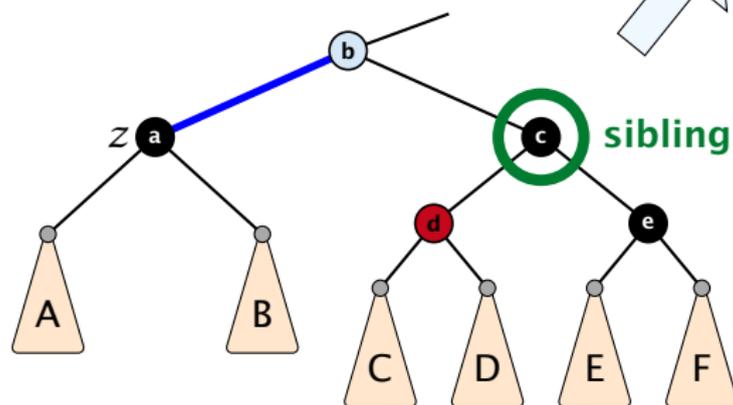
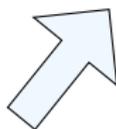
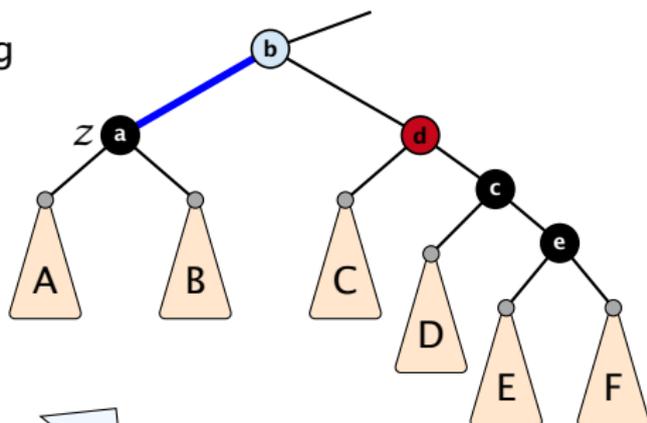
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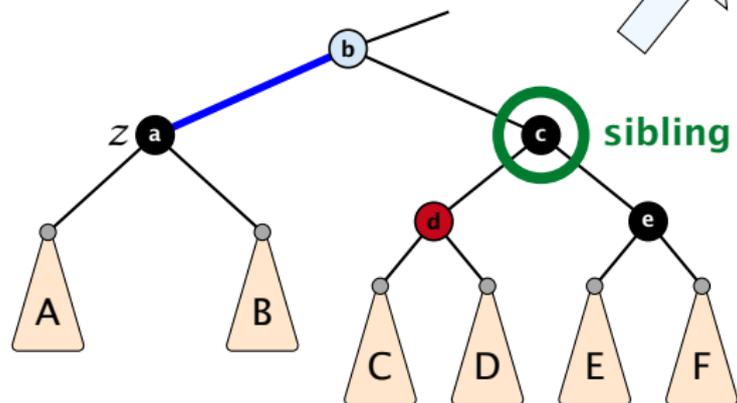
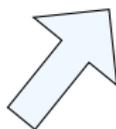
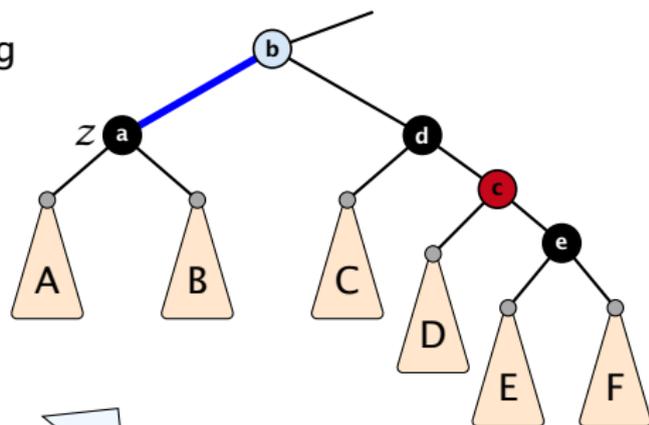
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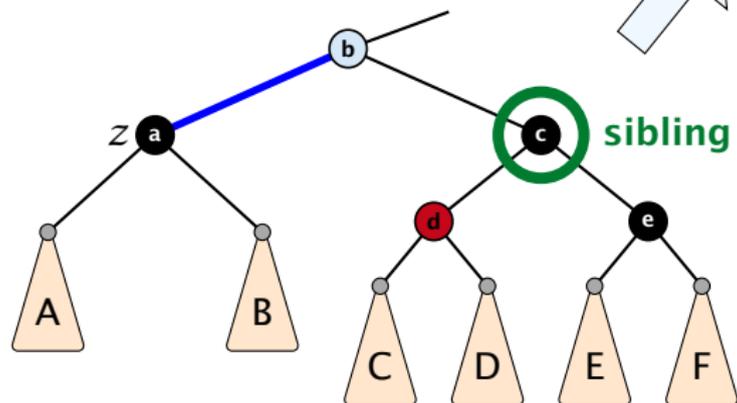
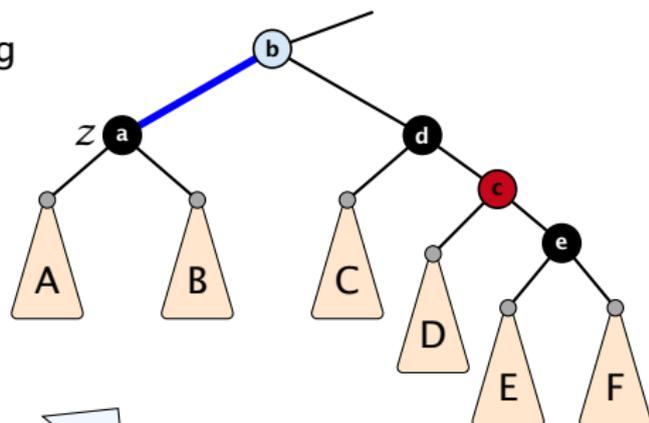
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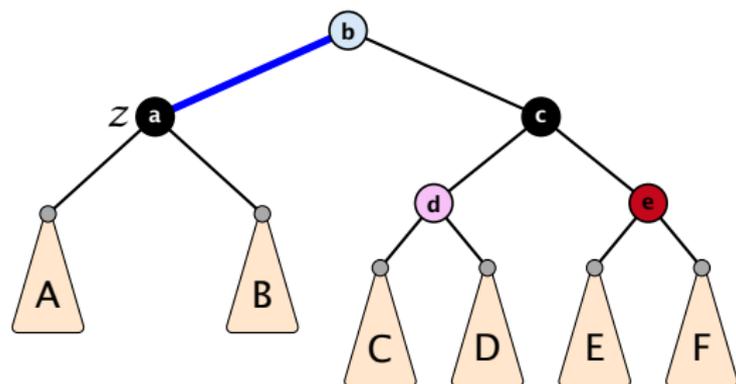


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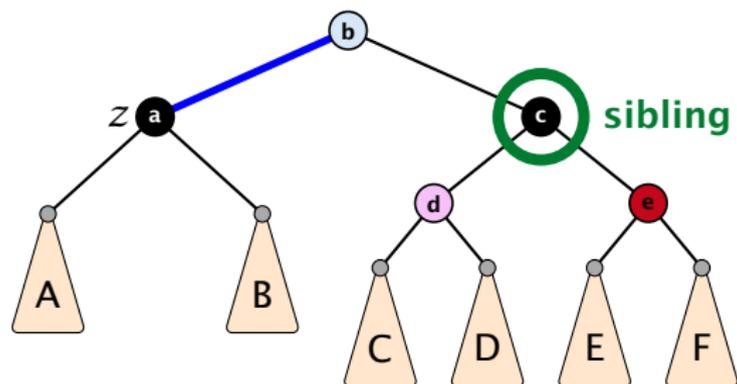
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1. left-rotate around b
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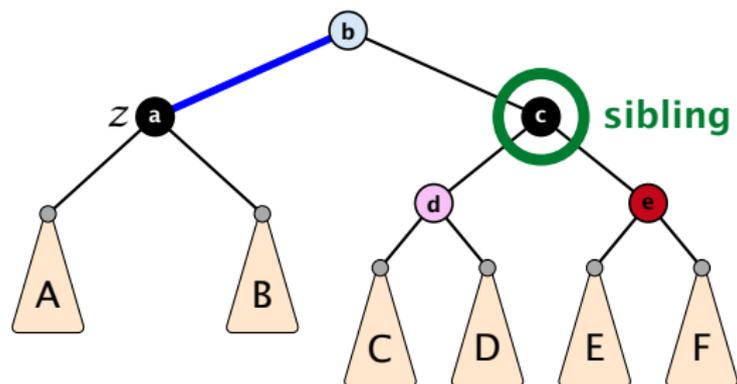
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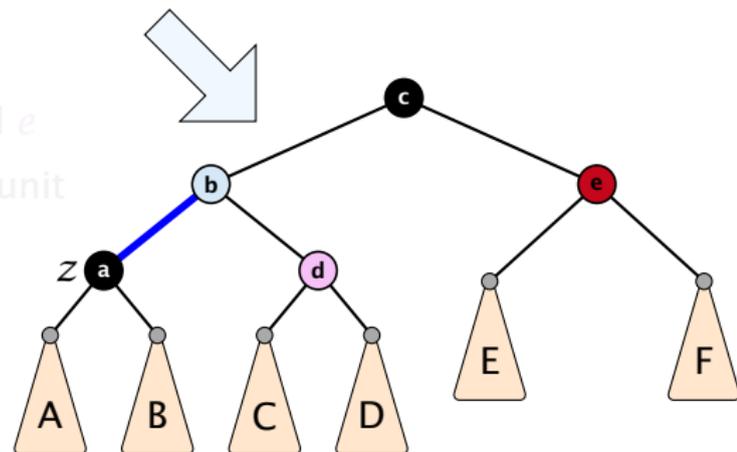
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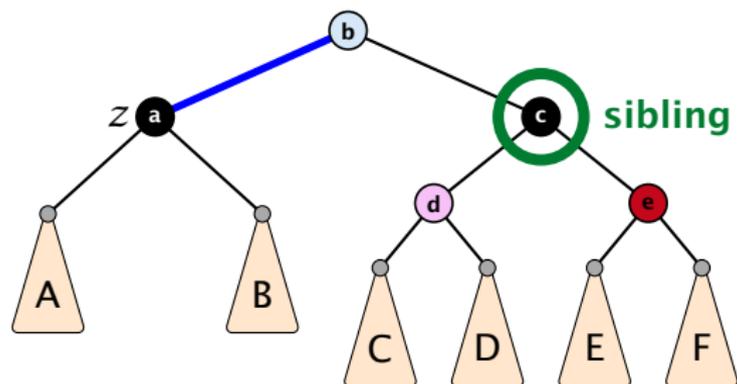
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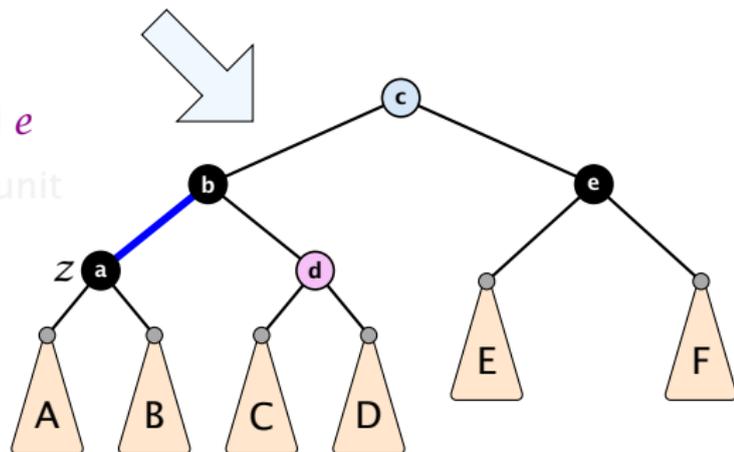
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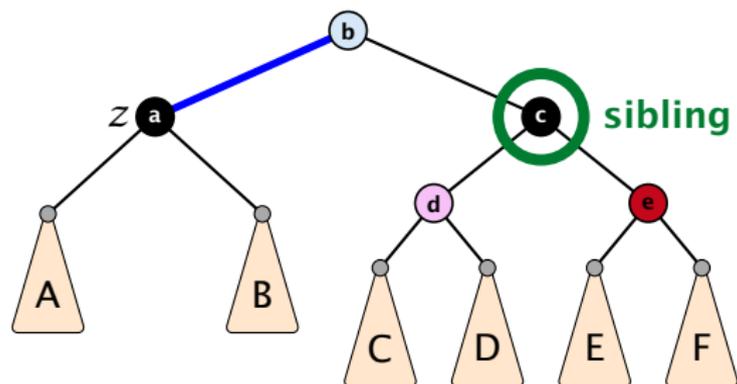
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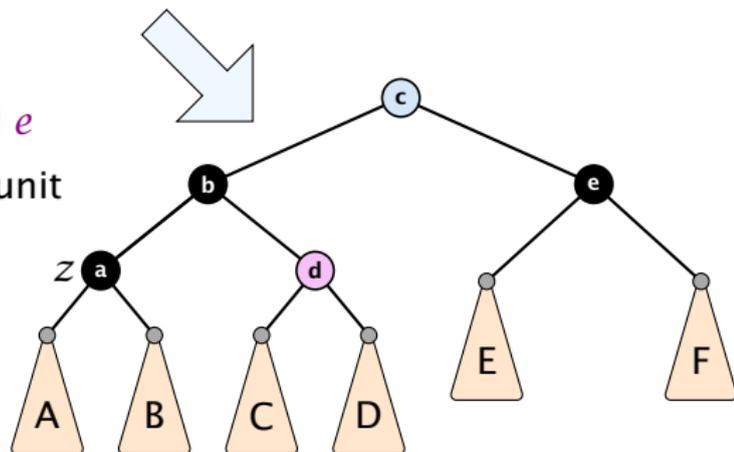
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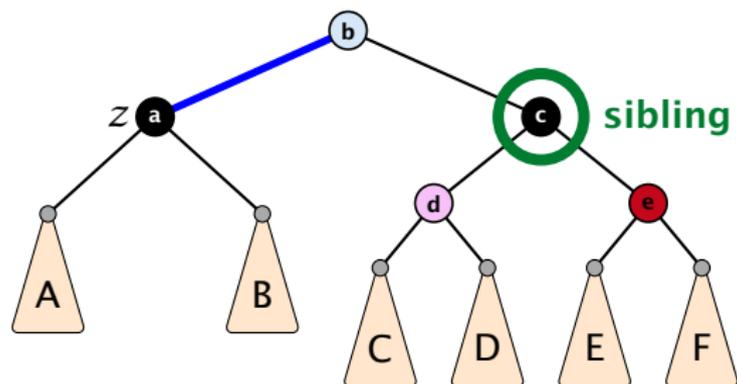
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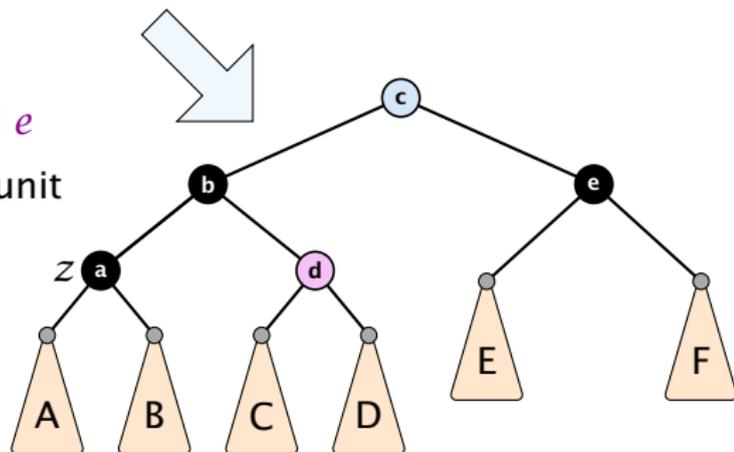
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Running time:

- ▶ only Case 2 can repeat; but only h many steps, where h is the height of the tree
- ▶ Case 1 → Case 2 (special) → red black tree
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Performing Case 2 at most $\mathcal{O}(\log n)$ times and every other step at most once, we get a red black tree. Hence, $\mathcal{O}(\log n)$ re-colorings and at most 3 rotations.

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7.3 AVL-Trees

Definition 5

AVL-trees are binary search trees that fulfill the following balance condition. For every node v

$$|\text{height}(\text{left sub-tree}(v)) - \text{height}(\text{right sub-tree}(v))| \leq 1 .$$

Lemma 6

An AVL-tree of height h contains at least $F_{h+2} - 1$ and at most $2^h - 1$ internal nodes, where F_n is the n -th Fibonacci number ($F_0 = 0$, $F_1 = 1$), and the height is the maximal number of edges from the root to an (empty) dummy leaf.

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AVL trees

Proof.

The upper bound is clear, as a binary tree of height h can only contain

$$\sum_{j=0}^{h-1} 2^j = 2^h - 1$$

internal nodes.

AVL trees

Proof (cont.)

Induction (base cases):

1. an AVL-tree of height $h = 1$ contains at least one internal node, $1 \geq F_3 - 1 = 2 - 1 = 1$.
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AVL trees

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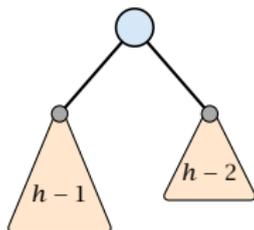


Induction step:

An AVL-tree of height $h \geq 2$ of minimal size has a root with sub-trees of height $h - 1$ and $h - 2$, respectively. Both, sub-trees have minimal node number.

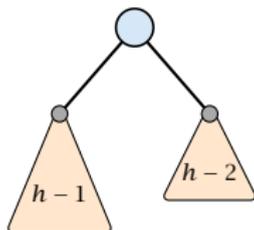
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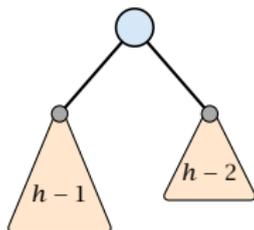


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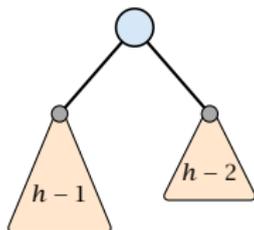
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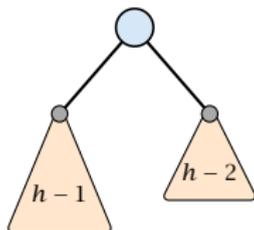
$g_h := 1 + \text{minimal size of AVL-tree of height } h$.

Then

$$g_1 = 2 \qquad \qquad \qquad = F_3$$

Induction step:

An AVL-tree of height $h \geq 2$ of minimal size has a root with sub-trees of height $h - 1$ and $h - 2$, respectively. Both, sub-trees have minimal node number.



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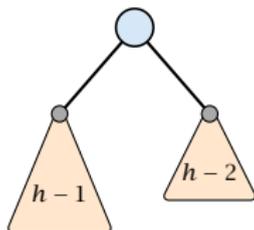
Then

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$$g_2 = 3 \qquad \qquad \qquad = F_4$$

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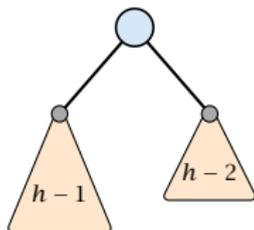
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$$g_{h-1} = 1 + g_{h-1-1} + g_{h-2-1}, \qquad \text{hence}$$

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$$g_{h-1} = 1 + g_{h-1-1} + g_{h-2-1}, \qquad \text{hence}$$

$$g_h = g_{h-1} + g_{h-2} \qquad = F_{h+2}$$

7.3 AVL-Trees

An AVL-tree of height h contains at least $F_{h+2} - 1$ internal nodes.

Since

$$n + 1 \geq F_{h+2} = \Omega \left(\left(\frac{1 + \sqrt{5}}{2} \right)^h \right),$$

we get

$$n \geq \Omega \left(\left(\frac{1 + \sqrt{5}}{2} \right)^h \right),$$

and, hence, $h = \mathcal{O}(\log n)$.

7.3 AVL-Trees

We need to maintain the balance condition through rotations.

For this we store in every internal tree-node v the **balance** of the node. Let v denote a tree node with left child c_ℓ and right child c_r .

$$\text{balance}[v] := \text{height}(T_{c_\ell}) - \text{height}(T_{c_r}) ,$$

where T_{c_ℓ} and T_{c_r} , are the sub-trees rooted at c_ℓ and c_r , respectively.

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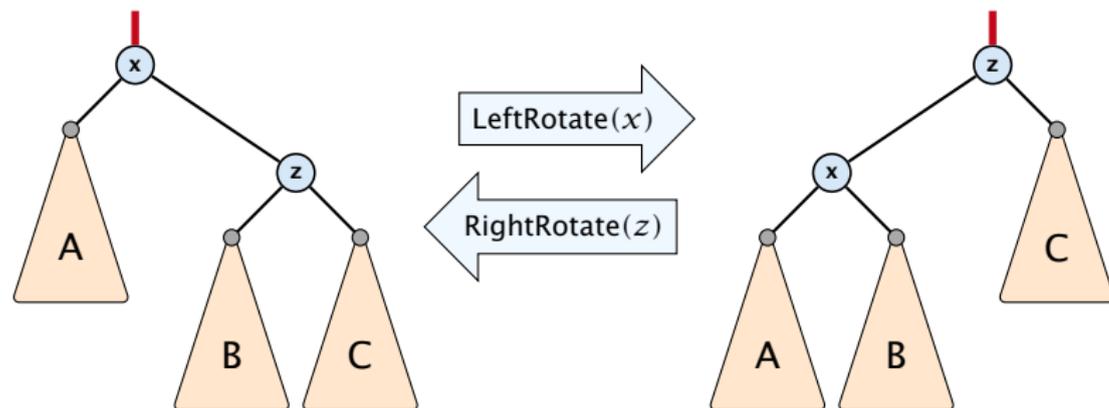
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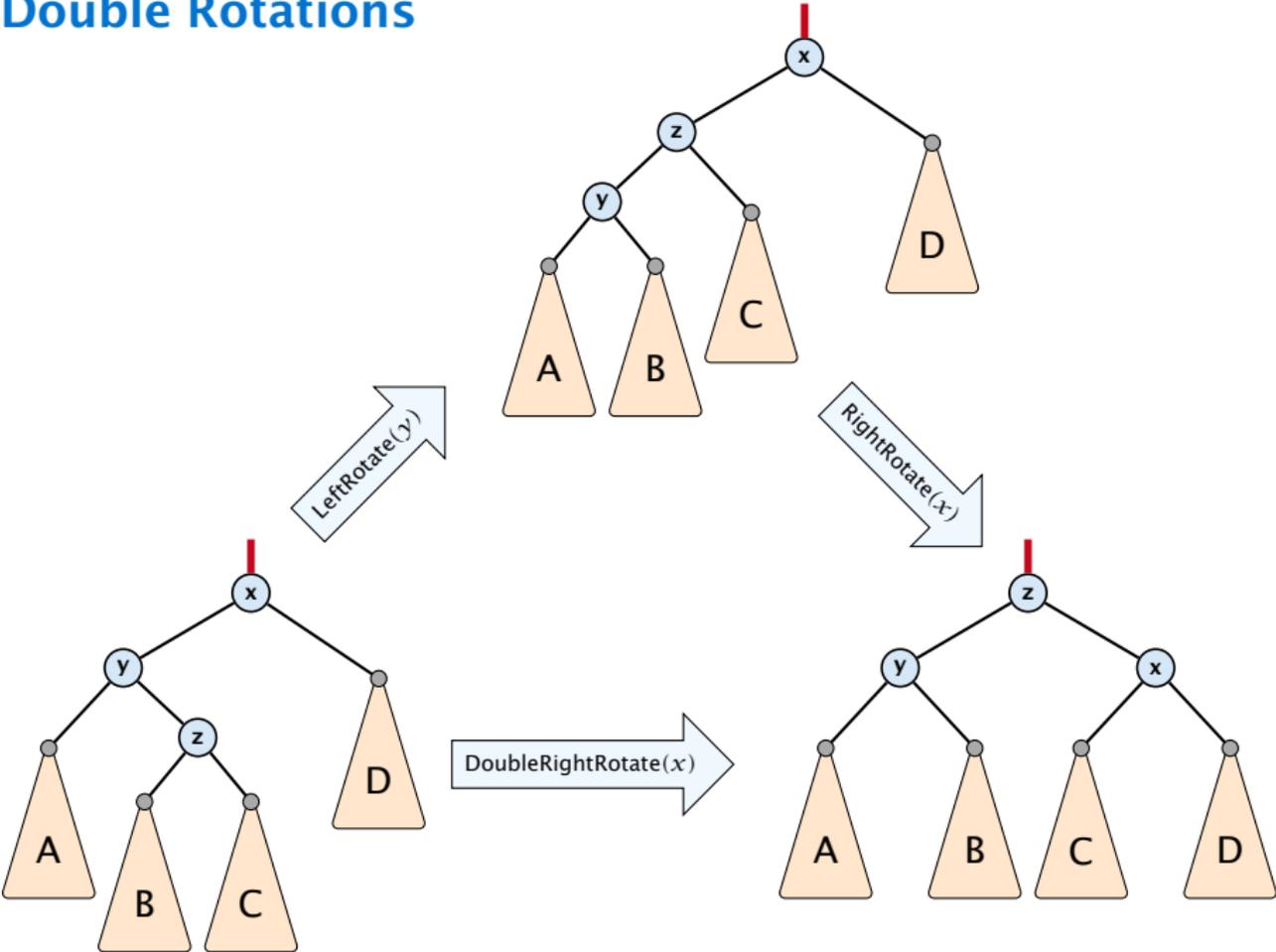
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Rotations

The properties will be maintained through rotations:



Double Rotations



AVL-trees: Insert

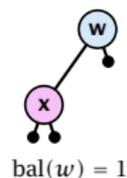
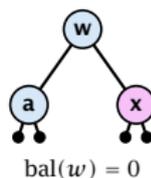
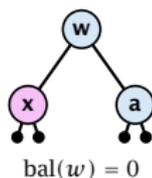
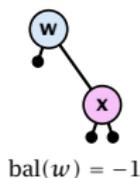
- ▶ Insert like in a binary search tree.

AVL-trees: Insert

- ▶ Insert like in a binary search tree.
- ▶ Let w denote the parent of the newly inserted node x .

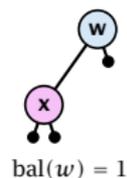
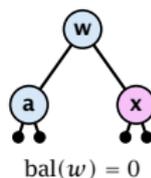
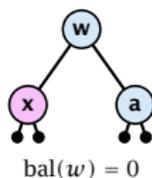
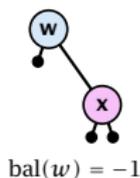
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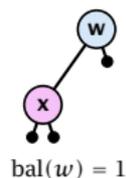
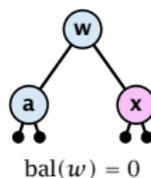
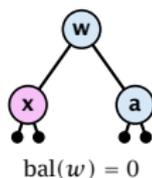
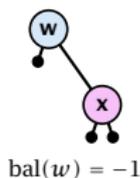
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- ▶ Insert like in a binary search tree.
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- ▶ One of the following cases holds:



- ▶ If $\text{bal}[w] \neq 0$, T_w has changed height; the balance-constraint may be violated at ancestors of w .
- ▶ Call $\text{AVL-fix-up-insert}(\text{parent}[w])$ to restore the balance-condition.

Invariant at the beginning of AVL-fix-up-insert(v):

1. The balance constraints hold at all descendants of v .
2. A node has been inserted into T_c , where c is either the right or left child of v .
3. T_c has increased its height by one (otw. we would already have aborted the fix-up procedure).
4. The balance at node c fulfills $\text{balance}[c] \in \{-1, 1\}$. This holds because if the balance of c is 0, then T_c did not change its height, and the whole procedure would have been aborted in the previous step.

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AVL-trees: Insert

Algorithm 11 AVL-fix-up-insert(v)

- 1: **if** $\text{balance}[v] \in \{-2, 2\}$ **then** DoRotationInsert(v);
- 2: **if** $\text{balance}[v] \in \{0\}$ **return**;
- 3: AVL-fix-up-insert(parent(v));

We will show that the above procedure is correct, and that it will do at most one rotation.

Algorithm 12 DoRotationInsert(v)

```
1: if balance[ $v$ ] = -2 then // insert in right sub-tree
2:     if balance[right[ $v$ ]] = -1 then
3:         LeftRotate( $v$ );
4:     else
5:         DoubleLeftRotate( $v$ );
6: else // insert in left sub-tree
7:     if balance[left[ $v$ ]] = 1 then
8:         RightRotate( $v$ );
9:     else
10:        DoubleRightRotate( $v$ );
```

AVL-trees: Insert

It is clear that the invariant for the fix-up routine holds as long as no rotations have been done.

We have to show that after doing one rotation all balance constraints are fulfilled.

We show that after doing a rotation at v :

- ▶ v fulfills balance condition.
- ▶ All children of v still fulfill the balance condition.
- ▶ The height of T_v is the same as before the insert-operation took place.

We only look at the case where the insert happened into the right sub-tree of v . The other case is symmetric.

AVL-trees: Insert

It is clear that the invariant for the fix-up routine holds as long as no rotations have been done.

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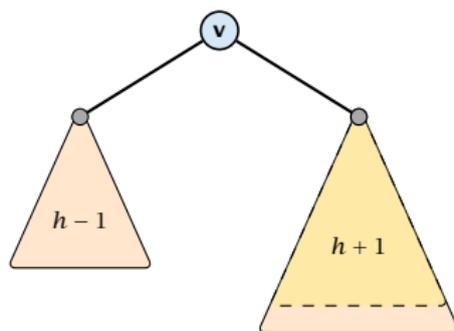
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We only look at the case where the insert happened into the right sub-tree of v . The other case is symmetric.

AVL-trees: Insert

We have the following situation:



The right sub-tree of v has increased its height which results in a balance of -2 at v .

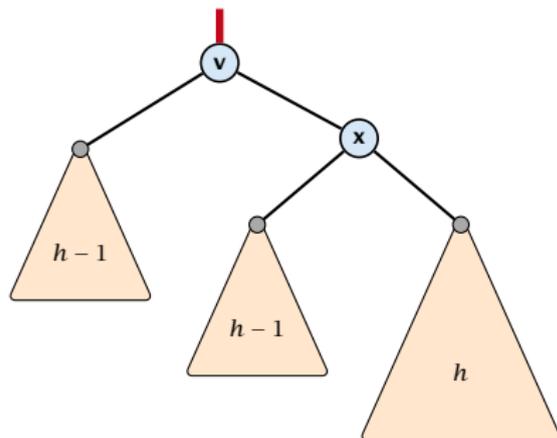
Before the insertion the height of T_v was $h+1$.

Case 1: $\text{balance}[\text{right}[v]] = -1$

We do a left rotation at v

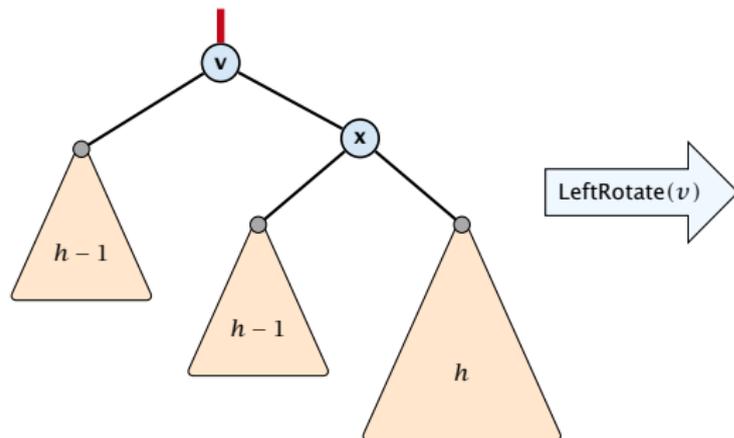
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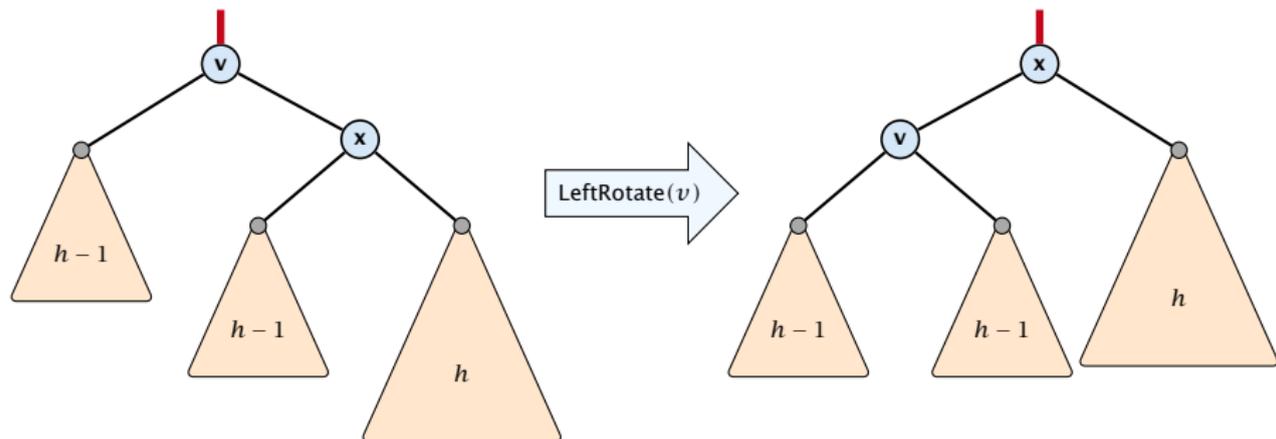
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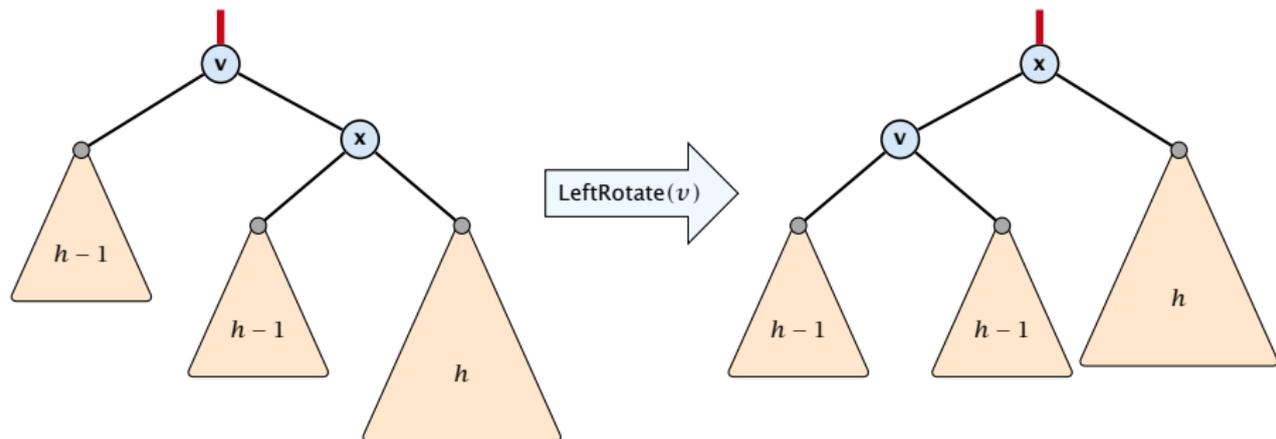
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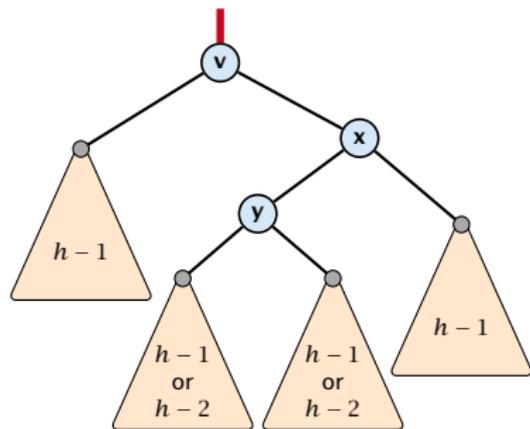
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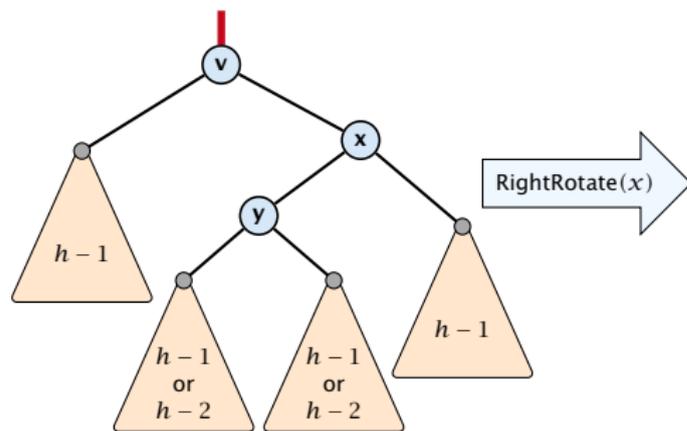
Now, the subtree has height $h + 1$ as before the insertion.
Hence, we do not need to continue.

Case 2: $\text{balance}[\text{right}[v]] = 1$

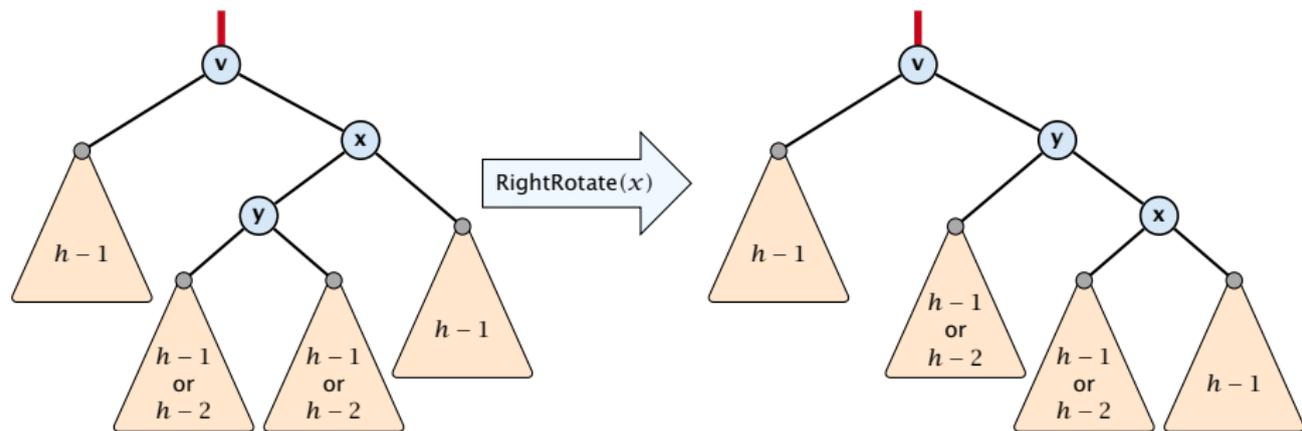
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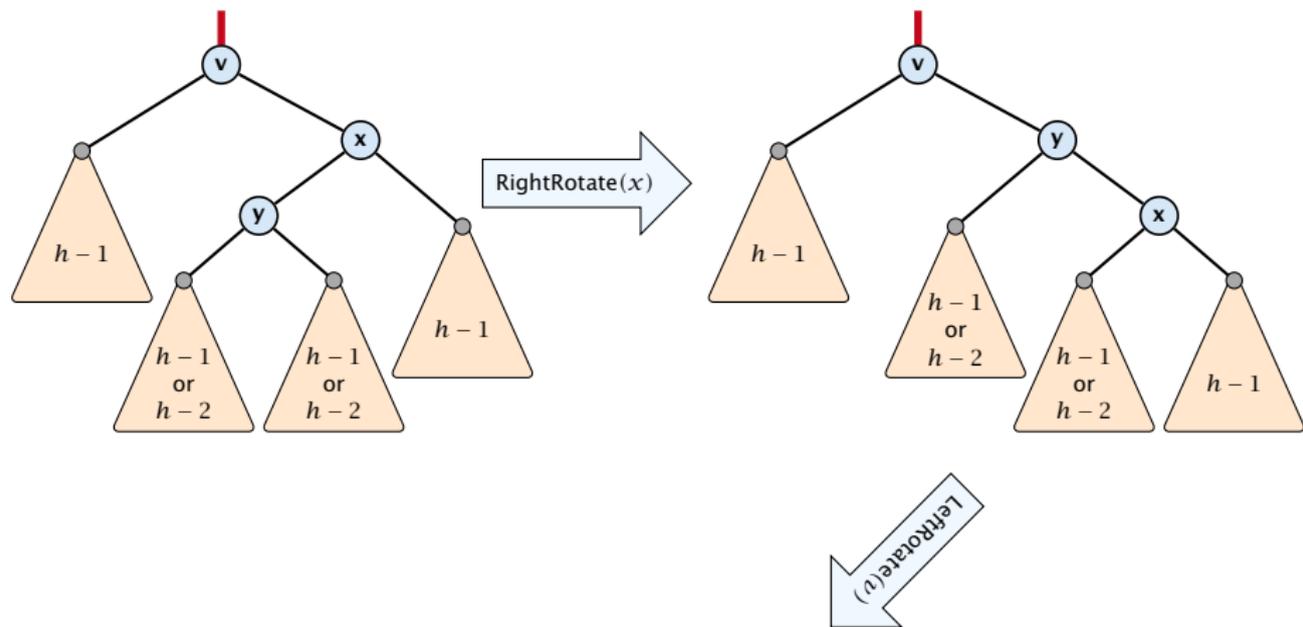
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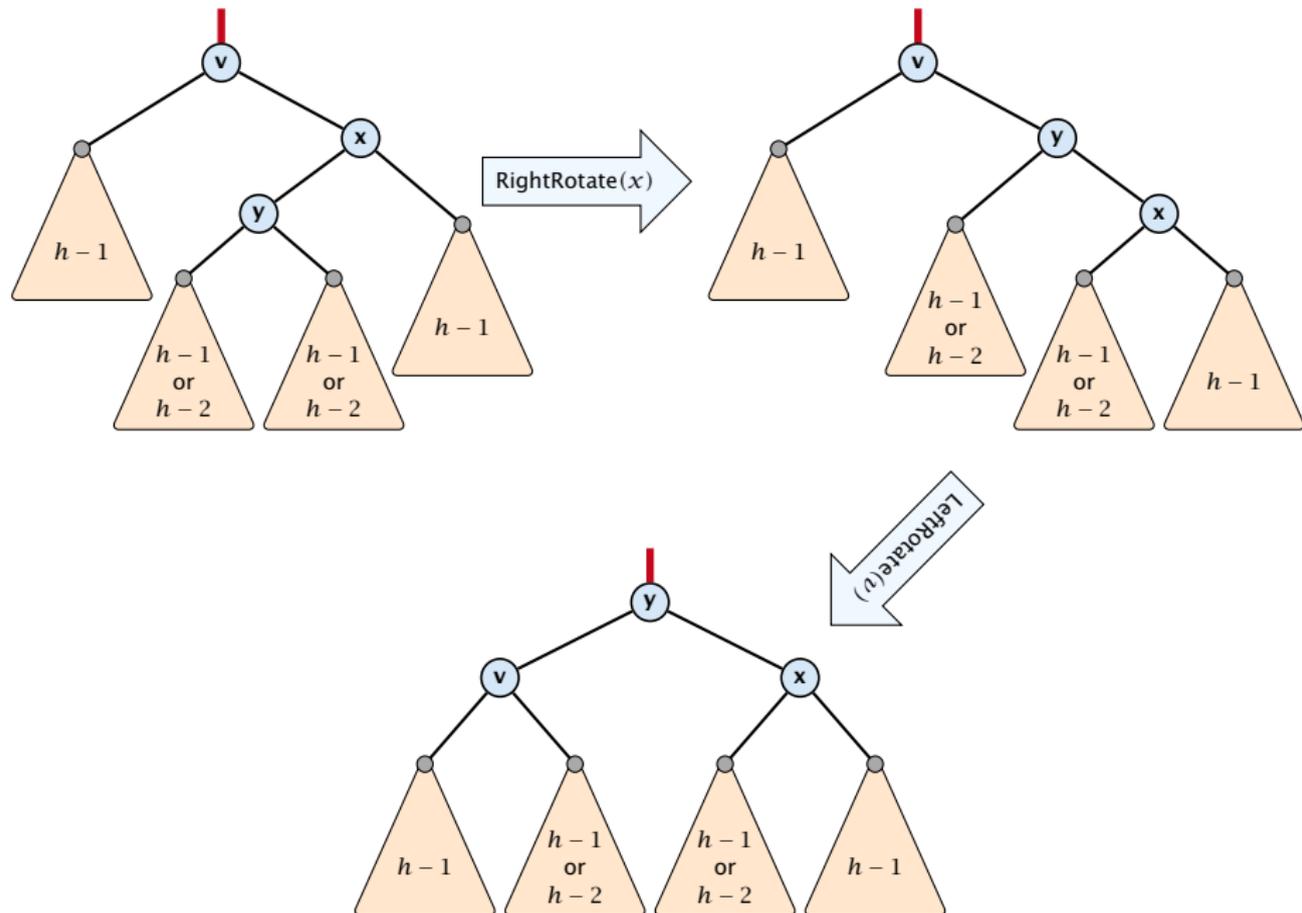
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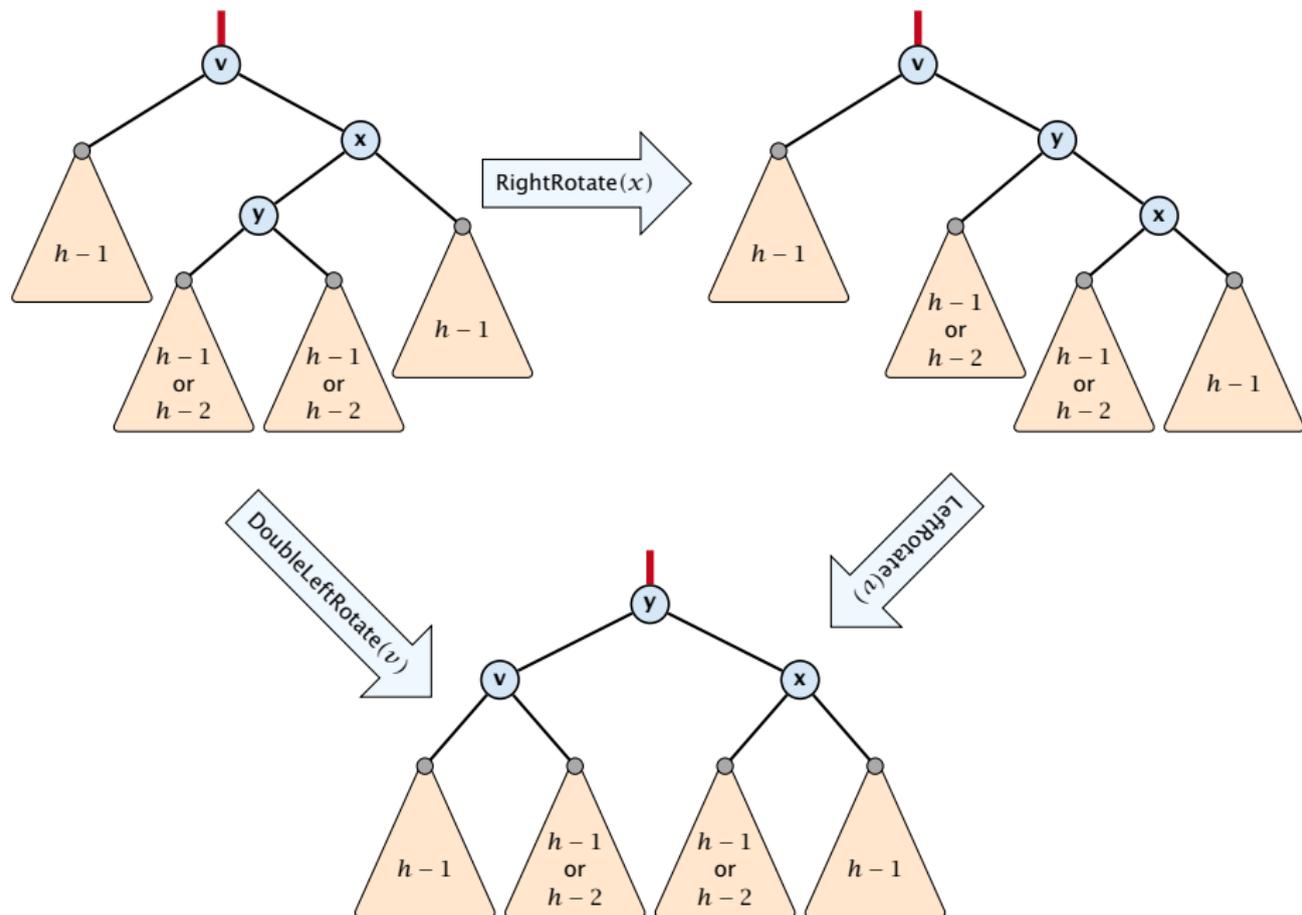
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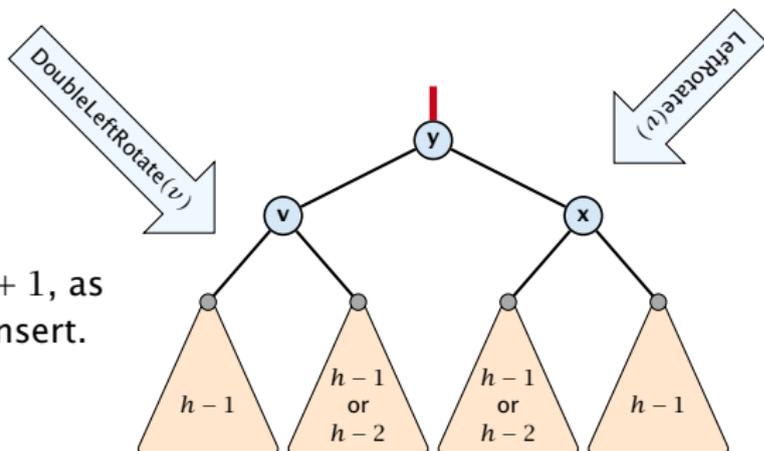
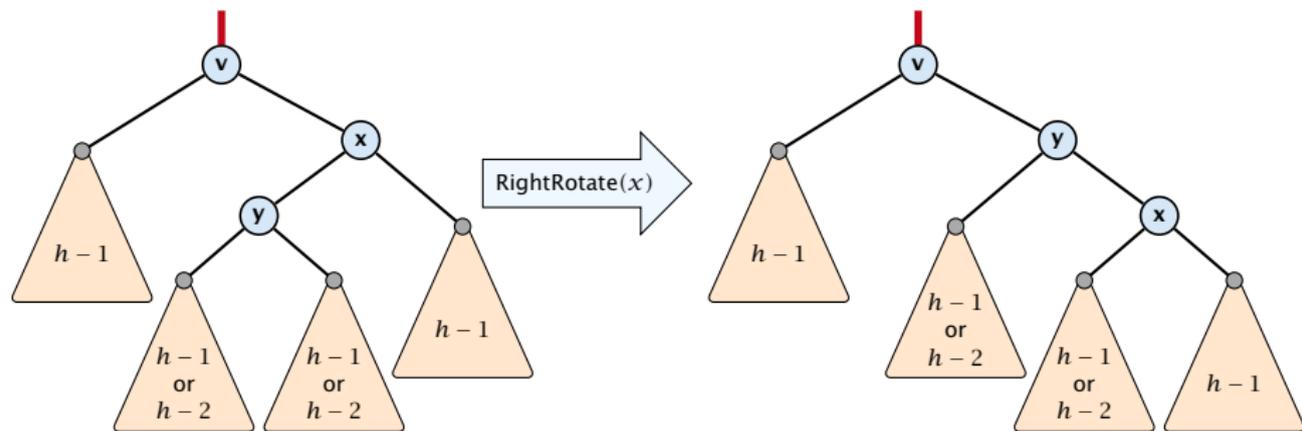
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Height is $h + 1$, as before the insert.

AVL-trees: Delete

- ▶ Delete like in a binary search tree.
- ▶ Let v denote the parent of the node that has been spliced out.
- ▶ The balance-constraint may be violated at v , or at ancestors of v , as a sub-tree of a child of v has reduced its height.
- ▶ Initially, the node c —the new root in the sub-tree that has changed—is either a dummy leaf or a node with two dummy leaves as children.



Case 1



Case 2

In both cases $\text{bal}[c] = 0$.

- ▶ Call $\text{AVL-fix-up-delete}(v)$ to restore the balance-condition.

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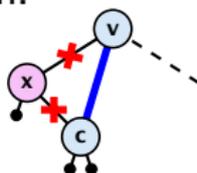
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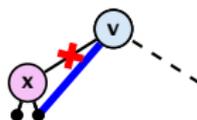
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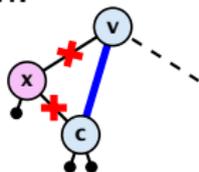
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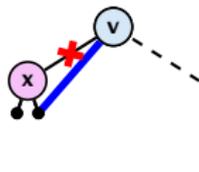
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In both cases $\text{bal}[c] = 0$.

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Invariant at the beginning $\text{AVL-fix-up-delete}(v)$:

1. The balance constraints holds at all descendants of v .
2. A node has been deleted from T_c , where c is either the right or left child of v .
3. T_c has decreased its height by one.
4. The balance at the node c fulfills $\text{balance}[c] = 0$. This holds because if the balance of c is in $\{-1, 1\}$, then T_c did not change its height, and the whole procedure would have been aborted in the previous step.

Invariant at the beginning AVL-fix-up-delete(v):

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AVL-trees: Delete

Algorithm 13 AVL-fix-up-delete(v)

- 1: **if** $\text{balance}[v] \in \{-2, 2\}$ **then** DoRotationDelete(v);
- 2: **if** $\text{balance}[v] \in \{-1, 1\}$ **return**;
- 3: AVL-fix-up-delete(parent(v));

We will show that the above procedure is correct. However, for the case of a delete there may be a logarithmic number of rotations.

AVL-trees: Delete

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We will show that the above procedure is correct. However, for the case of a delete there may be a logarithmic number of rotations.

Algorithm 14 DoRotationDelete(v)

```
1: if balance[ $v$ ] = -2 then // deletion in left sub-tree
2:     if balance[right[ $v$ ]]  $\in$  {0, -1} then
3:         LeftRotate( $v$ );
4:     else
5:         DoubleLeftRotate( $v$ );
6: else // deletion in right sub-tree
7:     if balance[left[ $v$ ]] = {0, 1} then
8:         RightRotate( $v$ );
9:     else
10:        DoubleRightRotate( $v$ );
```

AVL-trees: Delete

It is clear that the invariant for the fix-up routine hold as long as no rotations have been done.

We show that after doing a rotation at v :

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We only look at the case where the deleted node was in the right sub-tree of v . The other case is symmetric.

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It is clear that the invariant for the fix-up routine hold as long as no rotations have been done.

We show that after doing a rotation at v :

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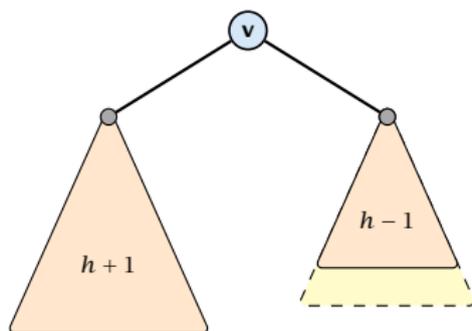
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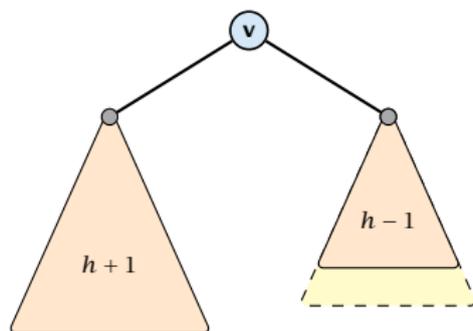


The right sub-tree of v has decreased its height which results in a balance of 2 at v .

Before the deletion the height of T_v was $h + 2$.

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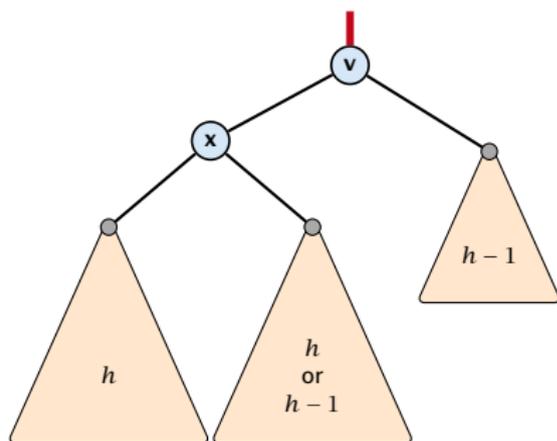


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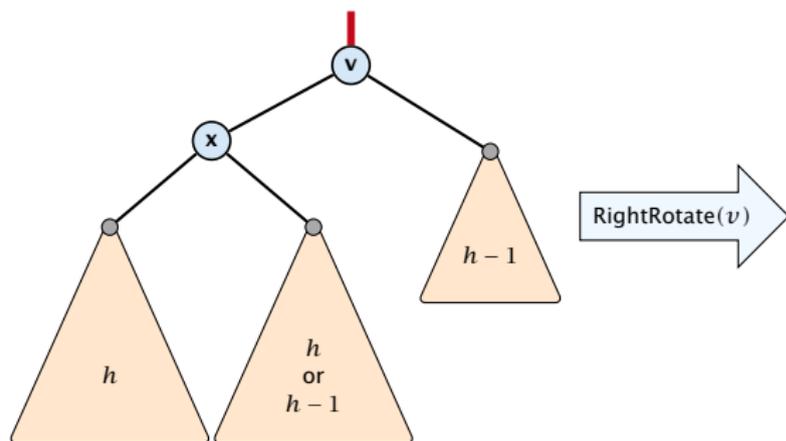
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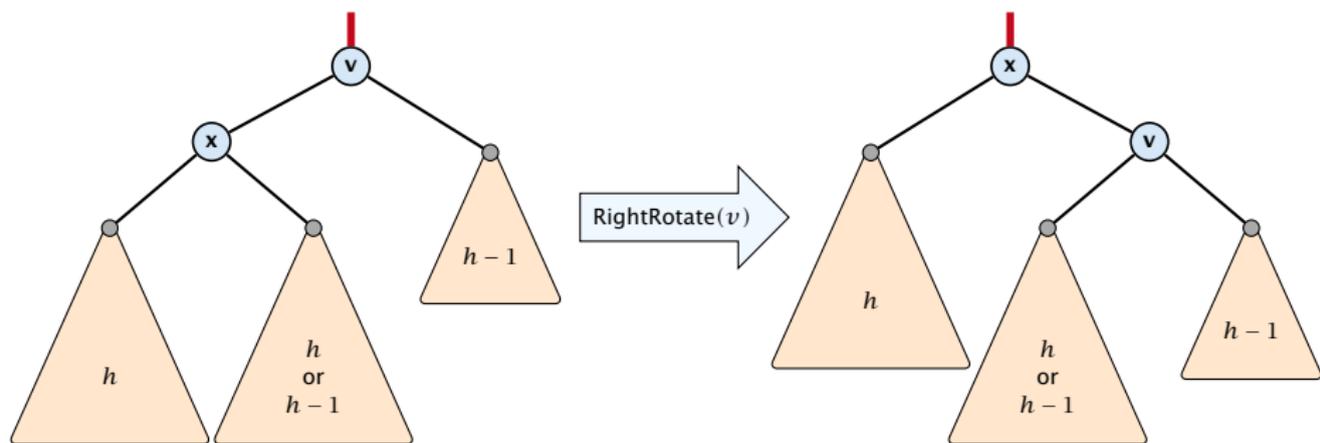
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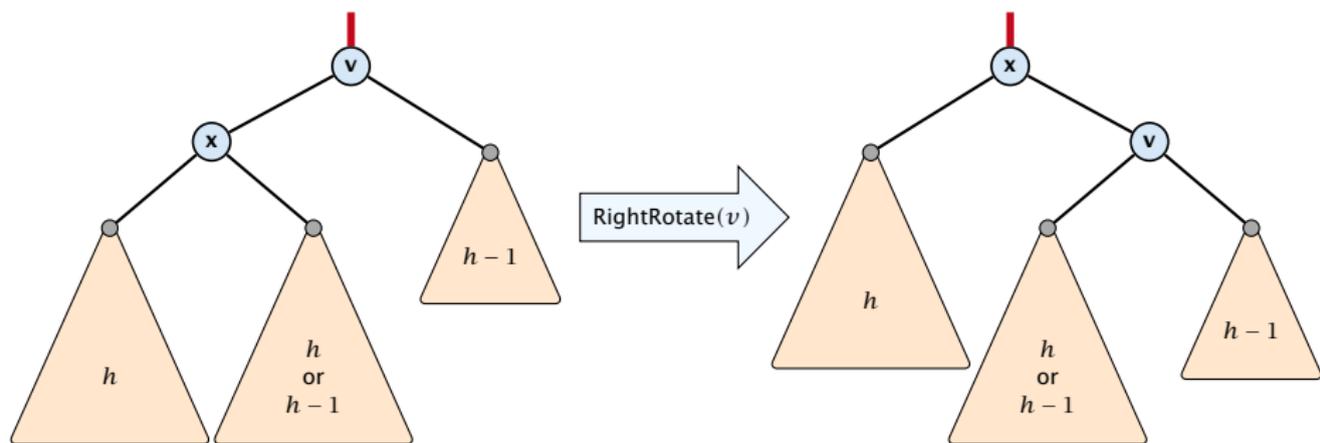
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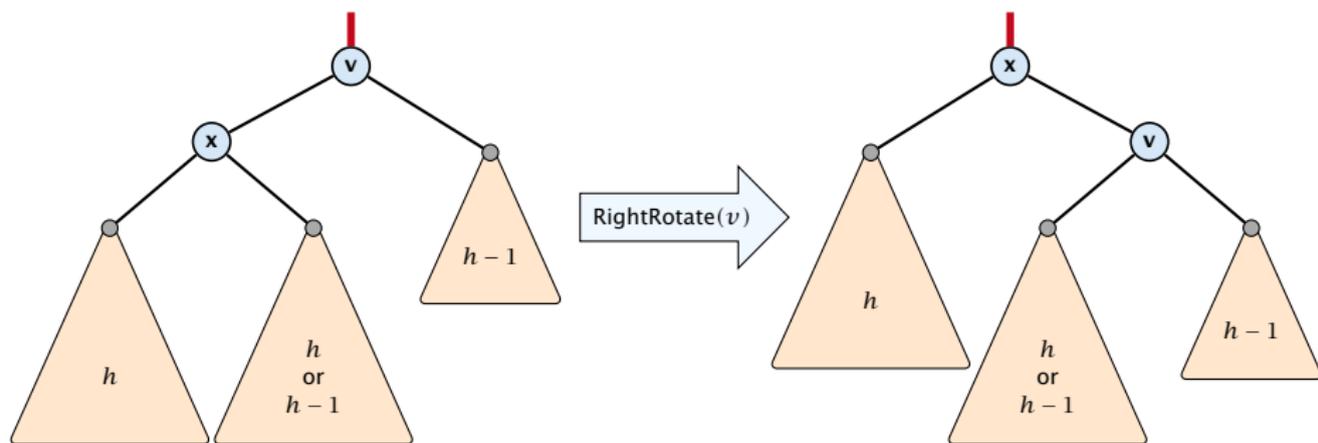


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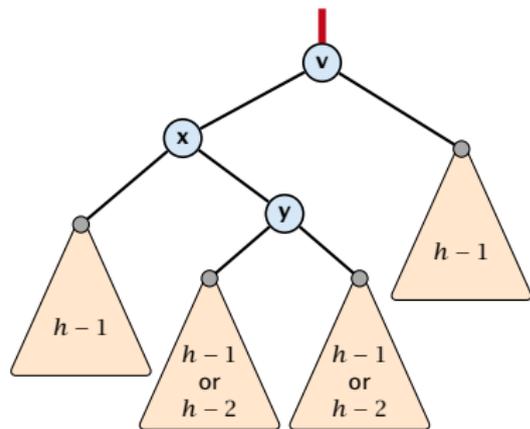


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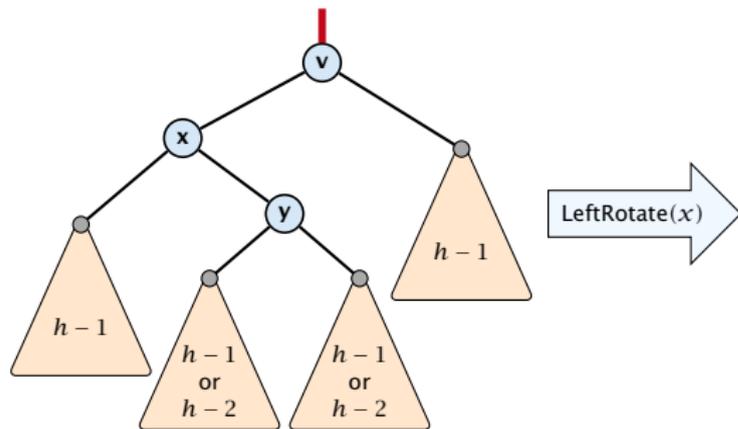
If the middle subtree has height $h - 1$ the whole tree has decreased its height from $h + 2$ to $h + 1$. We do continue the fix-up procedure as the balance at the root is zero.

Case 2: $\text{balance}[\text{left}[v]] = -1$

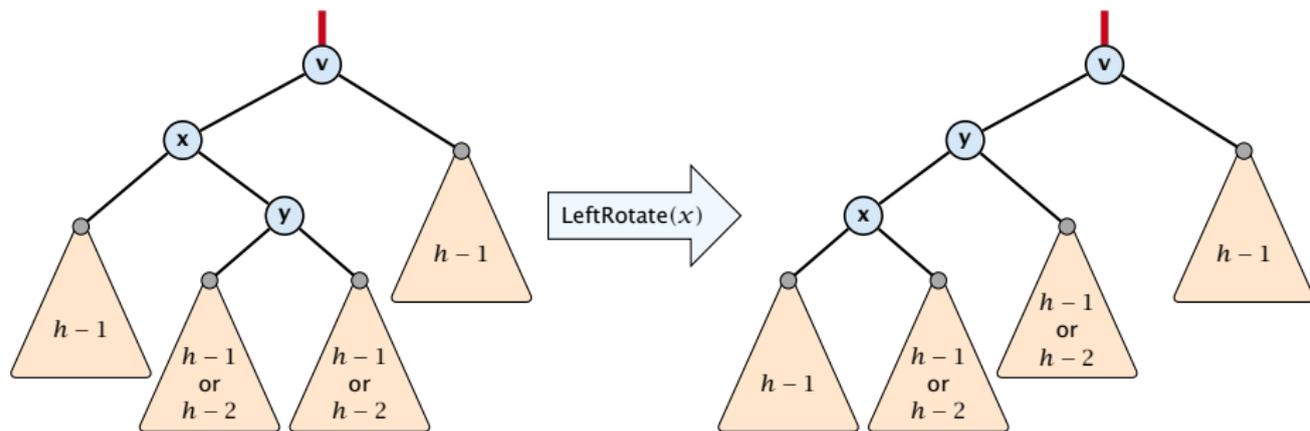
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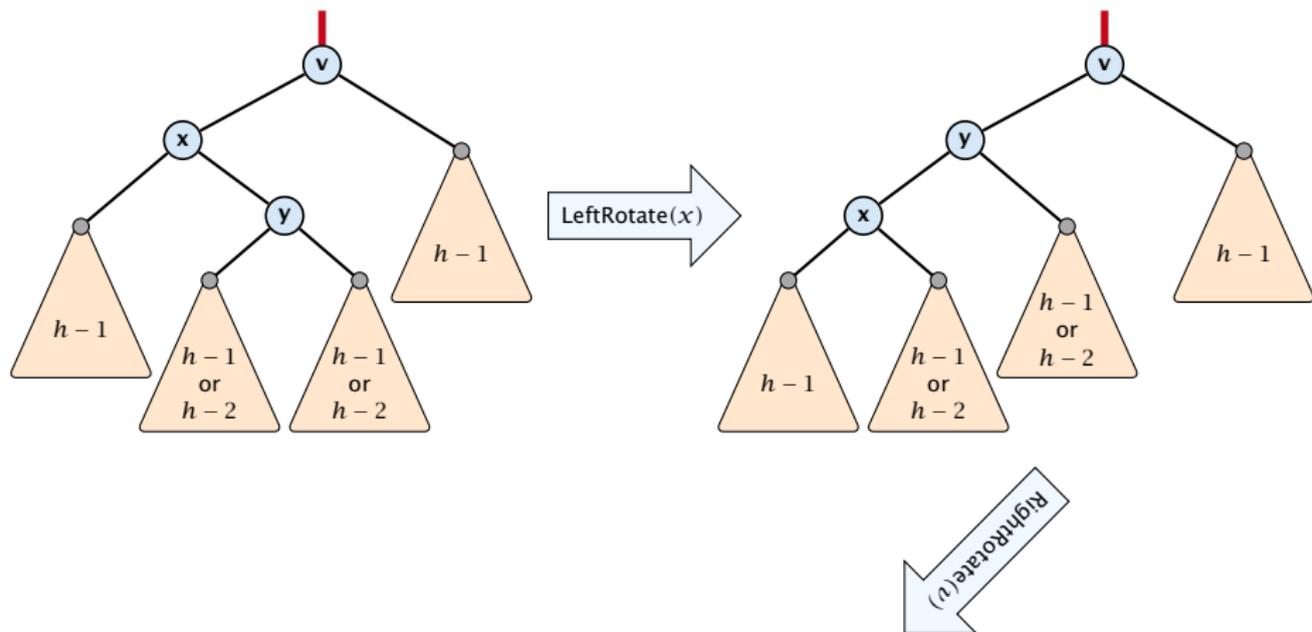
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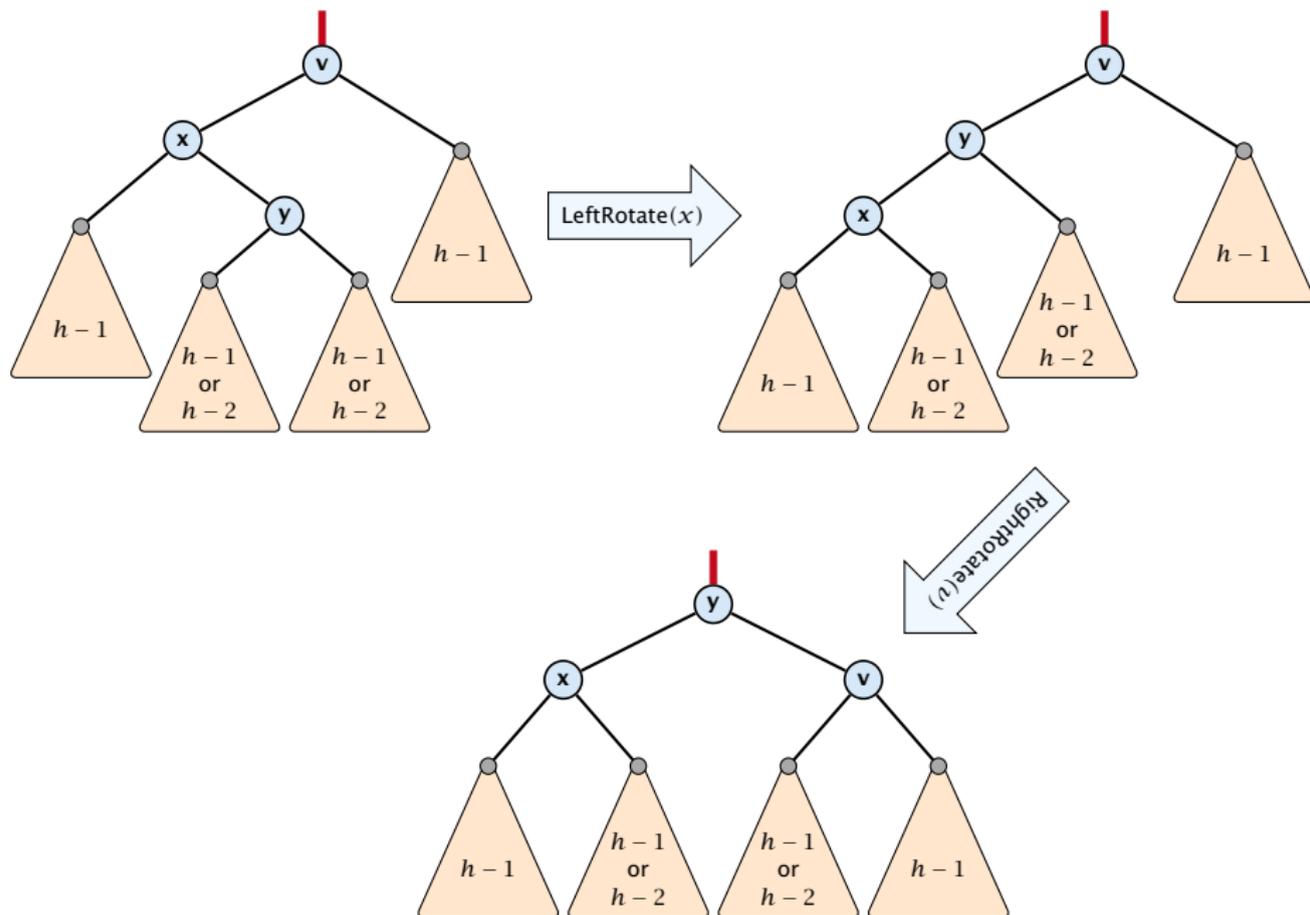
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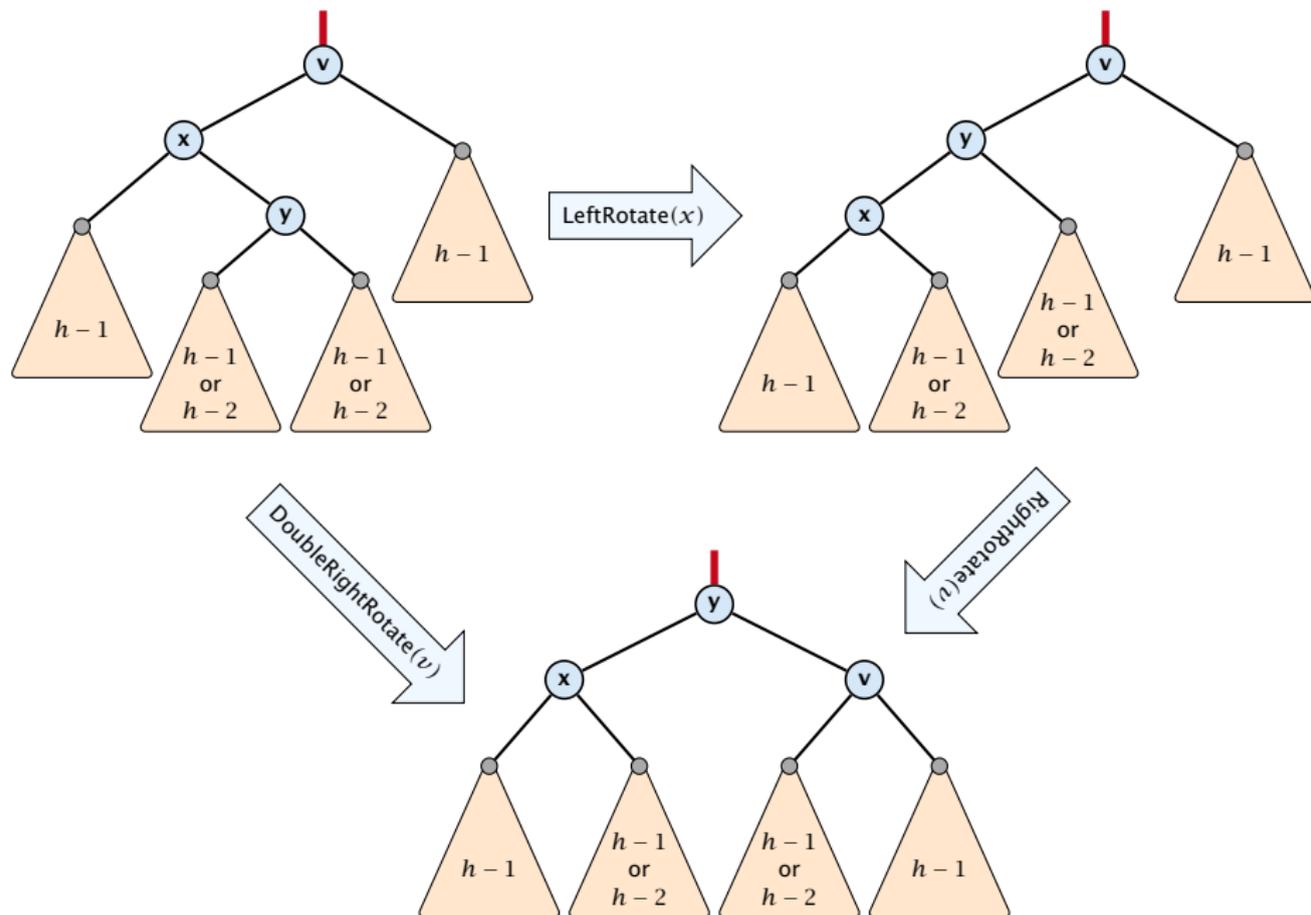
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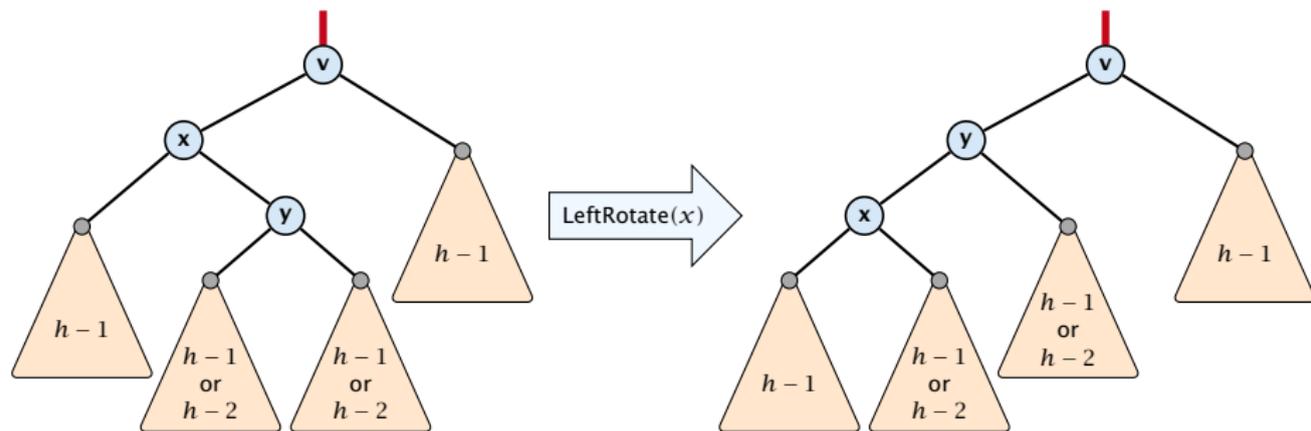
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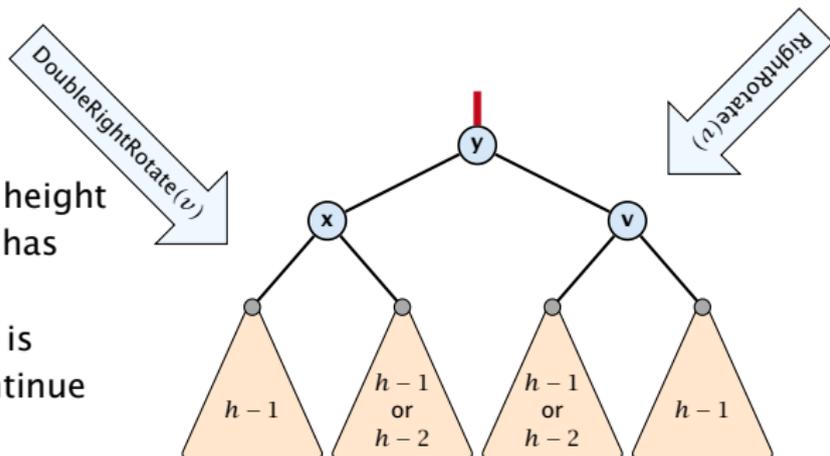
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Sub-tree has height $h + 1$, i.e., it has shrunk. The balance at y is zero. We continue the iteration.



7.4 Augmenting Data Structures

Suppose you want to develop a data structure with:

- ▶ **Insert(x)**: insert element x .
- ▶ **Search(k)**: search for element with key k .
- ▶ **Delete(x)**: delete element referenced by pointer x .
- ▶ **find-by-rank(ℓ)**: return the ℓ -th element; return “error” if the data-structure contains less than ℓ elements.

Augment an existing data-structure instead of developing a new one.

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How to augment a data-structure

1. choose an underlying data-structure
2. determine additional information to be stored in the underlying structure
3. verify/show how the additional information can be maintained for the basic modifying operations on the underlying structure.
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Goal: Design a data-structure that supports insert, delete, search, and find-by-rank in time $\mathcal{O}(\log n)$.

1. We choose a red-black tree as the underlying data-structure.
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3. We need to be able to update the size-field in each node without asymptotically affecting the running time of insert, delete, and search. We come back to this step later...

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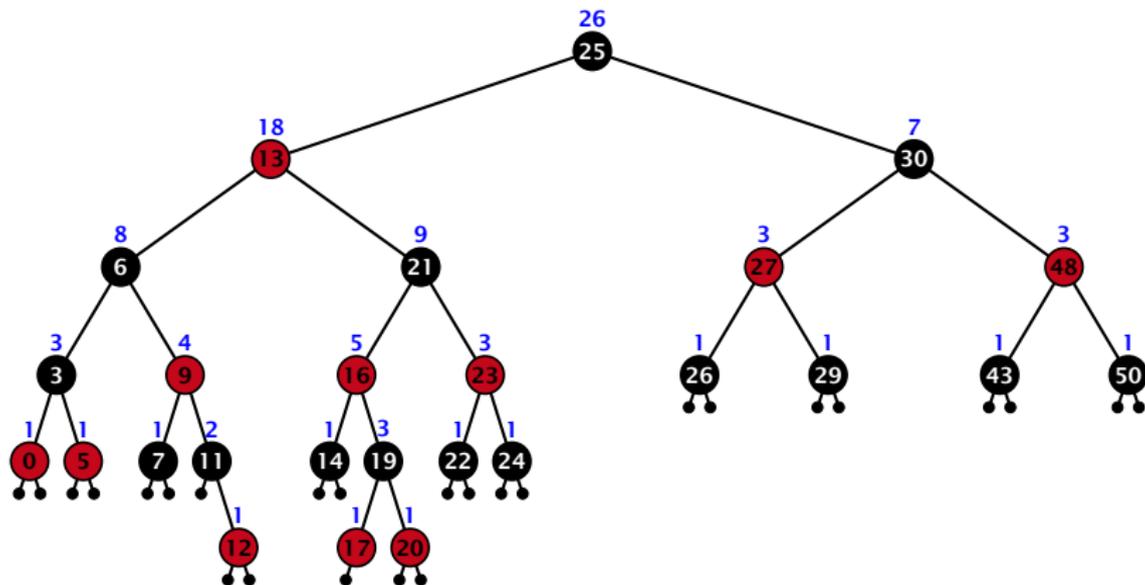
4. How does find-by-rank work?

Find-by-rank(k) := Select(root, k) with

Algorithm 15 Select(x, i)

```
1: if  $x = \text{null}$  then return error
2: if  $\text{left}[x] \neq \text{null}$  then  $r \leftarrow \text{left}[x].\text{size} + 1$  else  $r \leftarrow 1$ 
3: if  $i = r$  then return  $x$ 
4: if  $i < r$  then
5:     return Select( $\text{left}[x], i$ )
6: else
7:     return Select( $\text{right}[x], i - r$ )
```

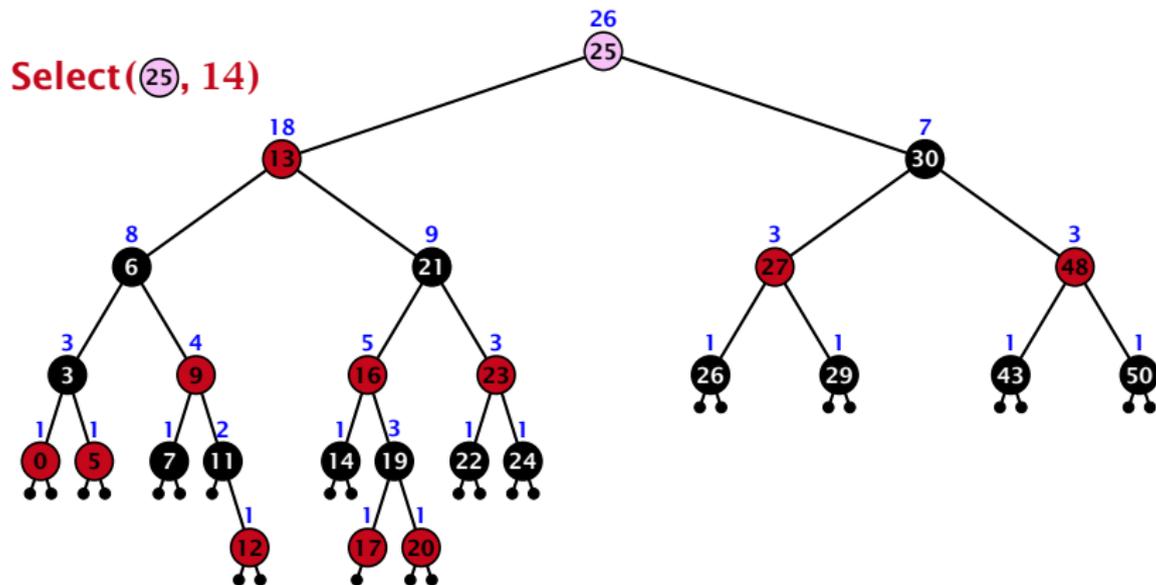
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Find-by-rank:

- ▶ decide whether you have to proceed into the left or right sub-tree
- ▶ adjust the rank that you are searching for if you go right

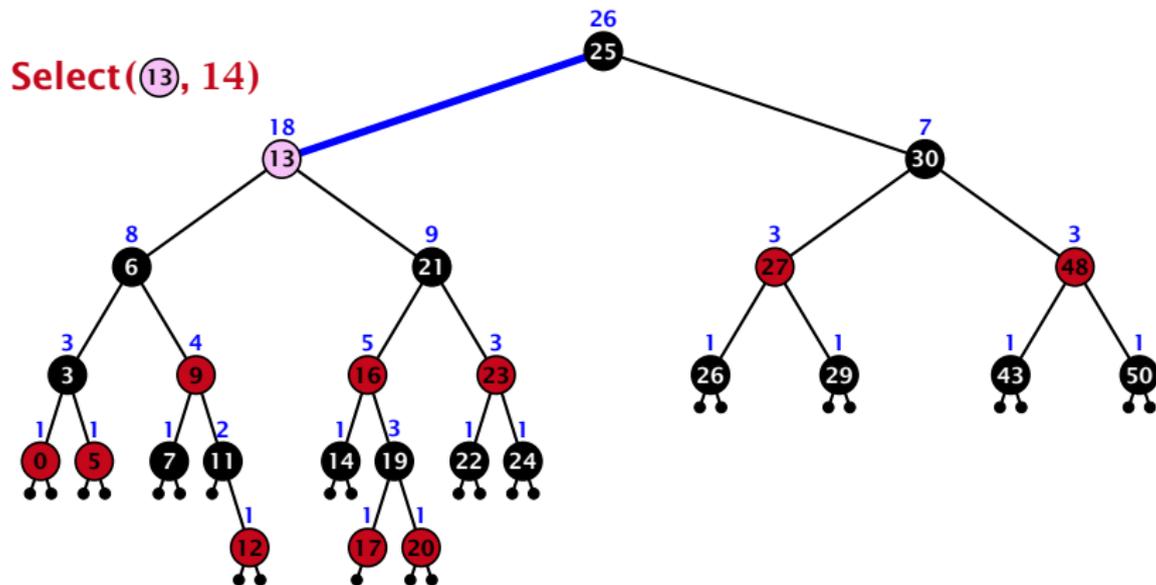
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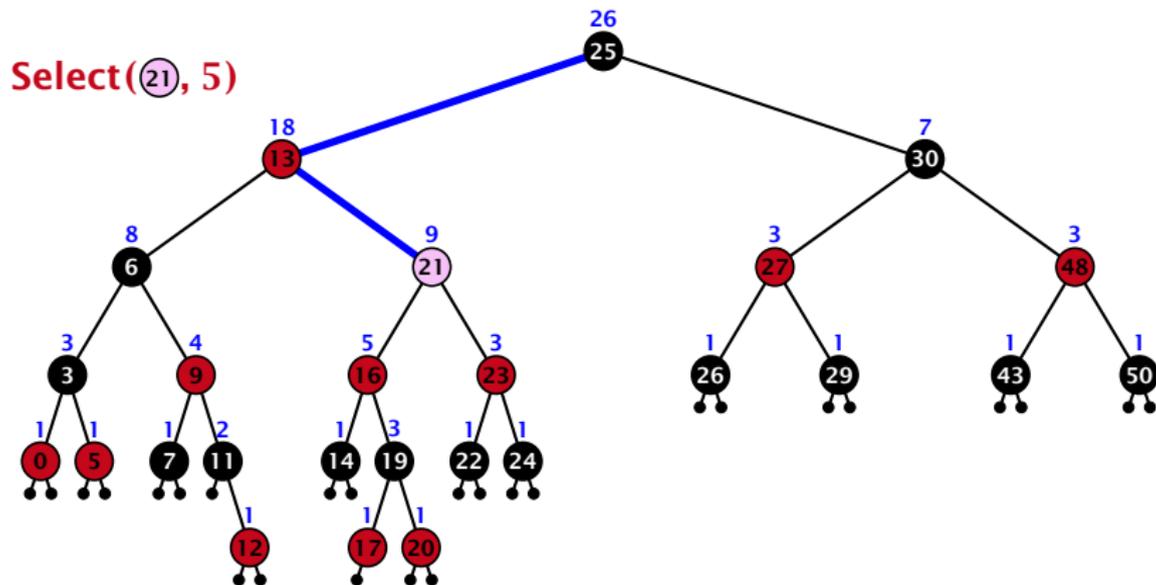
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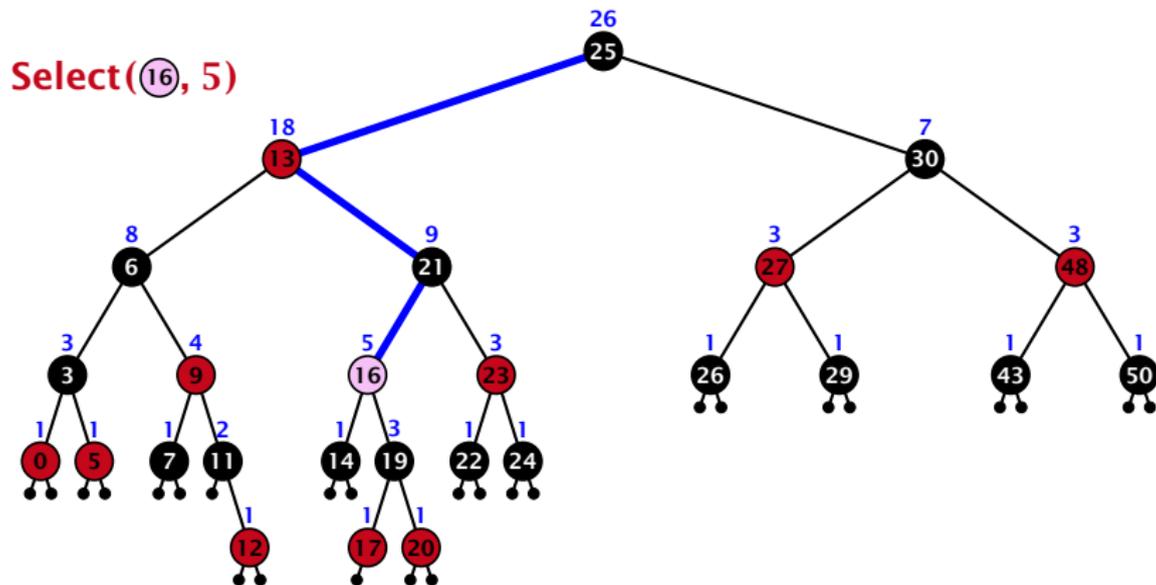
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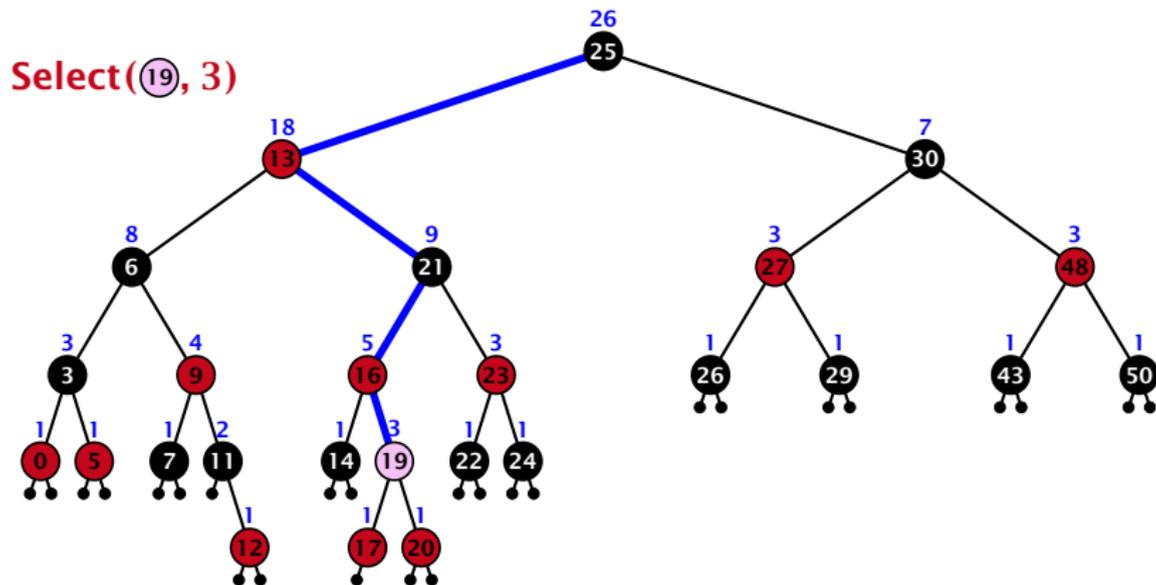
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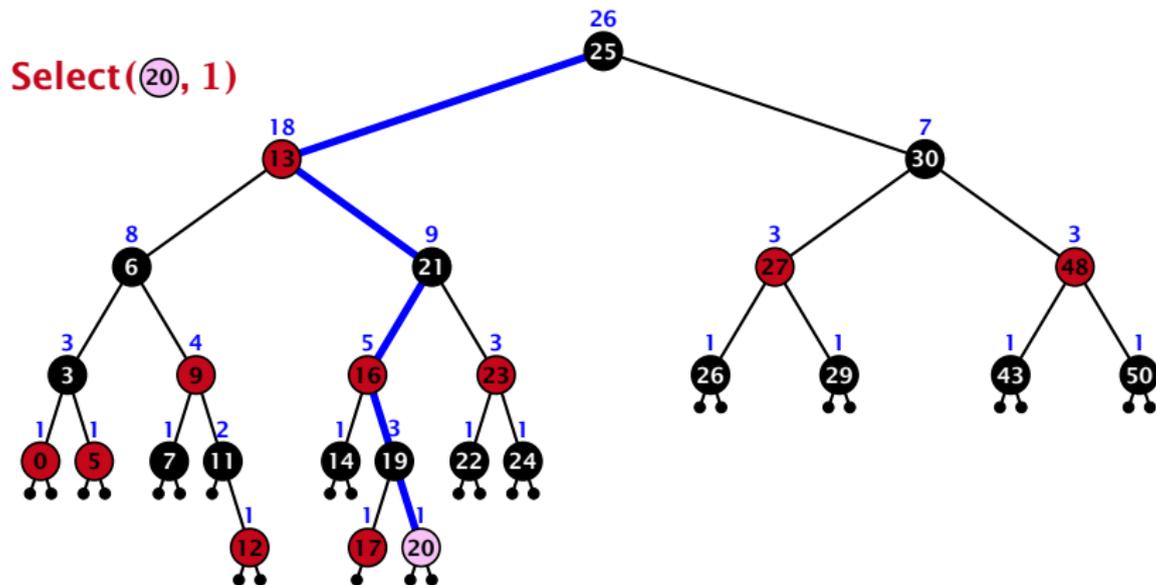
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Goal: Design a data-structure that supports insert, delete, search, and find-by-rank in time $\mathcal{O}(\log n)$.

3. How do we maintain information?

Search(k): Nothing to do.

Insert(x): When going down the search path increase the size field for each visited node. Maintain the size field during rotations.

Delete(x): Directly after splicing out a node traverse the path from the spliced out node upwards, and decrease the size counter on every node on this path. Maintain the size field during rotations.

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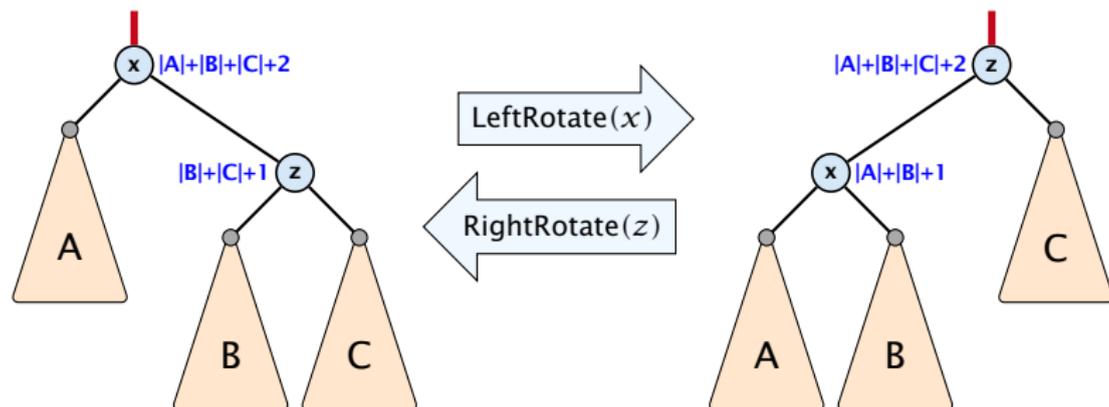
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Rotations

The only operation during the fix-up procedure that alters the tree and requires an update of the size-field:



The nodes x and z are the only nodes changing their size-fields.

The new size-fields can be computed **locally** from the size-fields of the children.

7.5 (a, b) -trees

Definition 7

For $b \geq 2a - 1$ an (a, b) -tree is a search tree with the following properties

1. all leaves have the same distance to the root
2. every internal non-root vertex v has at least a and at most b children
3. the root has degree at least 2 if the tree is non-empty
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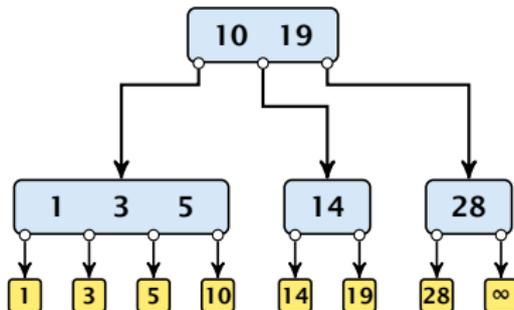
Each internal node v with $d(v)$ children stores $d - 1$ keys k_1, \dots, k_{d-1} . The i -th subtree of v fulfills

$$k_{i-1} < \text{key in } i\text{-th sub-tree} \leq k_i ,$$

where we use $k_0 = -\infty$ and $k_d = \infty$.

7.5 (a, b)-trees

Example 8



7.5 (a, b)-trees

Variants

- ▶ The dummy leaf element may not exist; it only makes implementation more convenient.
- ▶ Variants in which $b = 2a$ are commonly referred to as B -trees.
- ▶ A B -tree usually refers to the variant in which keys and data are stored at internal nodes.
- ▶ A B^+ tree stores the data only at leaf nodes as in our definition. Sometimes the leaf nodes are also connected in a linear list data structure to speed up the computation of successors and predecessors.
- ▶ A B^* tree requires that a node is at least $2/3$ -full as opposed to $1/2$ -full (the requirement of a B -tree).

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Lemma 9

Let T be an (a, b) -tree for $n > 0$ elements (i.e., $n + 1$ leaf nodes) and height h (number of edges from root to a leaf vertex). Then

1. $2a^{h-1} \leq n + 1 \leq b^h$
2. $\log_b(n + 1) \leq h \leq 1 + \log_a\left(\frac{n+1}{2}\right)$

Proof.

Since the root has degree at least a and all other nodes have degree at least a , this gives that there are at least $2a^{h-1}$ leaf nodes. Since the root has degree at most b and all other nodes have degree at most b , this gives that there are at most b^h leaf nodes.



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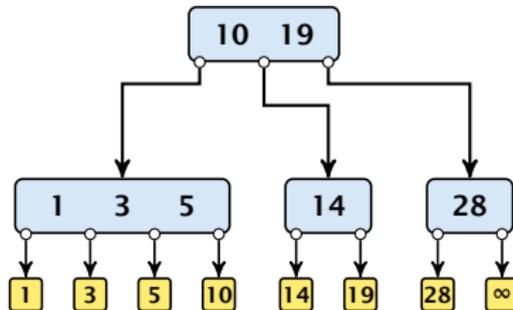
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- ▶ Analogously, the degree of any node is at most b and, hence, the number of leaf nodes at most b^h .

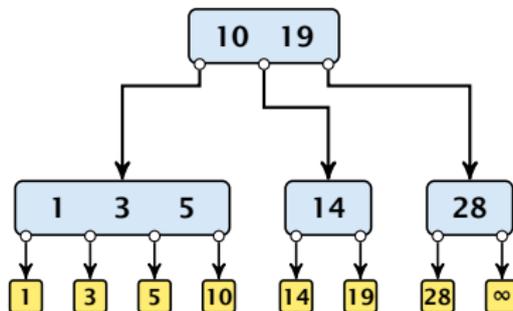


Search



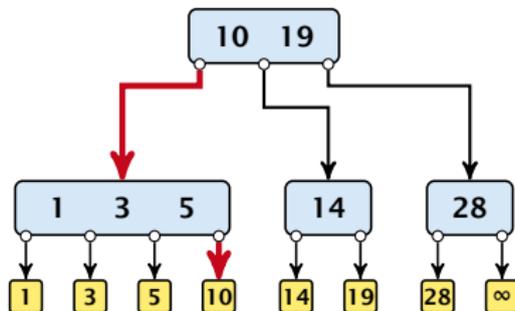
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Search(8)



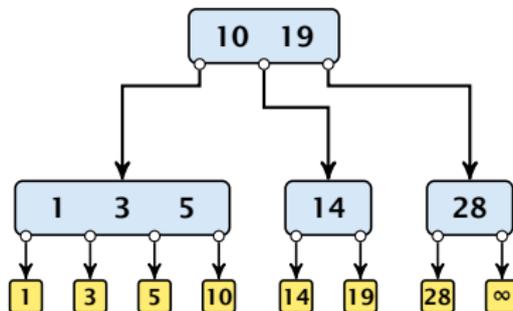
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Search(8)



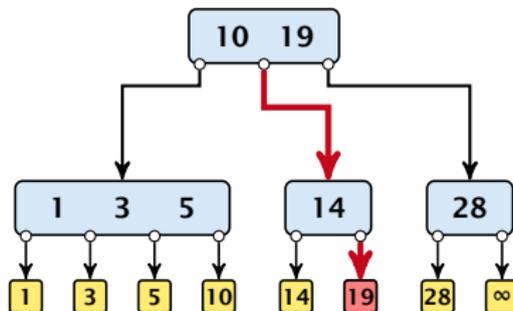
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Search(19)

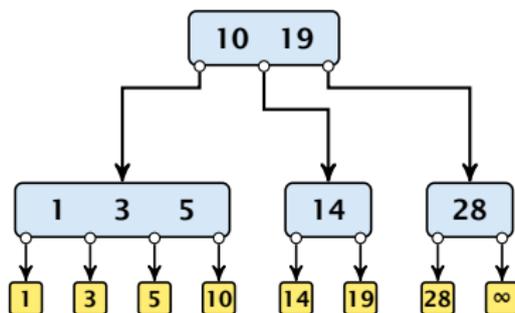


Search

Search(19)

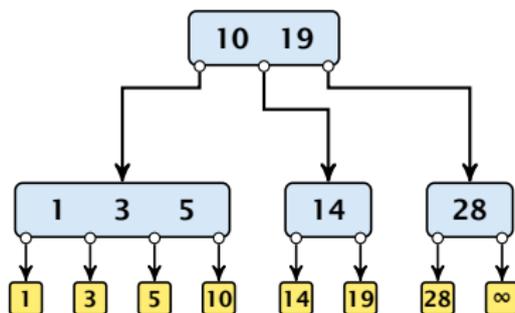


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Time: $\mathcal{O}(b \cdot h) = \mathcal{O}(b \cdot \log n)$, if the individual nodes are organized as linear lists.

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Insert element x :

- ▶ Follow the path as if searching for $\text{key}[x]$.
- ▶ If this search ends in leaf ℓ , insert x before this leaf.
- ▶ For this add $\text{key}[x]$ to the key-list of the last internal node v on the path.
- ▶ If after the insert v contains b nodes, do $\text{Rebalance}(v)$.

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Rebalance(v):

- ▶ Let k_i , $i = 1, \dots, b$ denote the keys stored in v .
- ▶ Let $j := \lfloor \frac{b+1}{2} \rfloor$ be the middle element.
- ▶ Create two nodes v_1 , and v_2 . v_1 gets all keys k_1, \dots, k_{j-1} and v_2 gets keys k_{j+1}, \dots, k_b .
- ▶ Both nodes get at least $\lfloor \frac{b-1}{2} \rfloor$ keys, and have therefore degree at least $\lfloor \frac{b-1}{2} \rfloor + 1 \geq a$ since $b \geq 2a - 1$.
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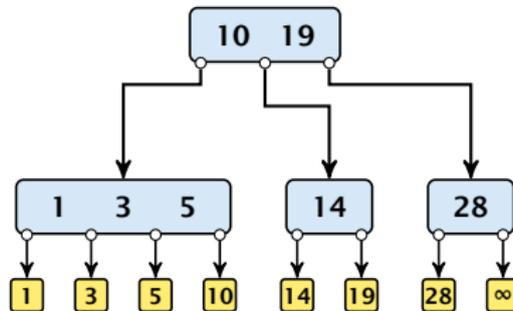
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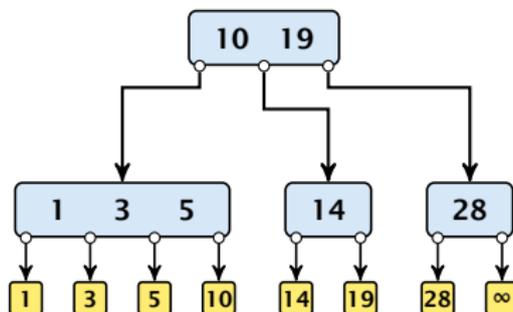
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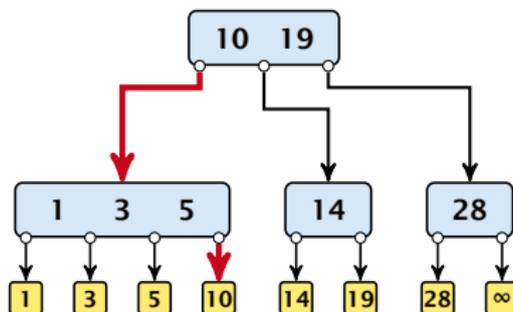
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Insert(8)



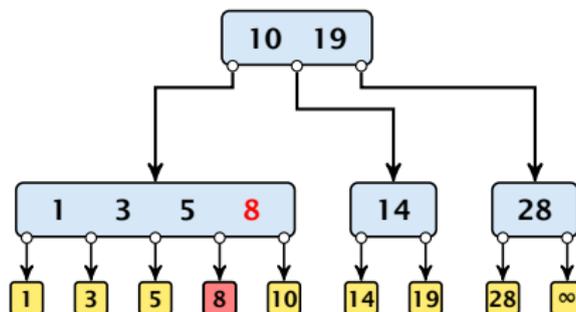
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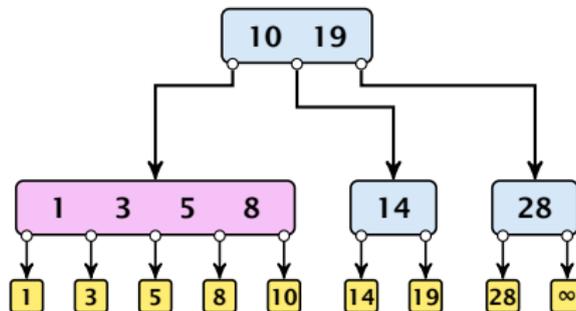
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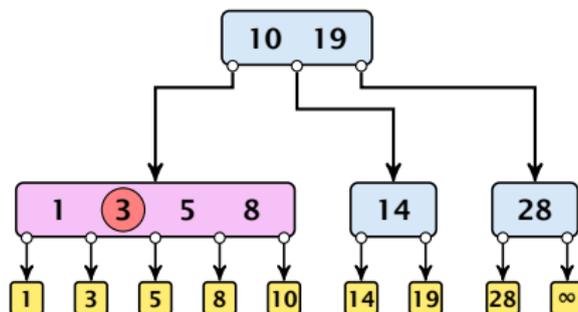
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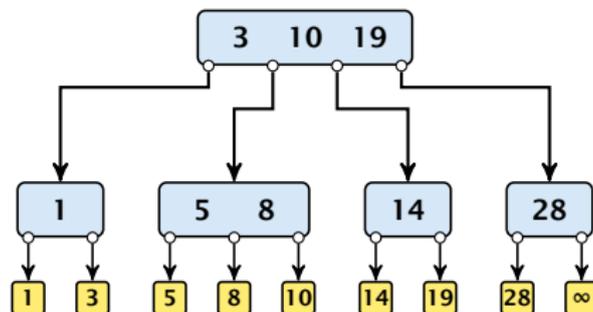


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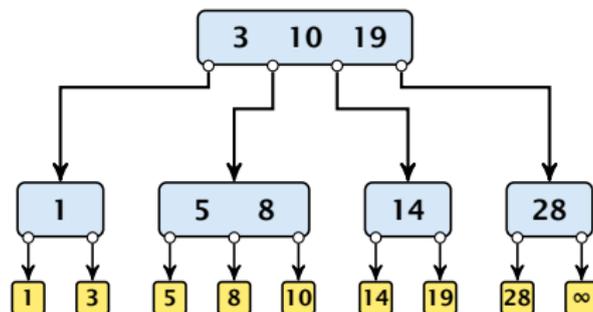


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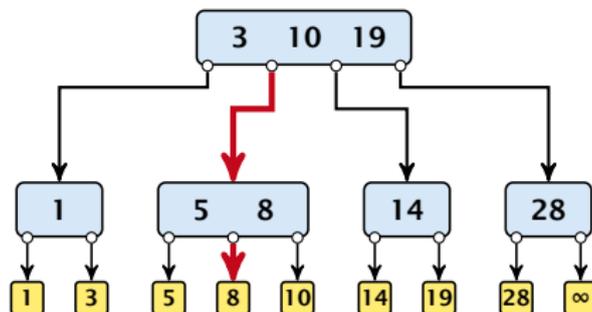
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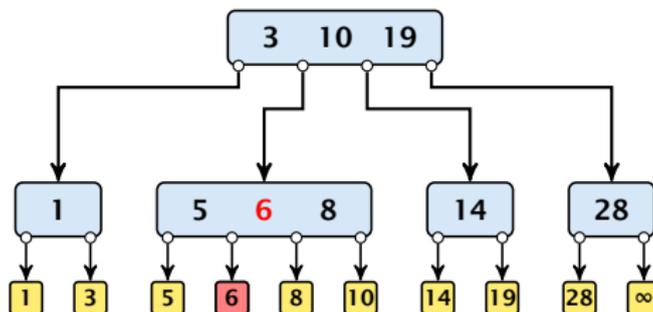
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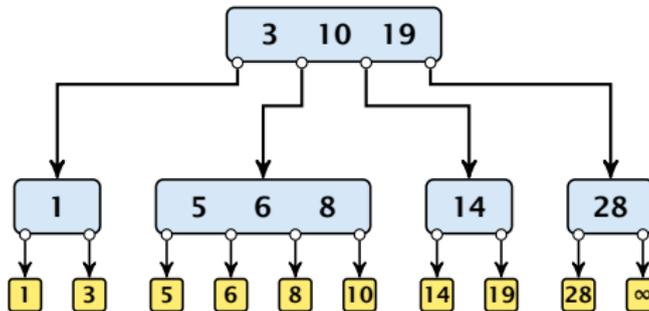
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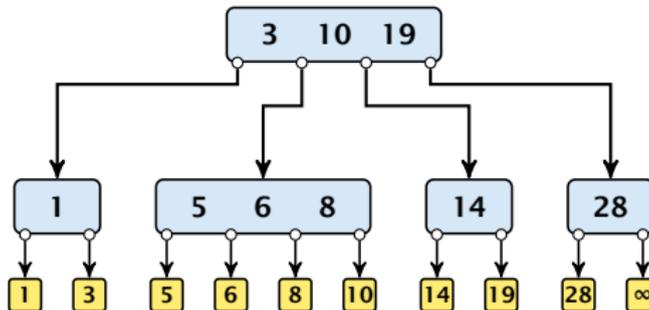
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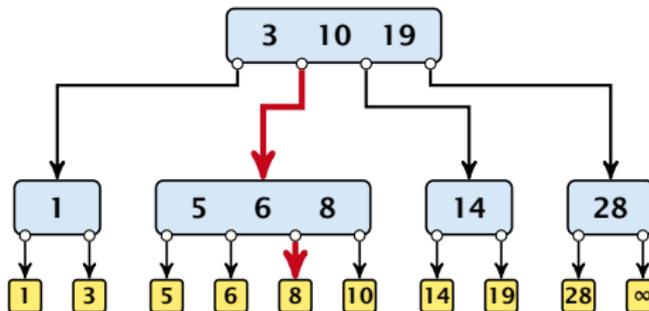
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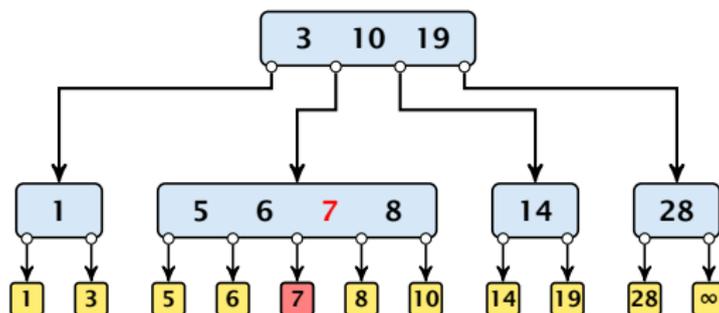
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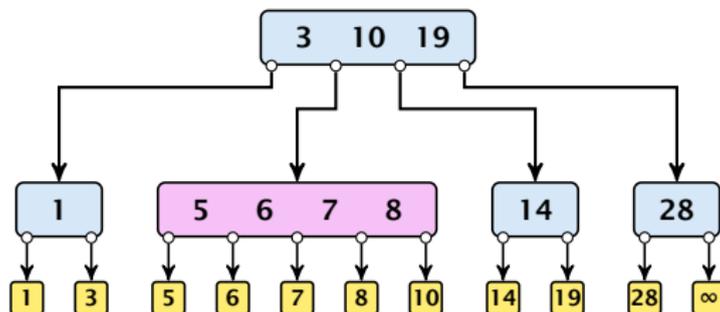
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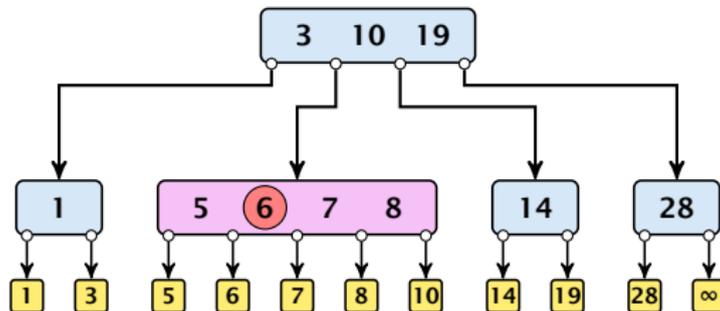
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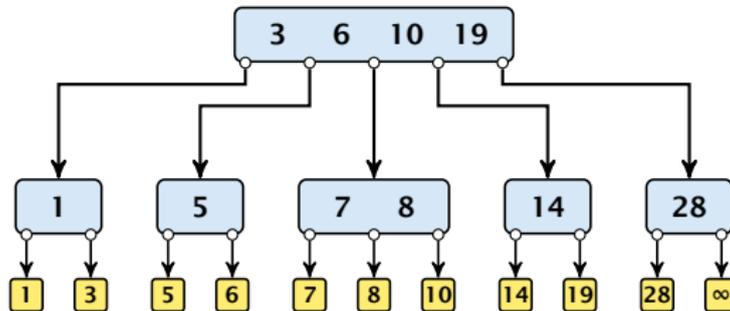
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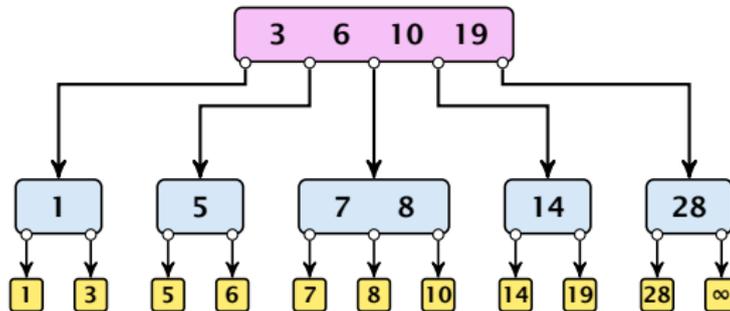
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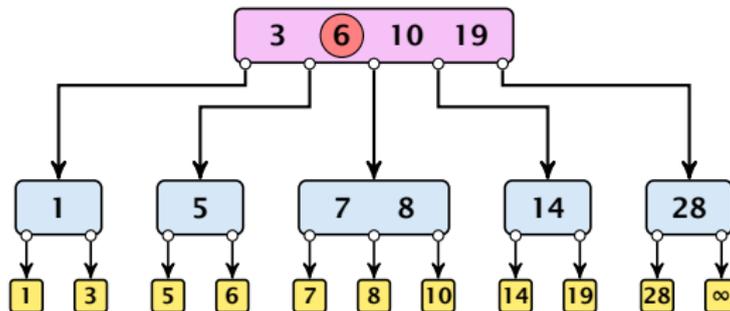
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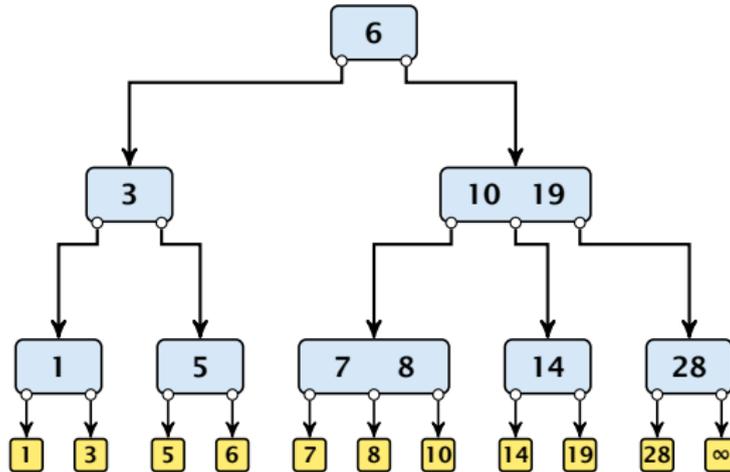
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Delete

Delete element x (pointer to leaf vertex):

- ▶ Let v denote the parent of x . If $\text{key}[x]$ is contained in v , remove the key from v , and delete the leaf vertex.
- ▶ Otherwise delete the key of the predecessor of x from v ; delete the leaf vertex; and replace the occurrence of $\text{key}[x]$ in internal nodes by the predecessor key. (Note that it appears in exactly one internal vertex).
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- ▶ If there is a neighbour of v that has at least a keys take over the largest (if right neighbor) or smallest (if left neighbour) and the corresponding sub-tree.
- ▶ If not: merge v with one of its neighbours.
- ▶ The merged node contains at most $(a - 2) + (a - 1) + 1$ keys, and has therefore at most $2a - 1 \leq b$ successors.
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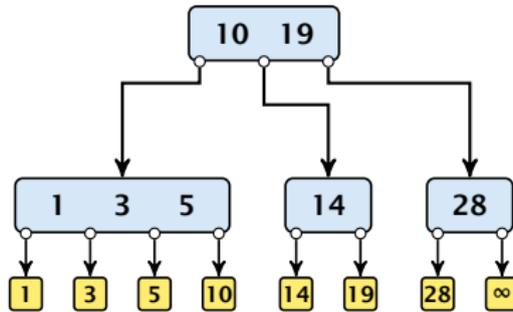
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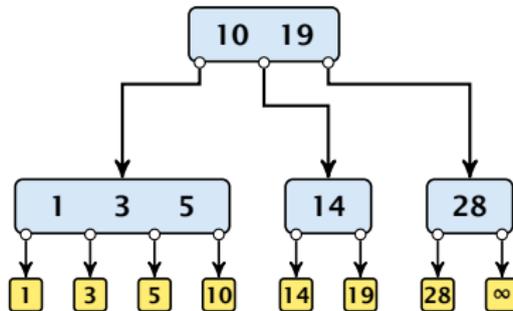
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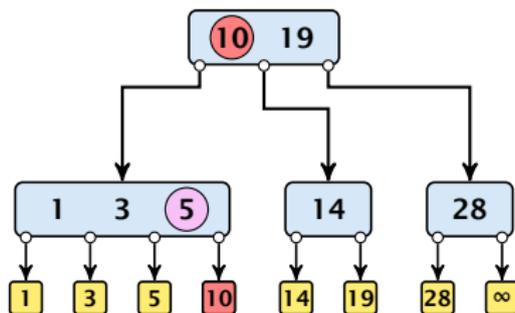
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Delete(10)



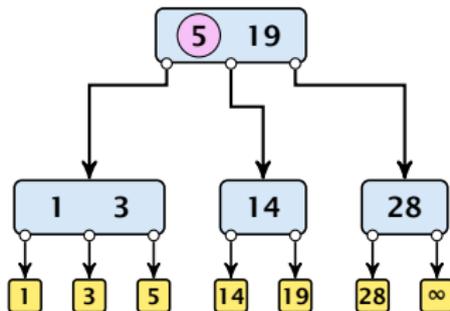
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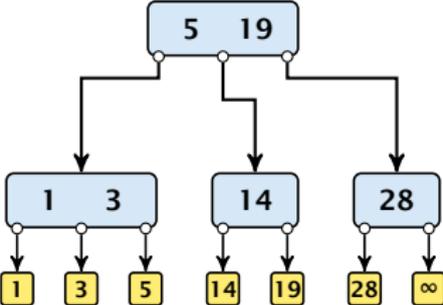


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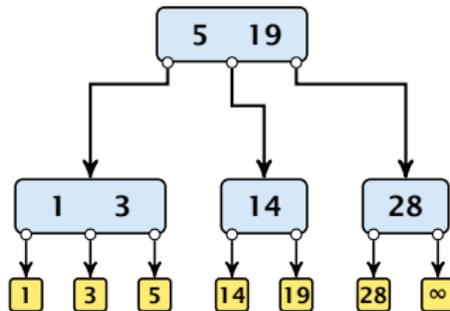


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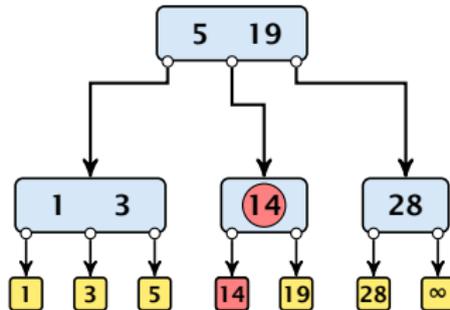
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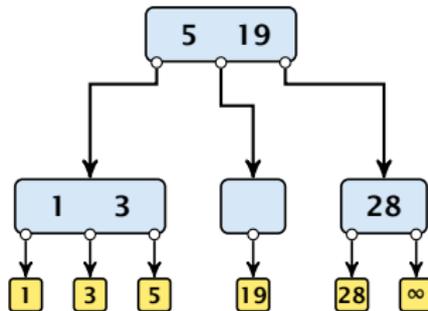
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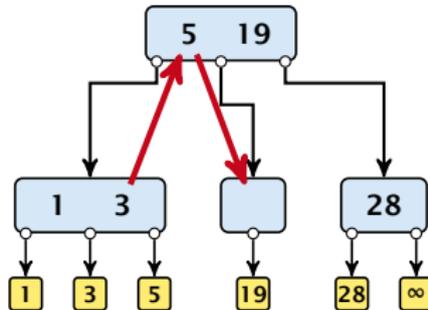
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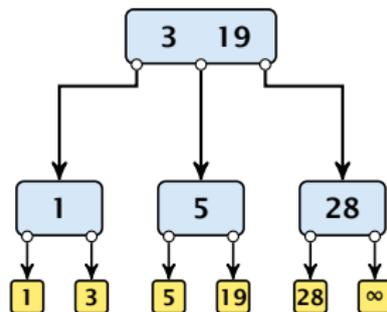
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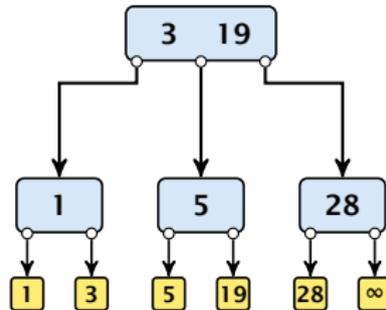


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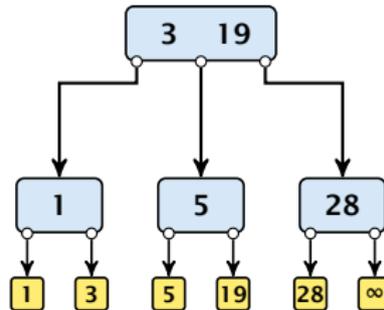


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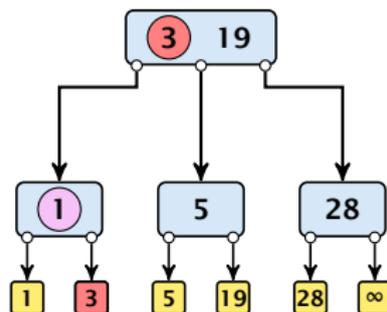
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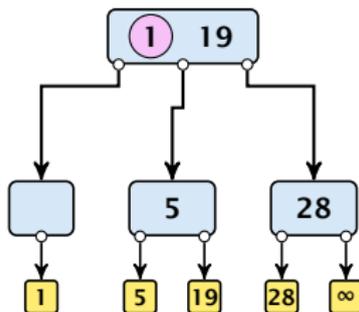
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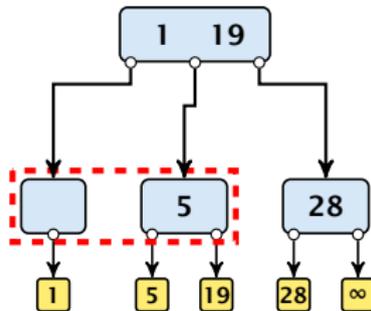
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Delete(3)



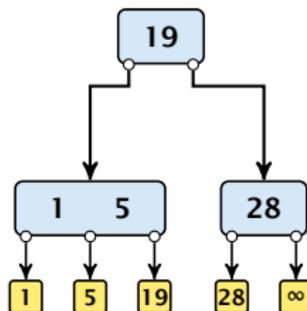
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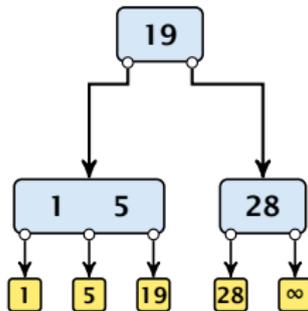


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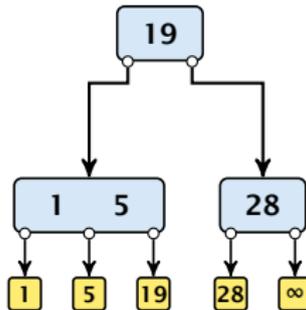


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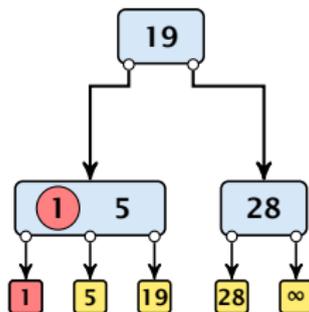
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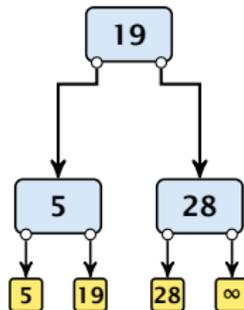
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Delete(1)

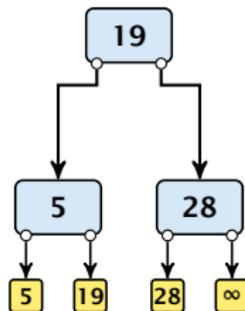


Delete

Delete(1)

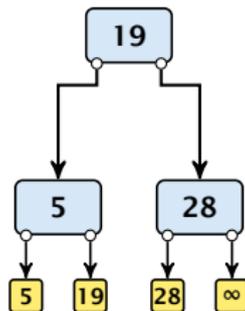


Delete



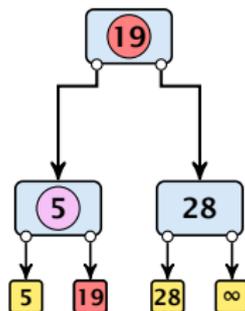
Delete

Delete(19)



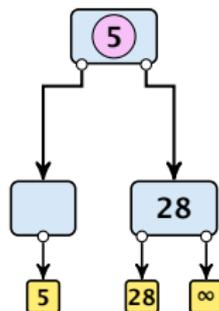
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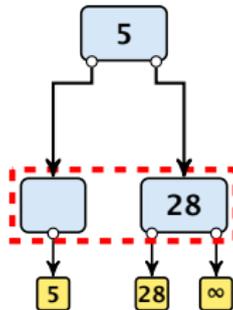
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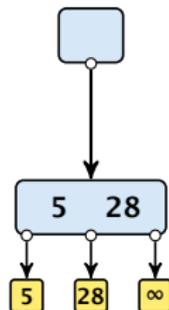
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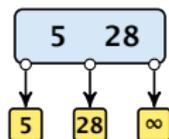
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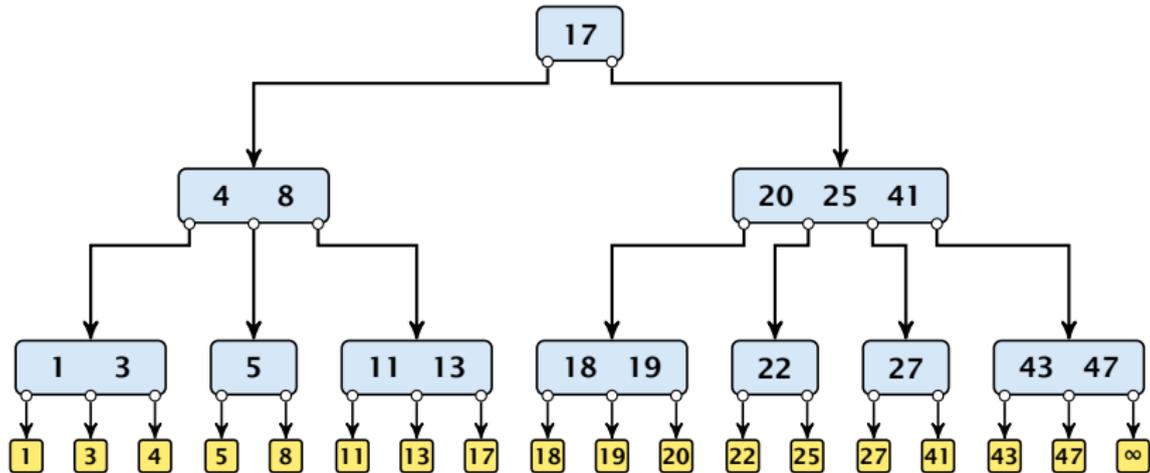
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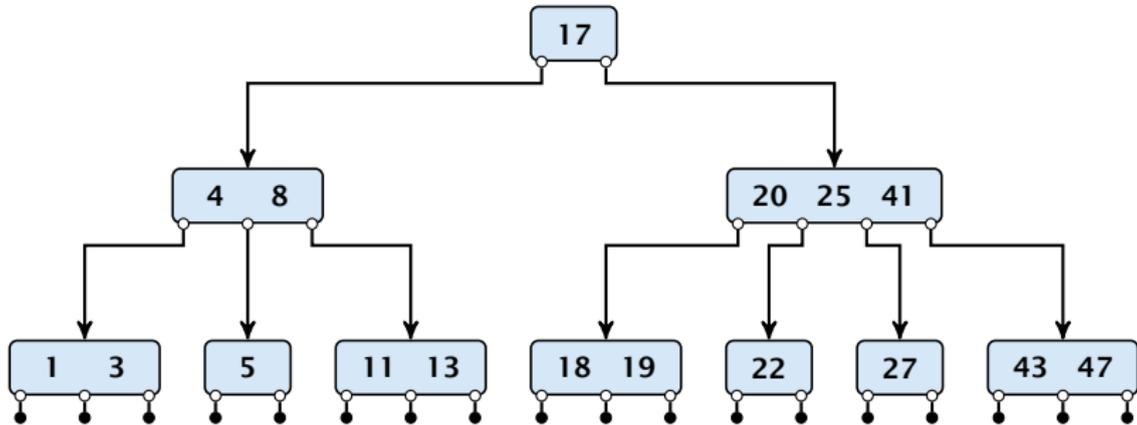
(2, 4)-trees and red black trees

There is a close relation between red-black trees and (2,4)-trees:



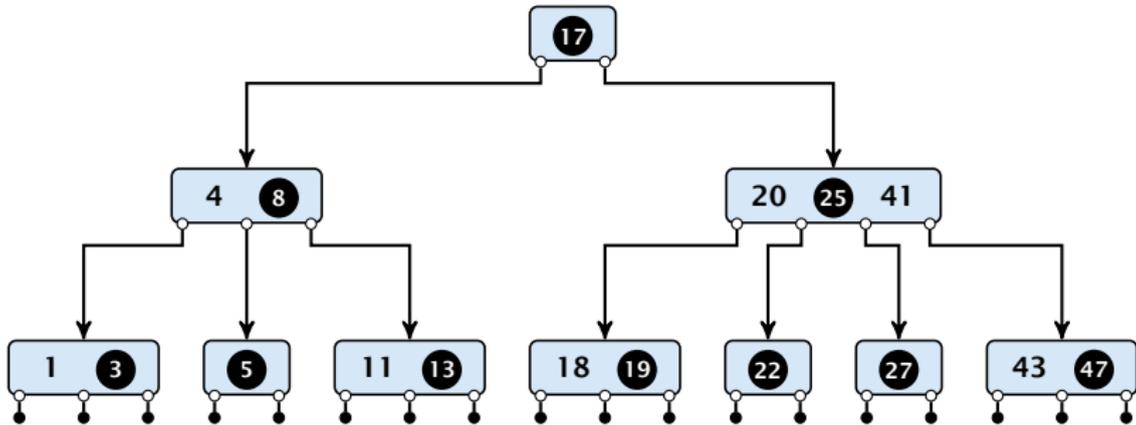
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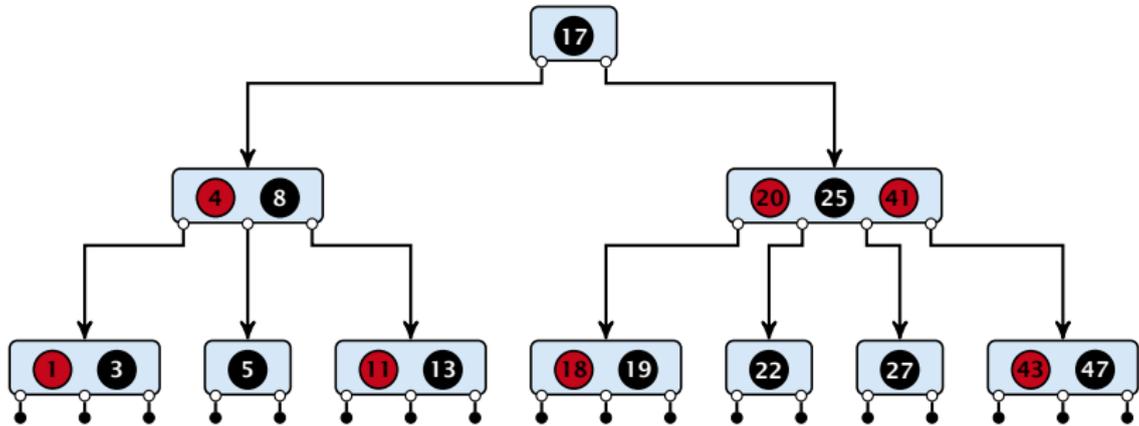
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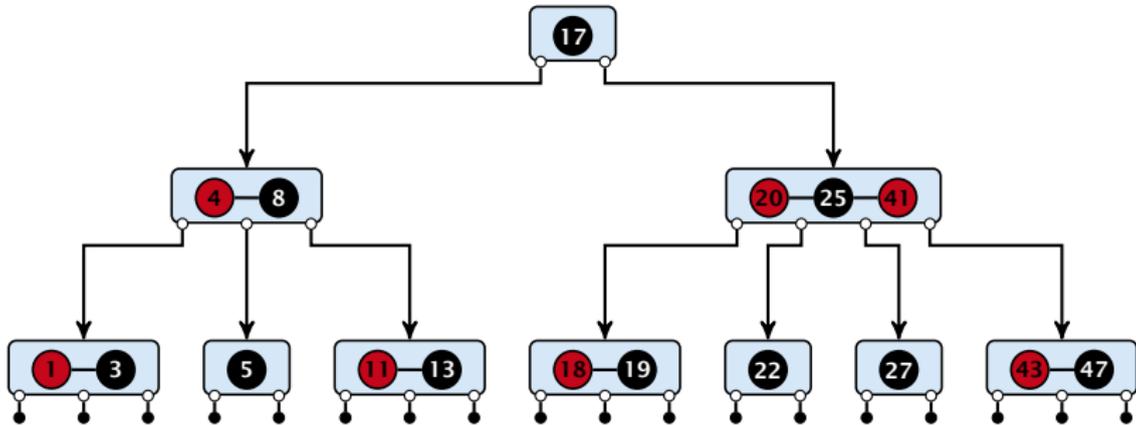
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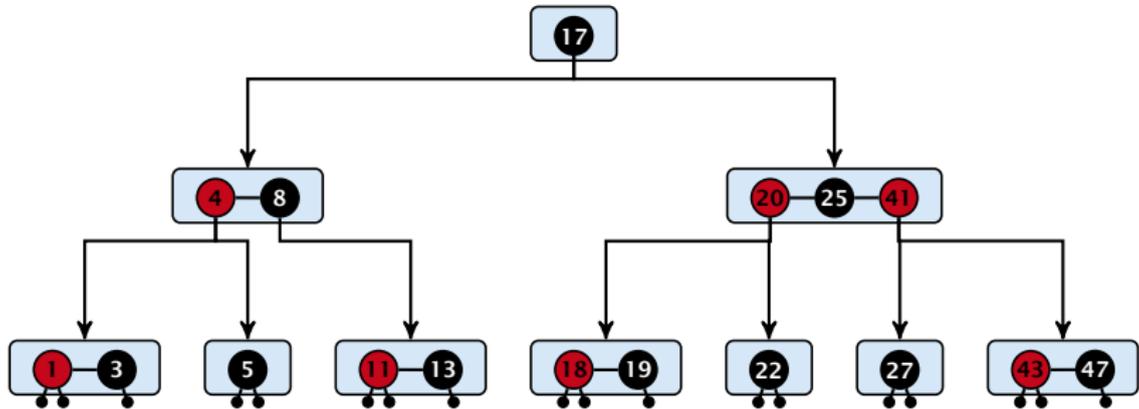
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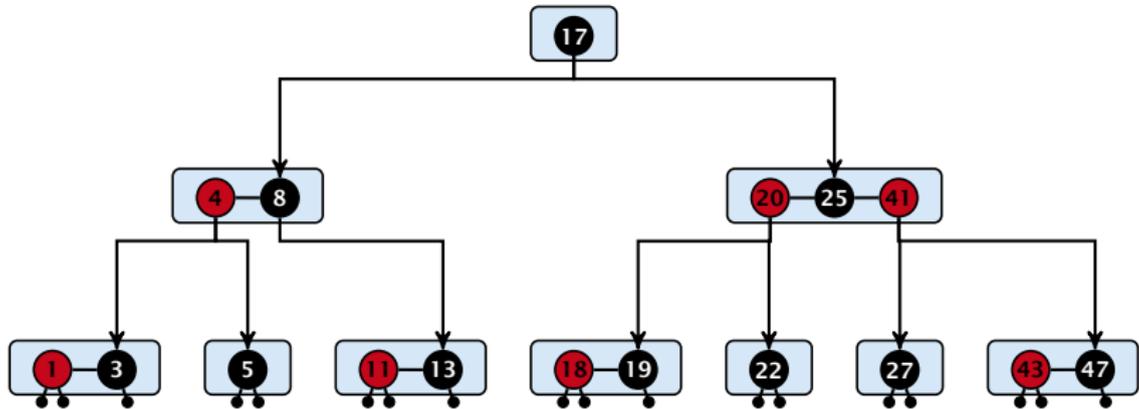
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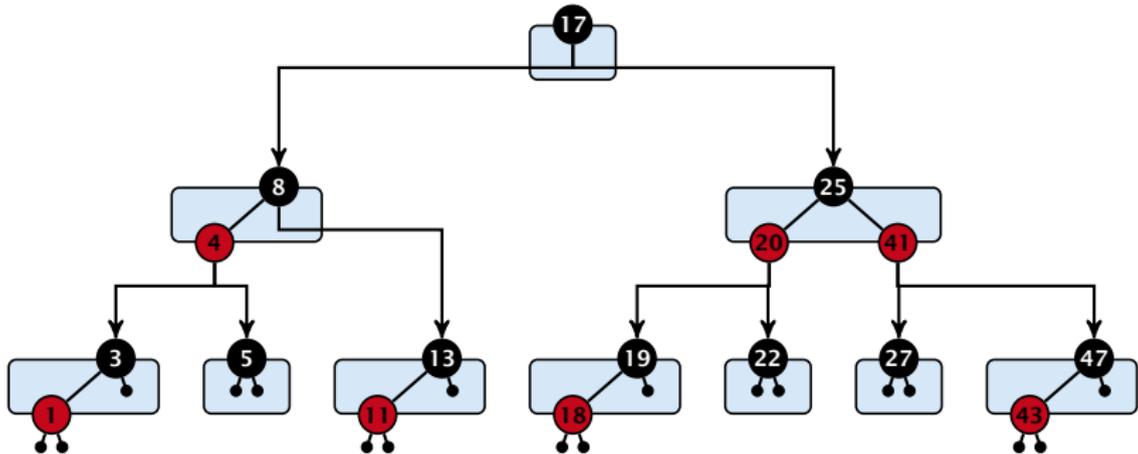
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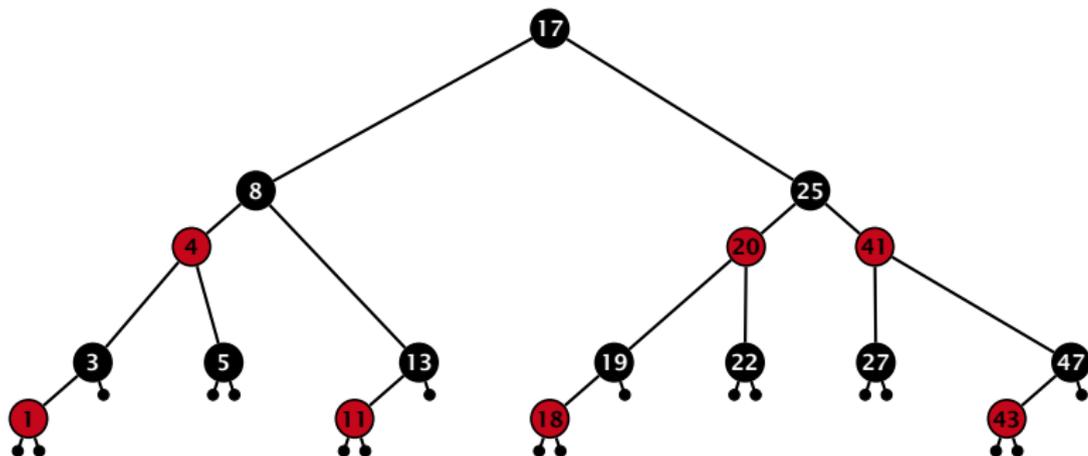
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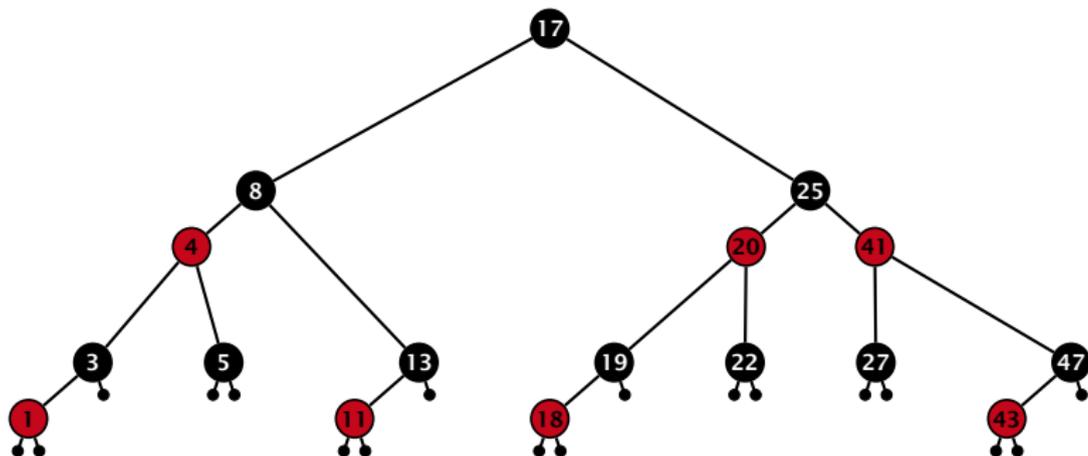
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Note that this correspondence is not unique. In particular, there are different red-black trees that correspond to the same (2, 4)-tree.

7.6 Skip Lists

Why do we not use a list for implementing the ADT Dynamic Set?

- ▶ time for search $\Theta(n)$
- ▶ time for insert $\Theta(n)$ (dominated by searching the item)
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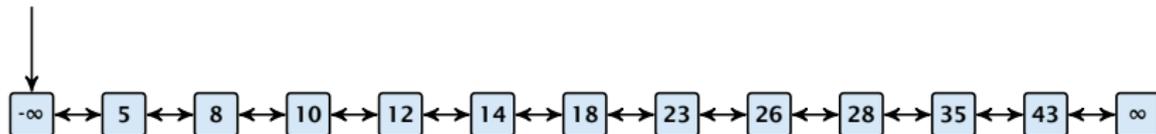
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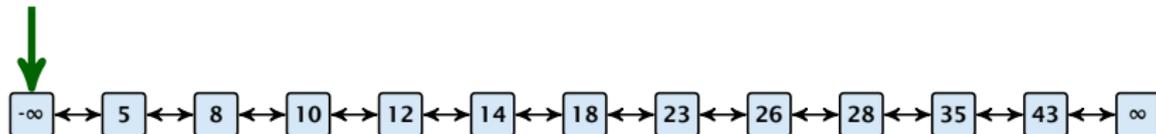
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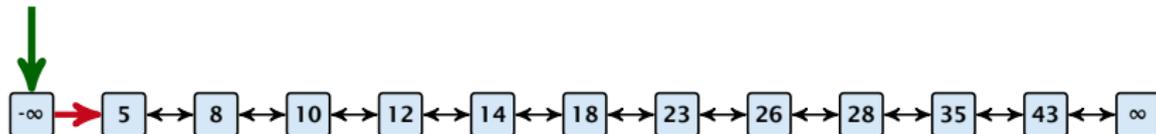
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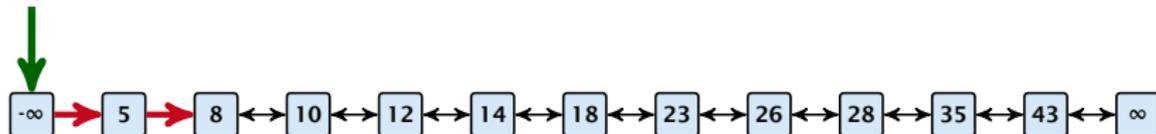
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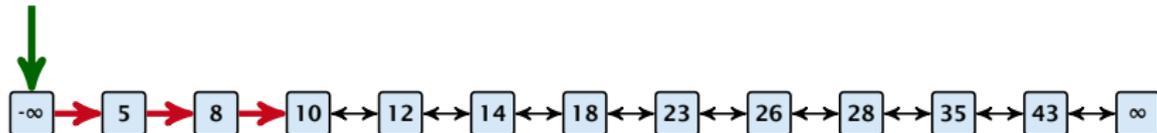
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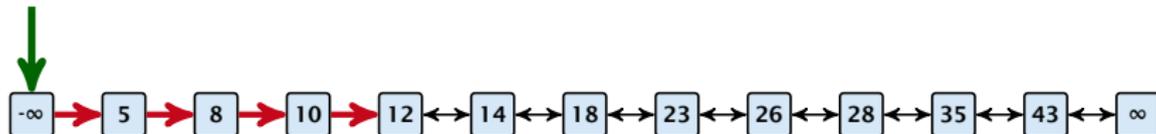
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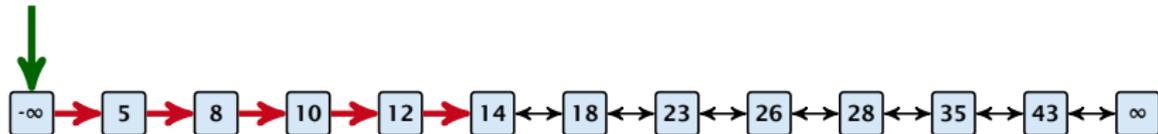
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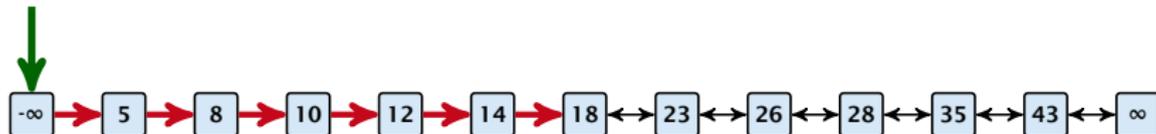
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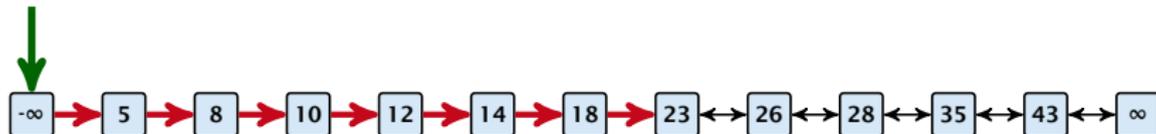
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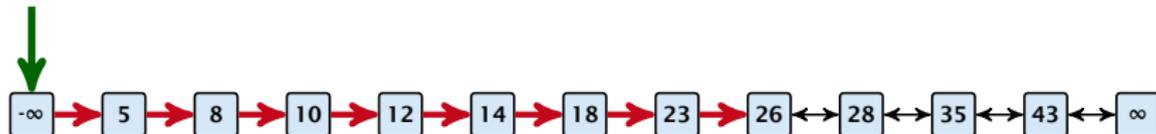
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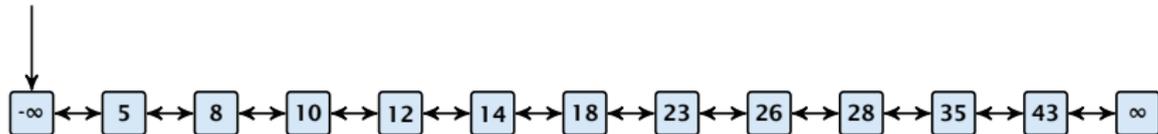
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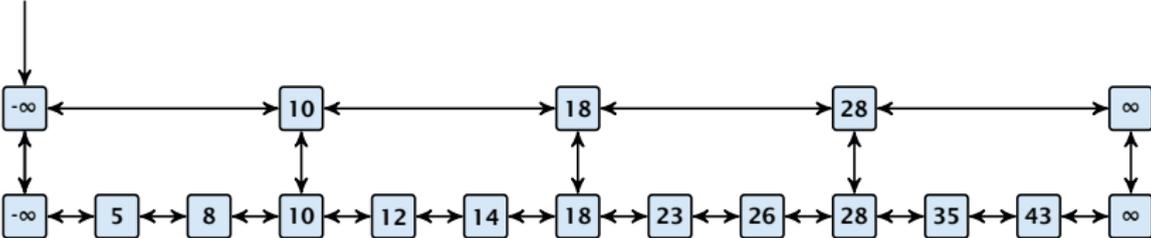
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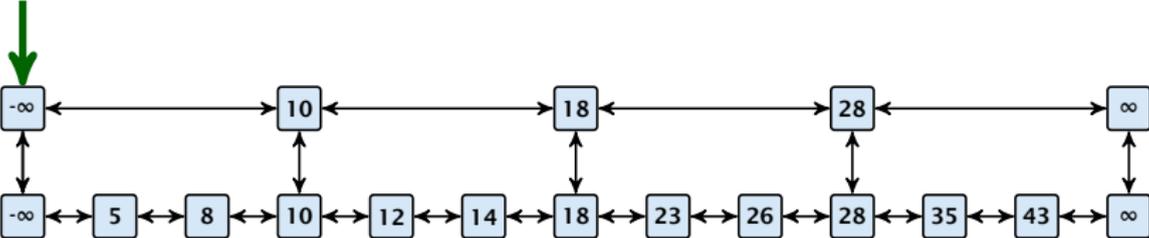
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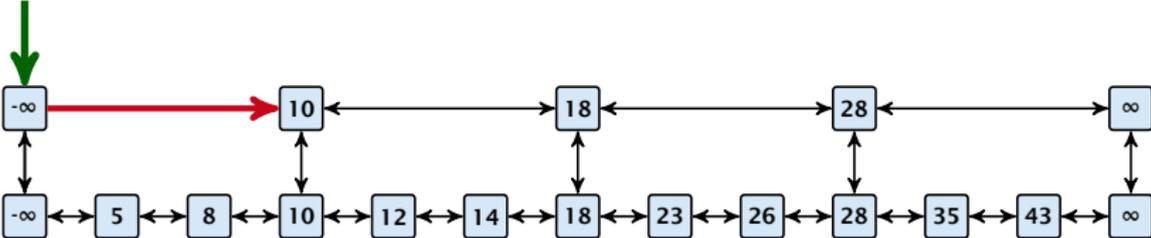
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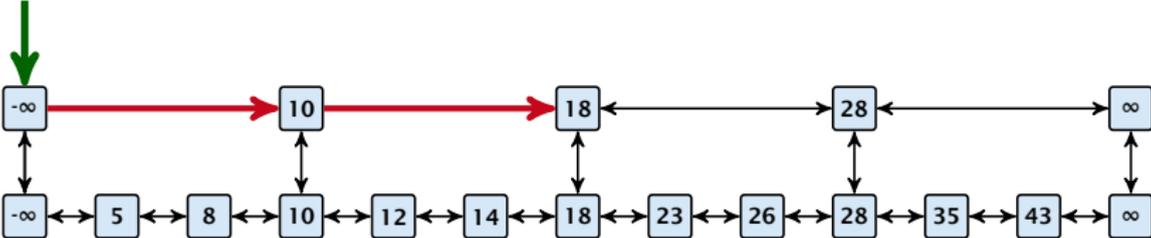
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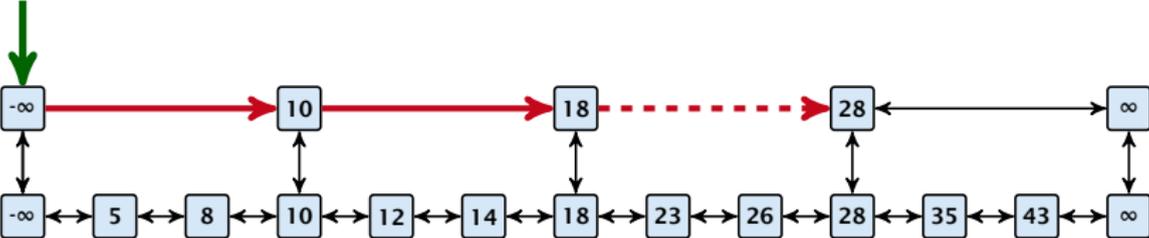
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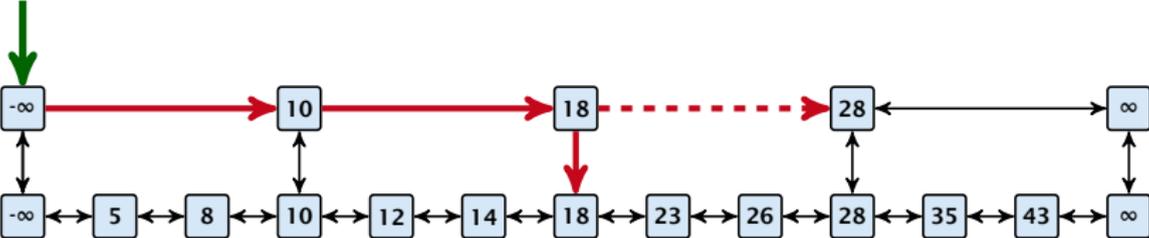
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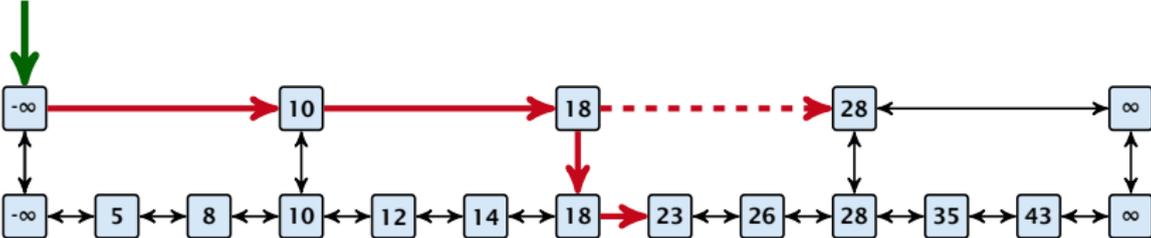
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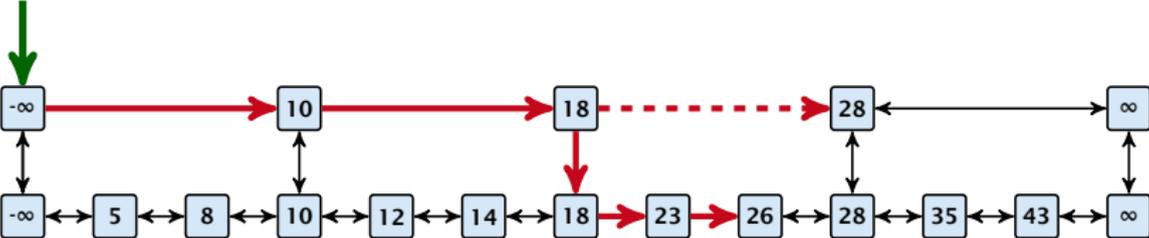
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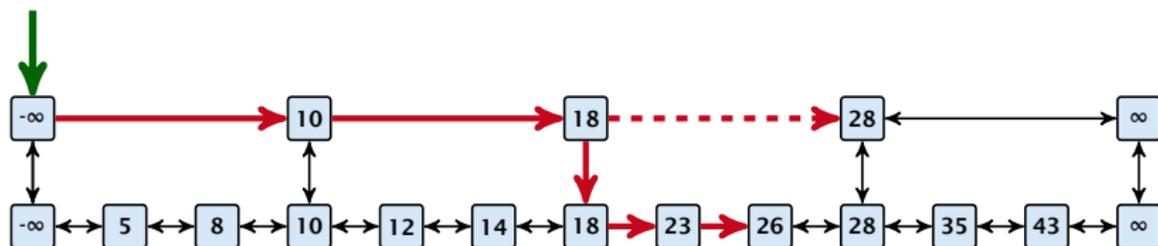
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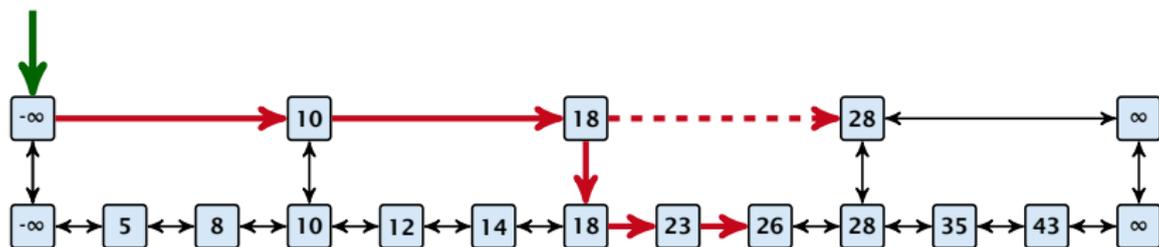


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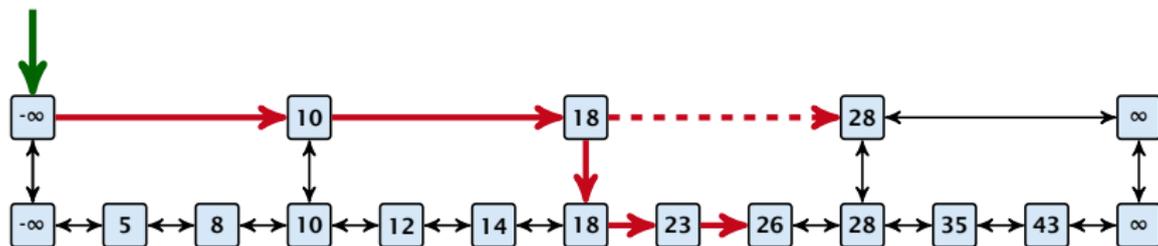
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Choose $|L_1| = \sqrt{n}$. Then search time $\Theta(\sqrt{n})$.

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- ▶ At most $|L_k| + \sum_{i=1}^k \frac{L_{i-1}}{L_i} + 3(k + 1)$ steps.

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Choosing $k = \Theta(\log n)$ gives a logarithmic running time.

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How to do insert and delete?

The cost of insert and delete is proportional to the number of elements in the list. Insert or delete may require a lot of reorganization.

Use randomization instead!

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Insert:

- ▶ A search operation gives you the insert position for element x in every list.
- ▶ Flip a coin until it shows head, and record the number $t \in \{1, 2, \dots\}$ of trials needed.
- ▶ Insert x into lists L_0, \dots, L_{t-1} .

Delete:

You get all predecessors via backward pointers.

Delete x in all lists it actually appears in.

The time for both operations is dominated by the search time.

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7.6 Skip Lists

Insert:

- ▶ A search operation gives you the insert position for element x in every list.
- ▶ Flip a coin until it shows head, and record the number $t \in \{1, 2, \dots\}$ of trials needed.
- ▶ Insert x into lists L_0, \dots, L_{t-1} .

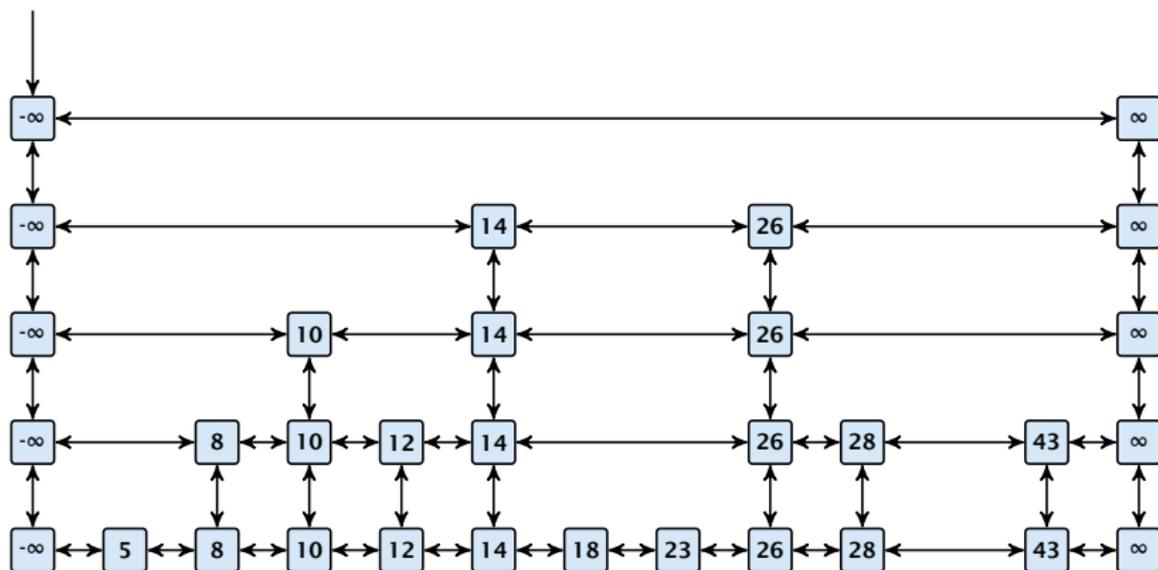
Delete:

- ▶ You get all predecessors via backward pointers.
- ▶ Delete x in all lists it actually appears in.

The time for both operations is dominated by the search time.

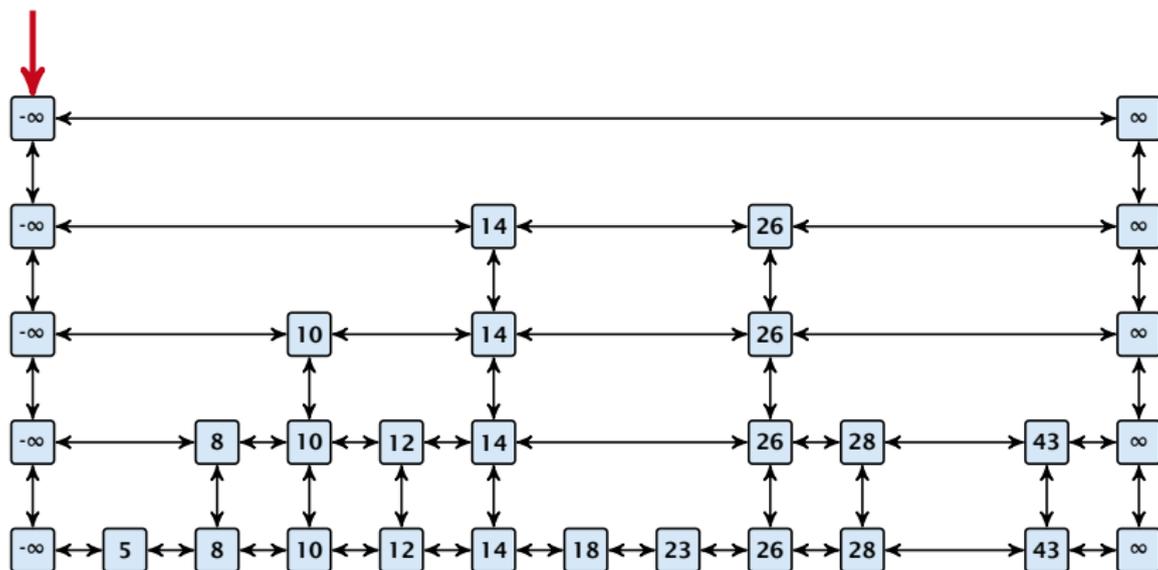
7.6 Skip Lists

Insert (35):



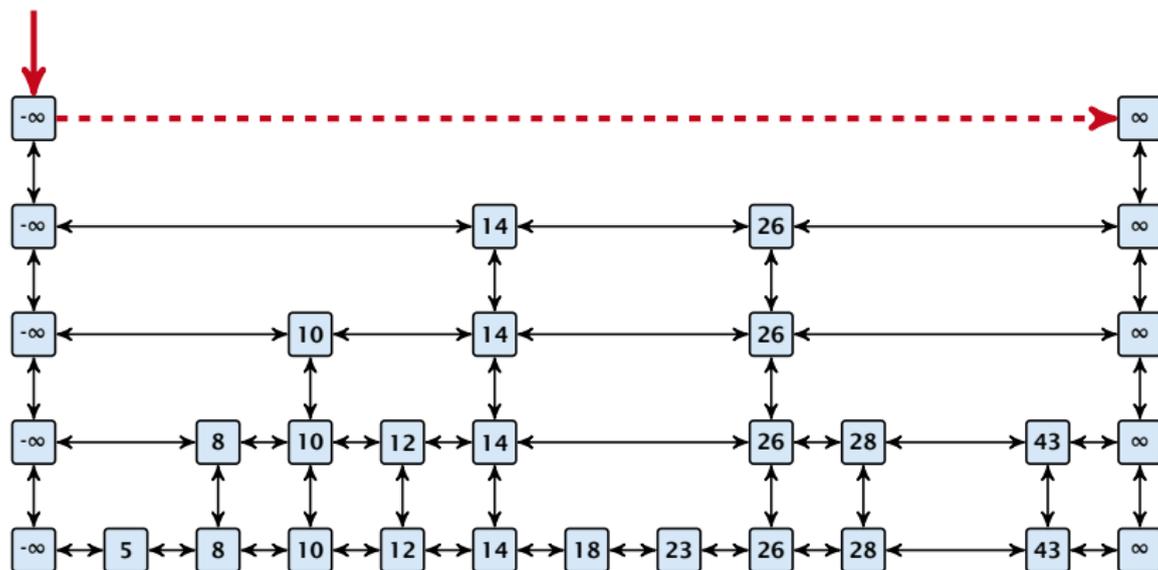
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Insert (35):



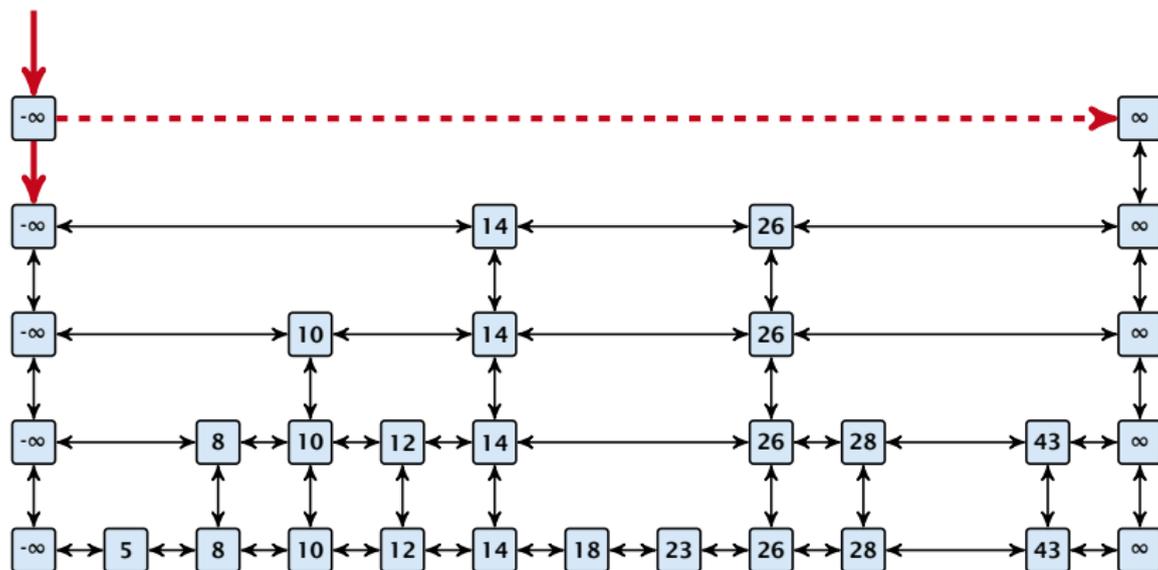
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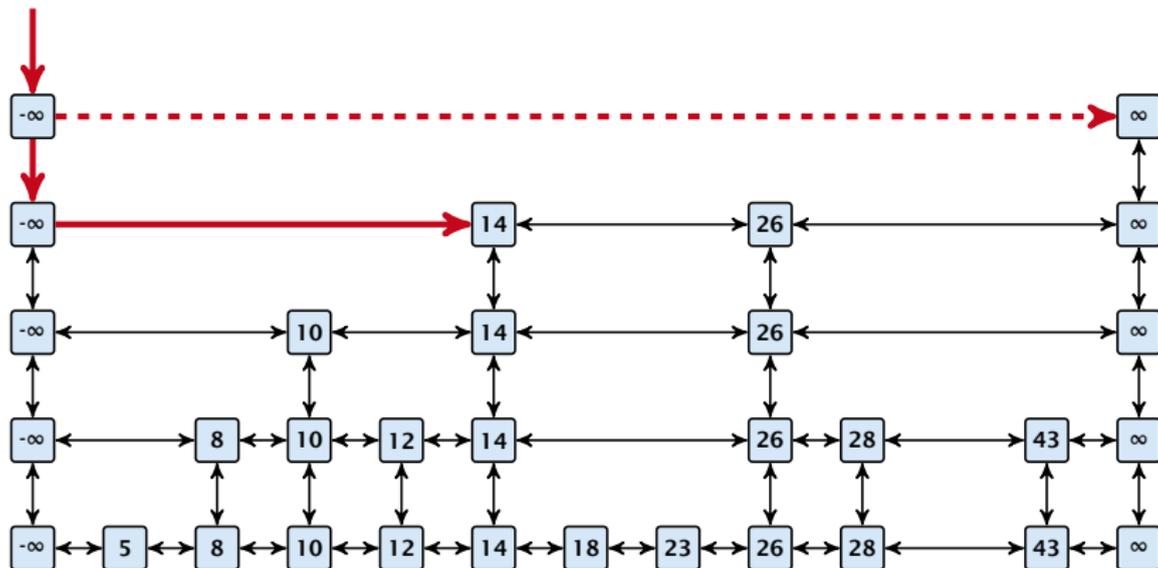
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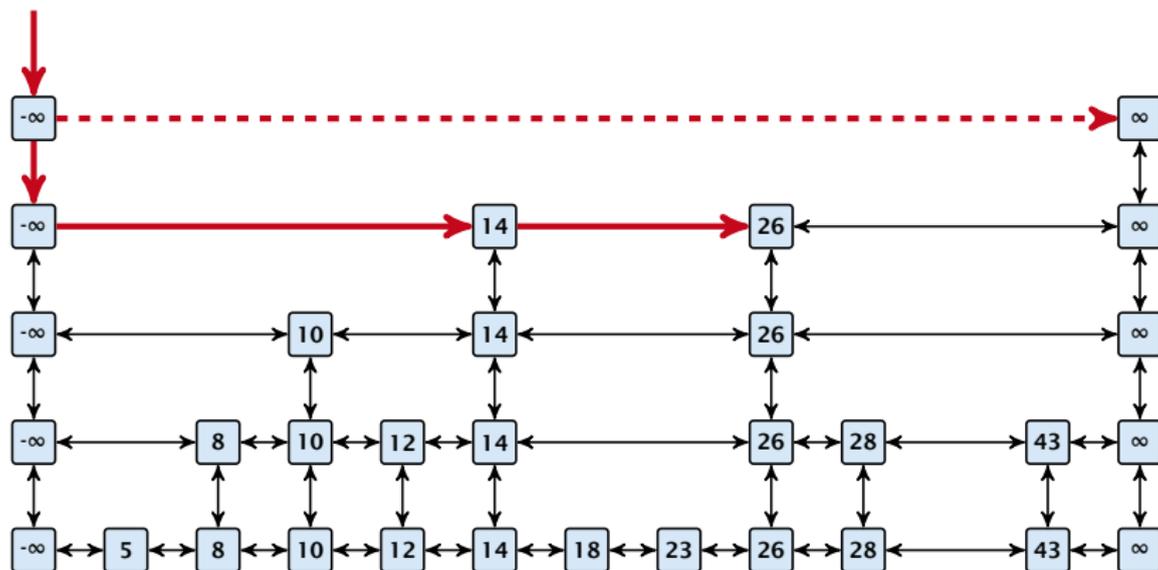
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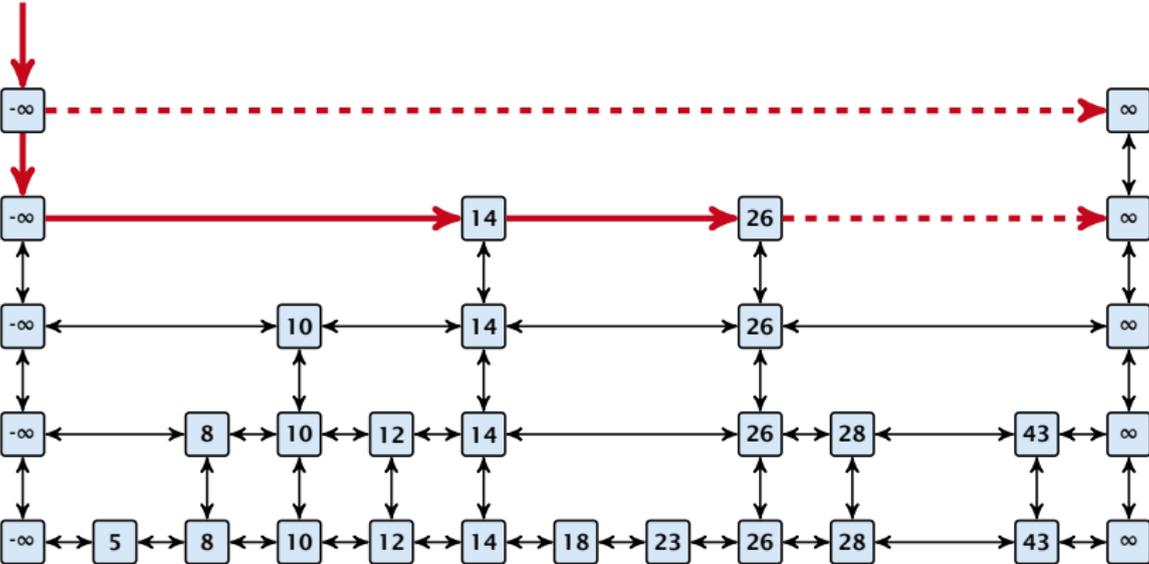
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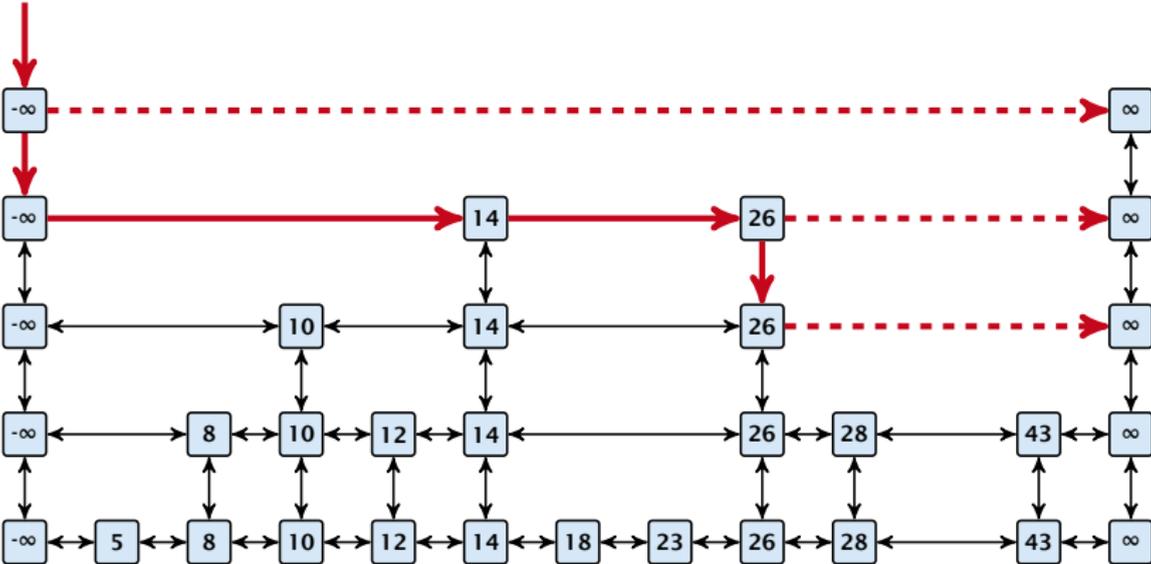
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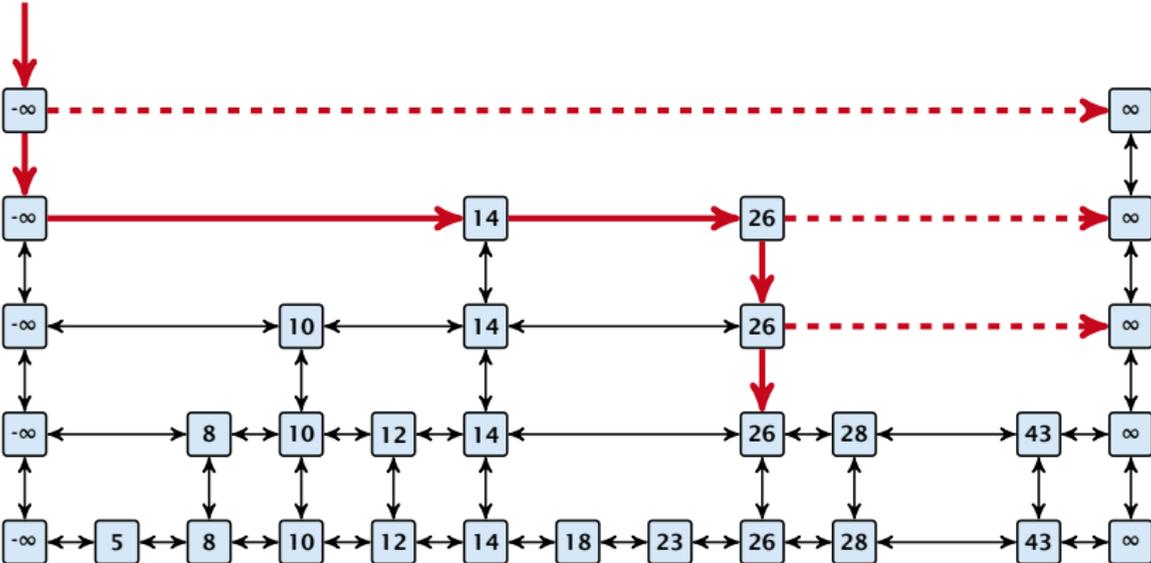
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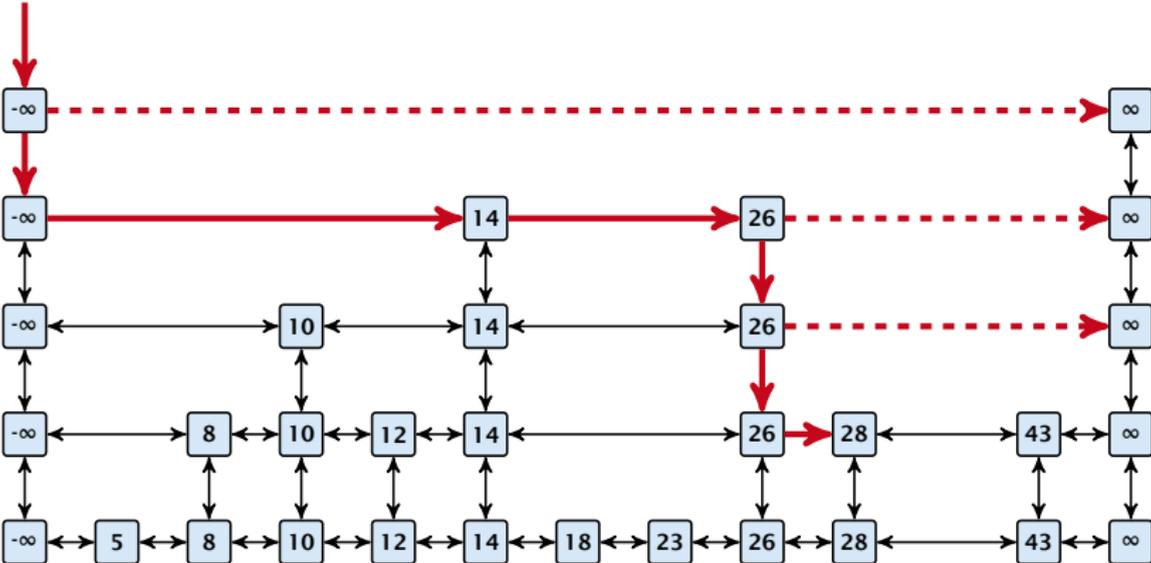
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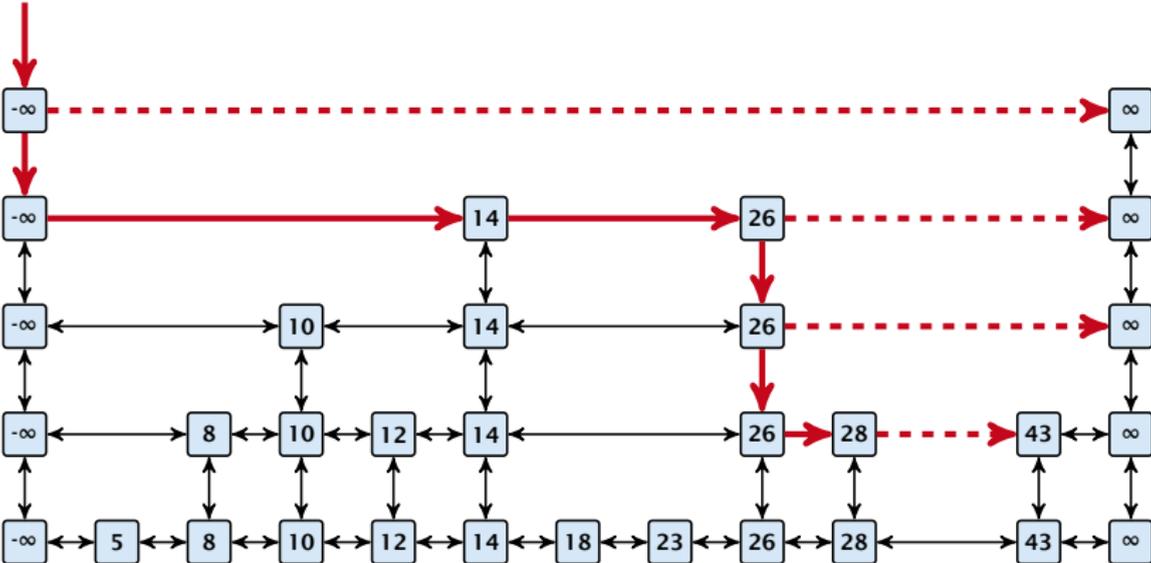
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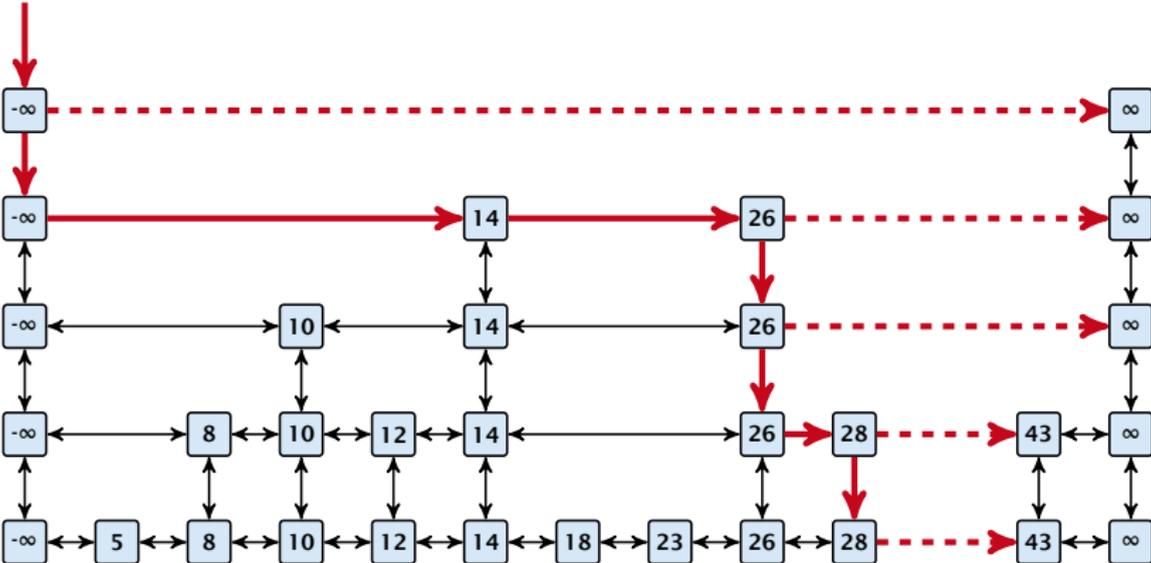
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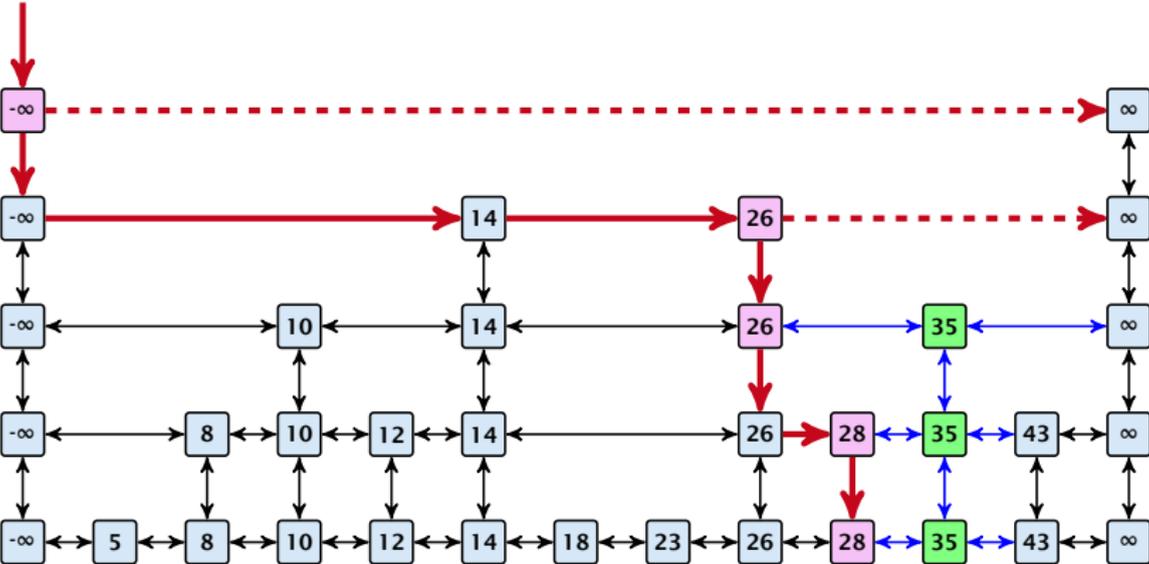
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High Probability

Definition 10 (High Probability)

We say a **randomized** algorithm has running time $\mathcal{O}(\log n)$ with **high probability** if for any constant α the running time is at most $\mathcal{O}(\log n)$ with probability at least $1 - \frac{1}{n^\alpha}$.

Here the \mathcal{O} -notation hides a constant that may depend on α .

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High Probability

Suppose there are a **polynomially** many events E_1, E_2, \dots, E_ℓ , $\ell = n^c$ each holding with high probability (e.g. E_i may be the event that the i -th search in a skip list takes time at most $\mathcal{O}(\log n)$).

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Then the probability that all E_i hold is at least

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This means $\Pr[E_1 \wedge \dots \wedge E_\ell]$ holds with high probability.

7.6 Skip Lists

Lemma 11

A search (and, hence, also insert and delete) in a skip list with n elements takes time $\mathcal{O}(\log n)$ with high probability (w. h. p.).

7.6 Skip Lists

Backward analysis:



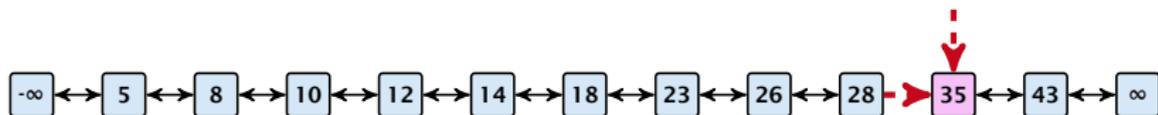
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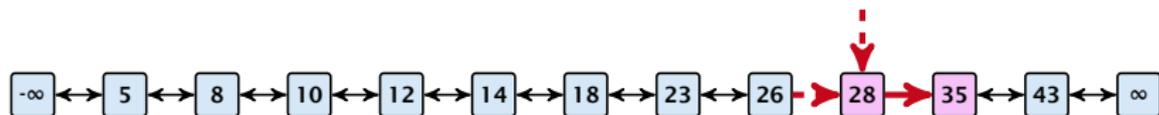
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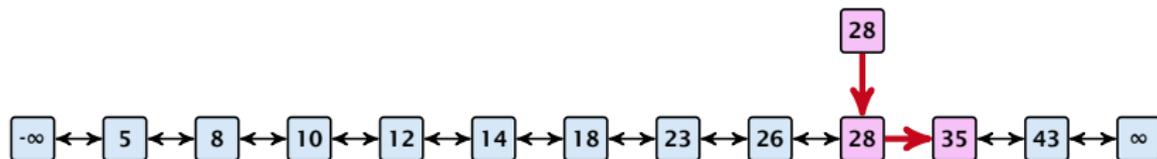
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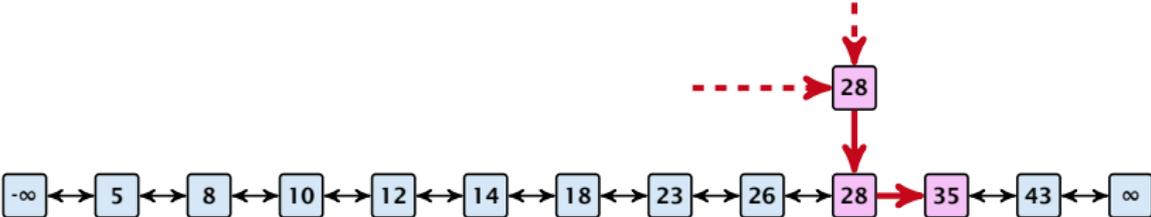
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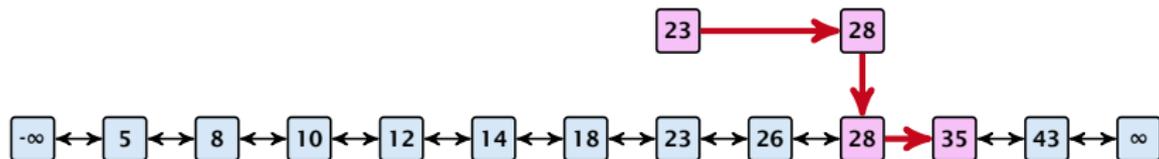
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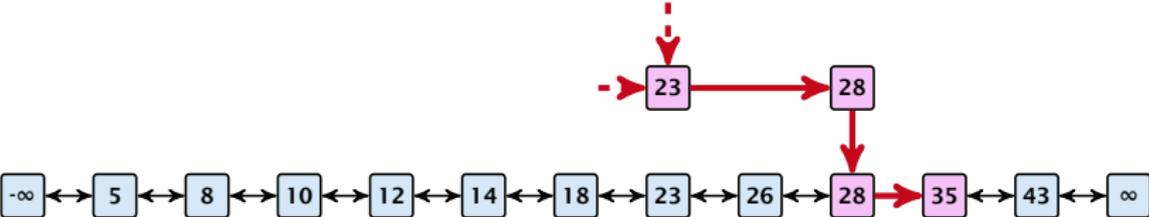
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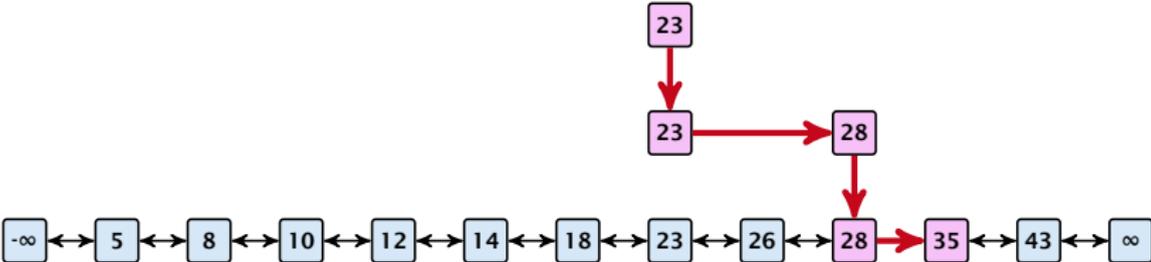
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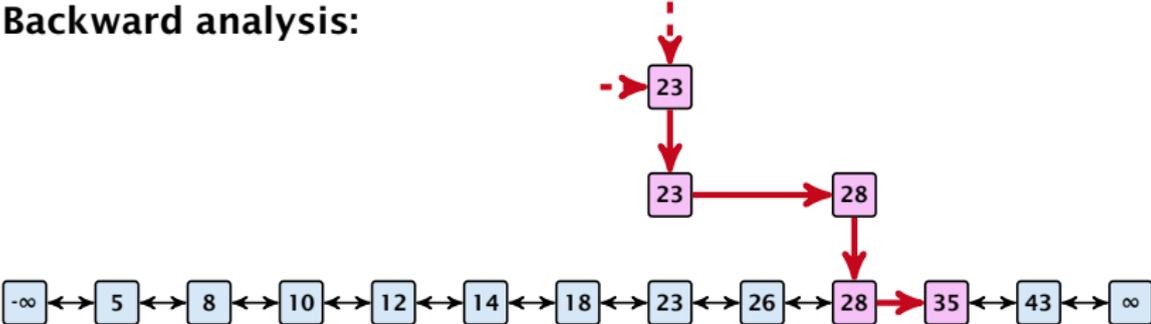
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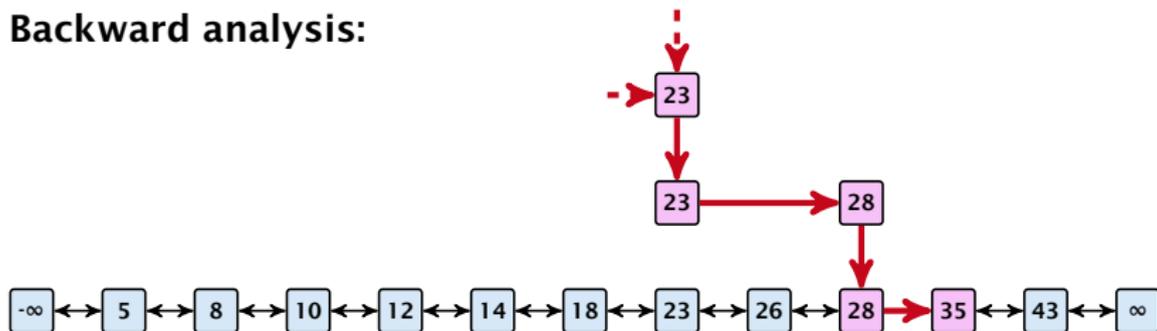
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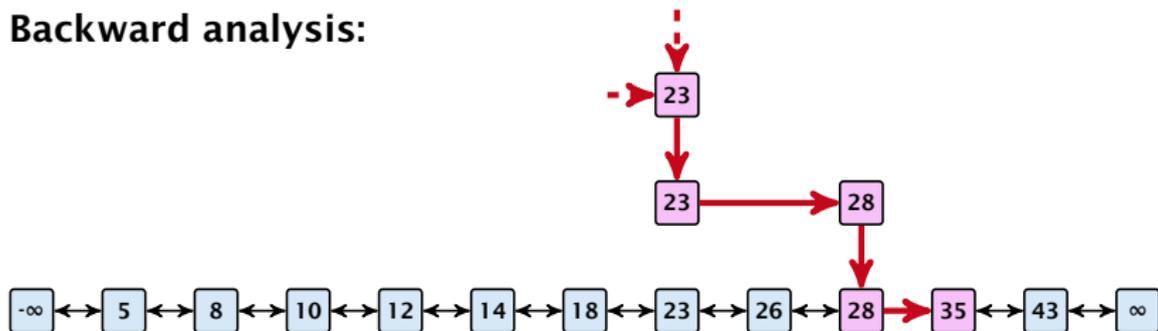
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At each point the path goes up with probability $1/2$ and left with probability $1/2$.

7.6 Skip Lists

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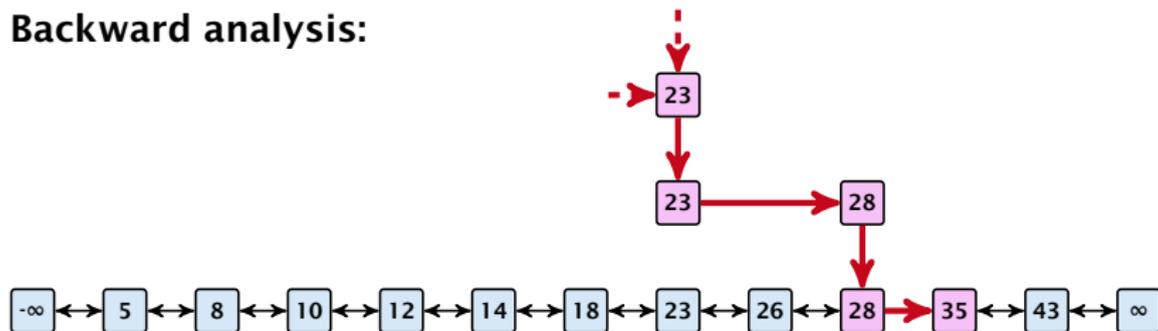
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We show that w.h.p:

- ▶ A “long” search path must also go very high.

7.6 Skip Lists

Backward analysis:



At each point the path goes up with probability $1/2$ and left with probability $1/2$.

We show that w.h.p.:

- ▶ A “long” search path must also go very high.
- ▶ There are no elements in high lists.

From this it follows that w.h.p. there are no long paths.

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{en}{k}\right)^k$$

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In particular, this means that during the construction in the backward analysis we see at most k heads (i.e., coin flips that tell you to go up) in z trials.

7.6 Skip Lists

$$\Pr[E_{z,k}]$$

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Hence,

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So far we fixed $k = \gamma \log n$, $\gamma \geq 1$, and $z = 7\alpha\gamma \log n$, $\alpha \geq 1$.

This means that a search path of length $\Omega(\log n)$ visits a list on a level $\Omega(\log n)$, w.h.p.

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This means, the search requires at most z steps, w. h. p.

7.7 Hashing

Dictionary:

- ▶ **S . insert(x)**: Insert an element x .
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So far we have implemented the search for a key by carefully choosing split-elements.

Then the memory location of an object x with key k is determined by successively comparing k to split-elements.

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Definitions:

- ▶ Universe U of keys, e.g., $U \subseteq \mathbb{N}_0$. U very large.
- ▶ Set $S \subseteq U$ of keys, $|S| = m \leq |U|$.
- ▶ Array $T[0, \dots, n-1]$ hash-table.
- ▶ Hash function $h : U \rightarrow [0, \dots, n-1]$.

The hash-function h should fulfill:

- ▶ Fast to evaluate.
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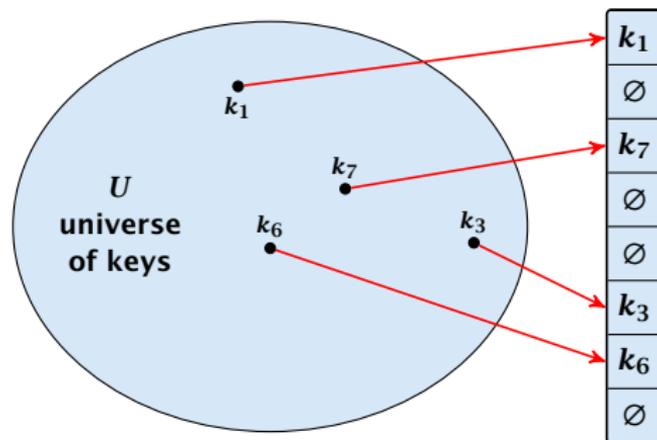
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Direct Addressing

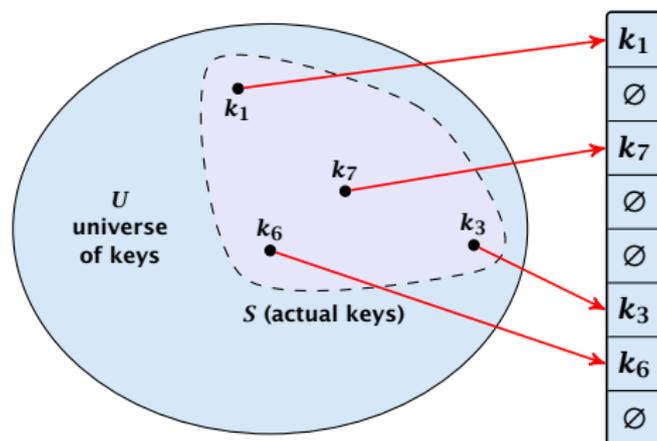
Ideally the hash function maps **all** keys to different memory locations.



This special case is known as **Direct Addressing**. It is usually very unrealistic as the universe of keys typically is quite large, and in particular larger than the available memory.

Perfect Hashing

Suppose that we **know** the set S of actual keys (no insert/no delete). Then we may want to design a **simple** hash-function that maps all these keys to different memory locations.



Such a hash function h is called a **perfect hash function** for set S .

Collisions

If we do not know the keys in advance, the best we can hope for is that the hash function distributes keys evenly across the table.

Problem: Collisions

Usually the universe U is much larger than the table-size n .

Hence, there may be two elements k_1, k_2 from the set S that map to the same memory location (i.e., $h(k_1) = h(k_2)$). This is called a **collision**.

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Typically, collisions do not appear once the size of the set S of actual keys gets close to n , but already when $|S| \geq \omega(\sqrt{n})$.

Lemma 12

The probability of having a collision when hashing m elements into a table of size n under uniform hashing is at least

$$1 - e^{-\frac{m(m-1)}{2n}} \approx 1 - e^{-\frac{m^2}{2n}}.$$

Uniform hashing:

Choose a hash function uniformly at random from all functions $f: U \rightarrow [0, \dots, n-1]$.

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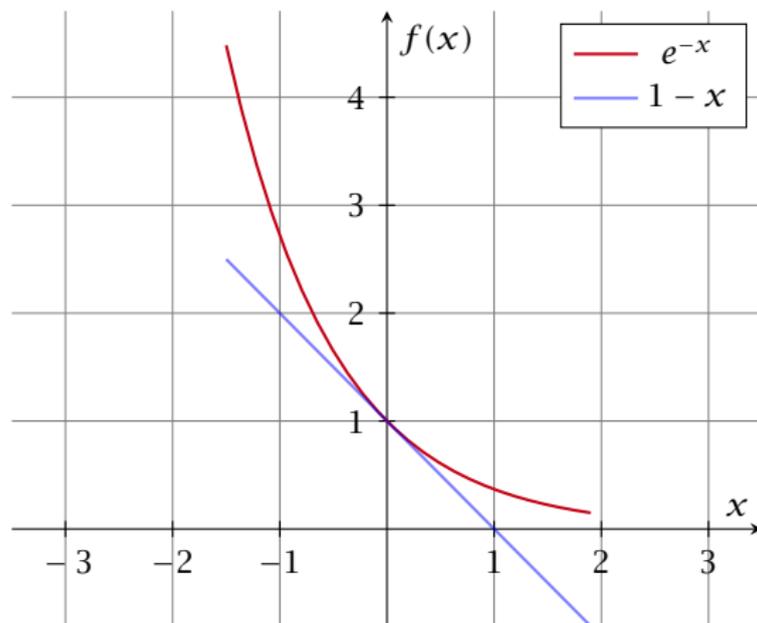
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Here the first equality follows since the ℓ -th element that is hashed has a probability of $\frac{n-\ell+1}{n}$ to not generate a collision under the condition that the previous elements did not induce collisions. □

Collisions



The inequality $1 - x \leq e^{-x}$ is derived by stopping the Taylor-expansion of e^{-x} after the second term.

Resolving Collisions

The methods for dealing with collisions can be classified into the two main types

- ▶ **open addressing**, aka. closed hashing
- ▶ **hashing with chaining**, aka. closed addressing, open hashing.

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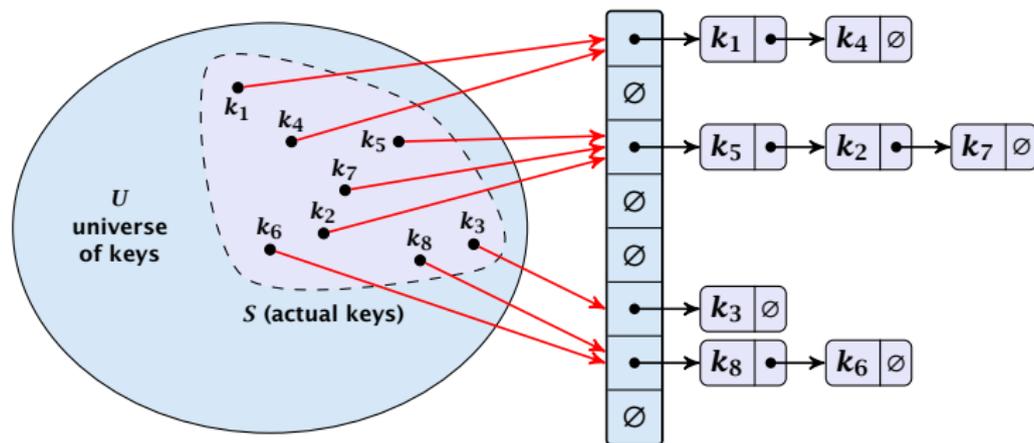
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Hashing with Chaining

Arrange elements that map to the same position in a linear list.

- ▶ Access: compute $h(x)$ and search list for $\text{key}[x]$.
- ▶ Insert: insert at the front of the list.



Hashing with Chaining

Let A denote a strategy for resolving collisions. We use the following notation:

- ▶ A^+ denotes the average time for a **successful** search when using A ;
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$$A^- = 1 + \alpha .$$

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For a successful search observe that we do **not** choose a list at random, but we consider a random key k in the hash-table and ask for the search-time for k .

This is 1 plus the number of elements that lie before k in k 's list.

Let k_ℓ denote the ℓ -th key inserted into the table.

Let for two keys k_i and k_j , X_{ij} denote the indicator variable for the event that k_i and k_j hash to the same position. Clearly, $\Pr[X_{ij} = 1] = 1/n$ for uniform hashing.

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Hence, the expected cost for a successful search is $A^+ \leq 1 + \frac{\alpha}{2}$.

Hashing with Chaining

Disadvantages:

- ▶ pointers increase memory requirements
- ▶ pointers may lead to bad cache efficiency

Advantages:

- ▶ no à priori limit on the number of elements
- ▶ deletion can be implemented efficiently
- ▶ by using balanced trees instead of linked list one can also obtain worst-case guarantees.

Open Addressing

All objects are stored in the table itself.

Define a function $h(k, j)$ that determines the table-position to be examined in the j -th step. The values $h(k, 0), \dots, h(k, n - 1)$ must form a permutation of $0, \dots, n - 1$.

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Choices for $h(k, j)$:

- ▶ **Linear probing:**

$$h(k, i) = h(k) + i \pmod n$$

(sometimes: $h(k, i) = h(k) + ci \pmod n$).

- ▶ Quadratic probing:

$$h(k, i) = h(k) + c_1 i + c_2 i^2 \pmod n.$$

- ▶ Double hashing:

$$h(k, i) = h_1(k) + ih_2(k) \pmod n.$$

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- ▶ Advantage: **Cache-efficiency**. The new probe position is very likely to be in the cache.
- ▶ Disadvantage: **Primary clustering**. Long sequences of occupied table-positions get longer as they have a larger probability to be hit. Furthermore, they can merge forming larger sequences.

Lemma 13

Let L be the method of linear probing for resolving collisions:

$$L^+ \approx \frac{1}{2} \left(1 + \frac{1}{1 - \alpha} \right)$$

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- ▶ Any probe into the hash-table usually creates a cache-miss.

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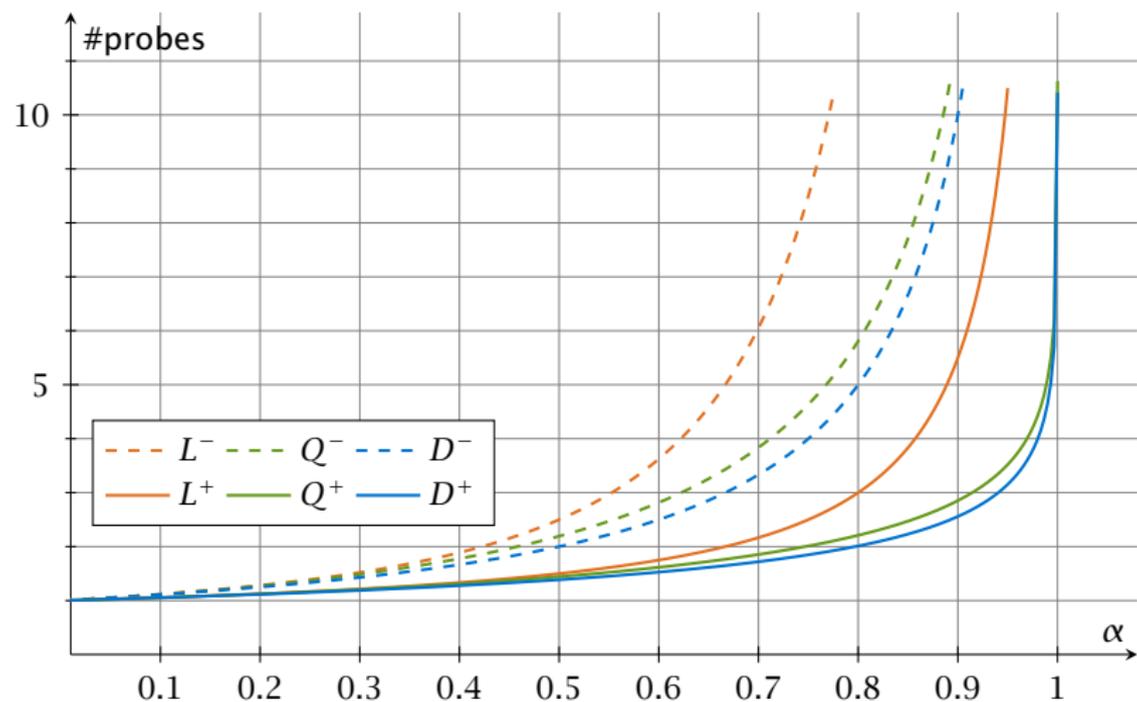
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Open Addressing

Some values:

α	<i>Linear Probing</i>		<i>Quadratic Probing</i>		<i>Double Hashing</i>	
	L^+	L^-	Q^+	Q^-	D^+	D^-
0.5	1.5	2.5	1.44	2.19	1.39	2
0.9	5.5	50.5	2.85	11.40	2.55	10
0.95	10.5	200.5	3.52	22.05	3.15	20

Open Addressing



Analysis of Idealized Open Address Hashing

We analyze the time for a search in a very idealized Open Addressing scheme.

- ▶ The probe sequence $h(k, 0), h(k, 1), h(k, 2), \dots$ is equally likely to be any permutation of $\langle 0, 1, \dots, n - 1 \rangle$.

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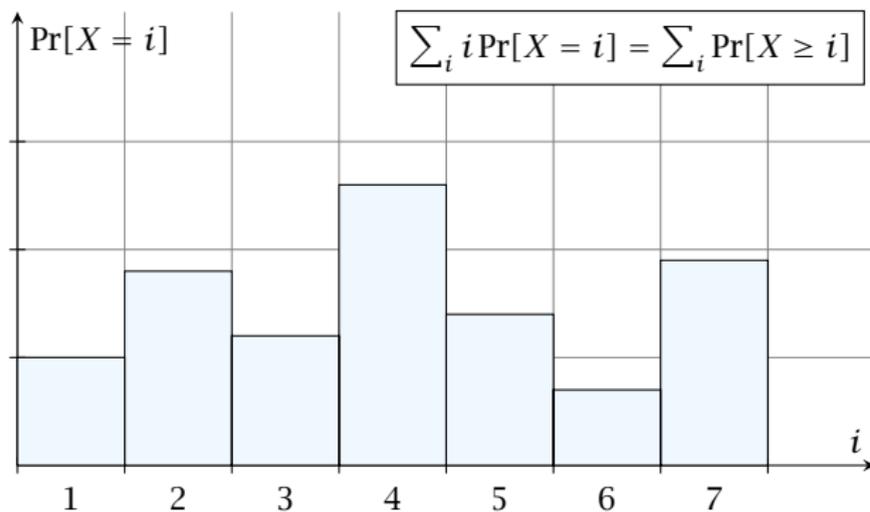
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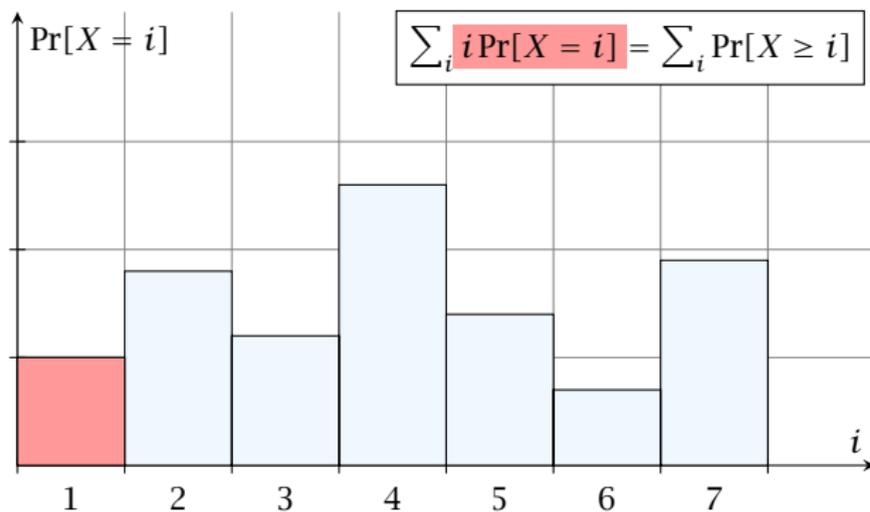
$$\frac{1}{1-\alpha} = 1 + \alpha + \alpha^2 + \alpha^3 + \dots$$

Analysis of Idealized Open Address Hashing



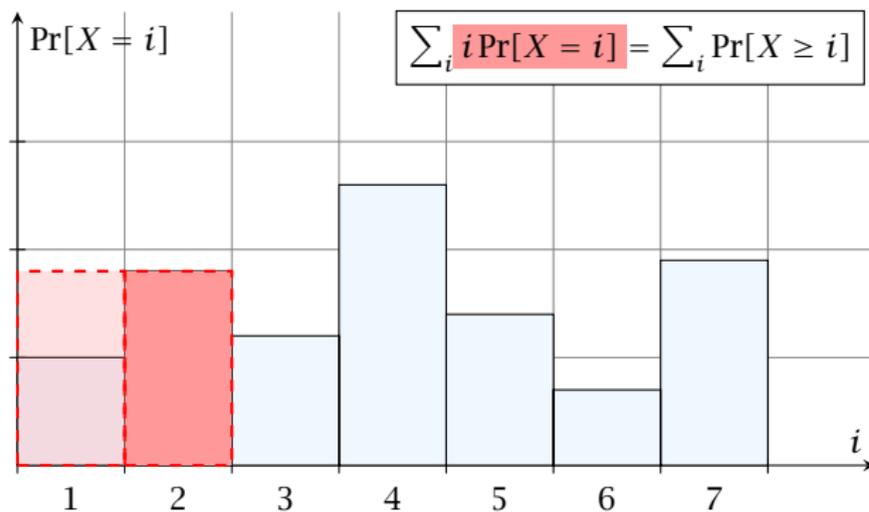
Analysis of Idealized Open Address Hashing

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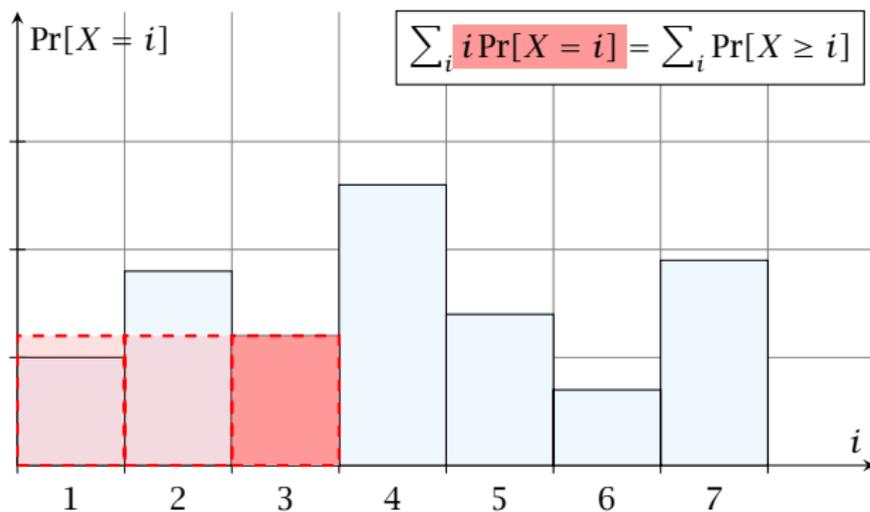
Analysis of Idealized Open Address Hashing

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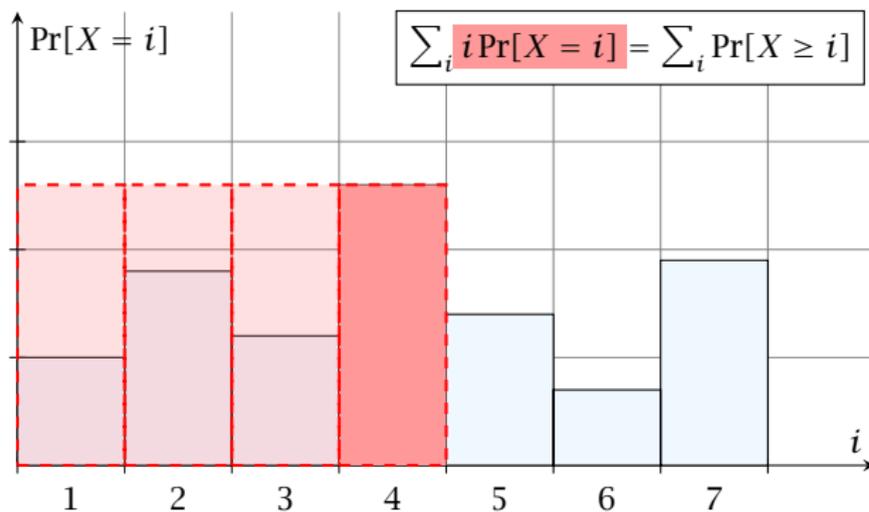
Analysis of Idealized Open Address Hashing

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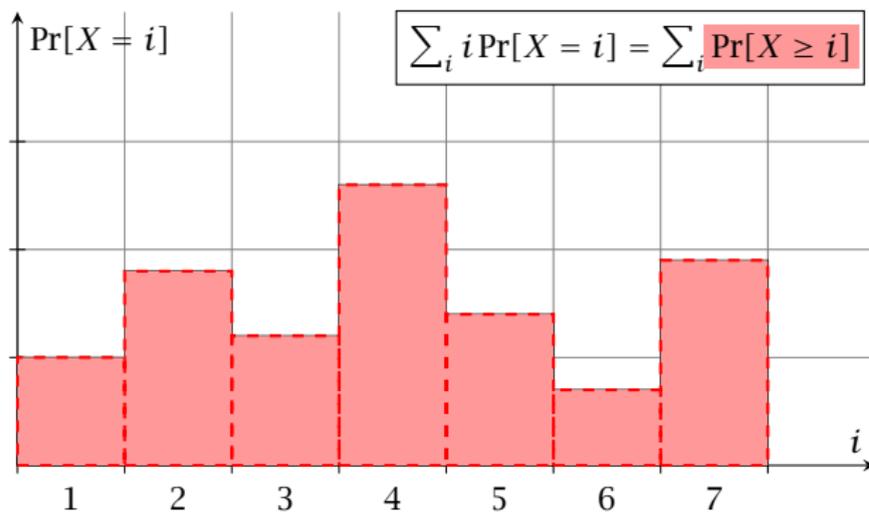
Analysis of Idealized Open Address Hashing

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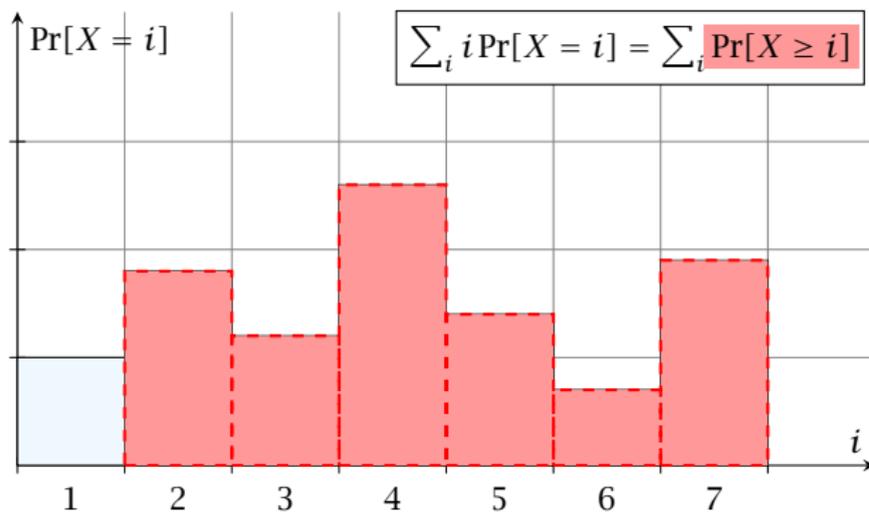
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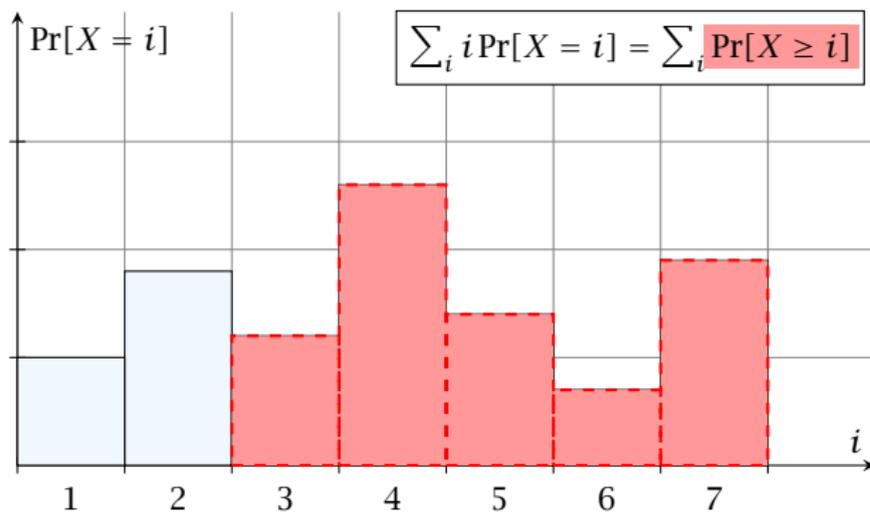
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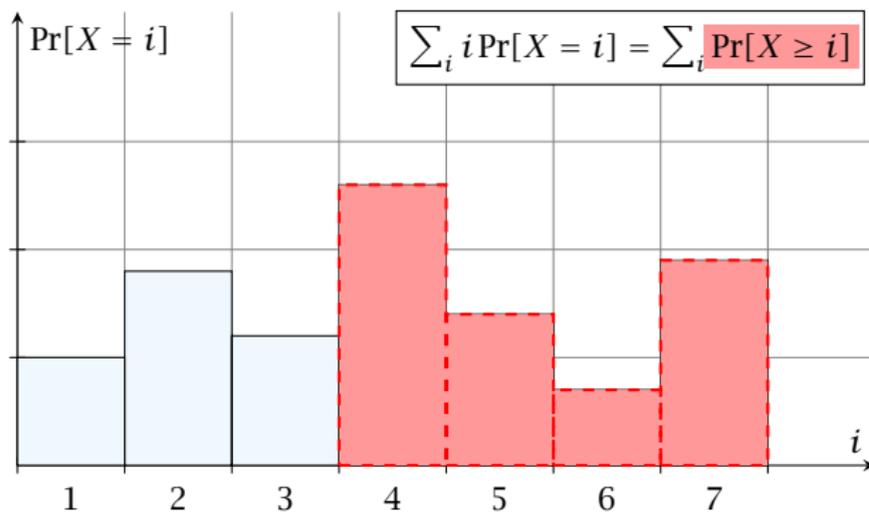
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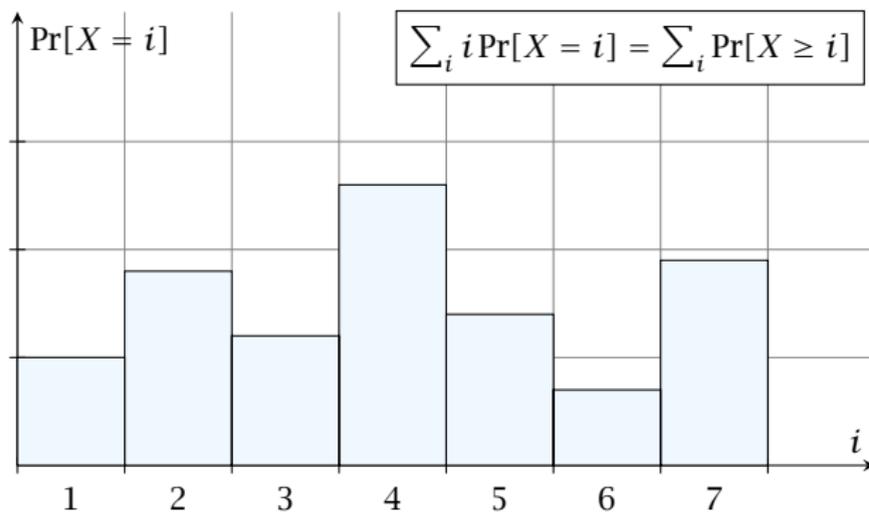


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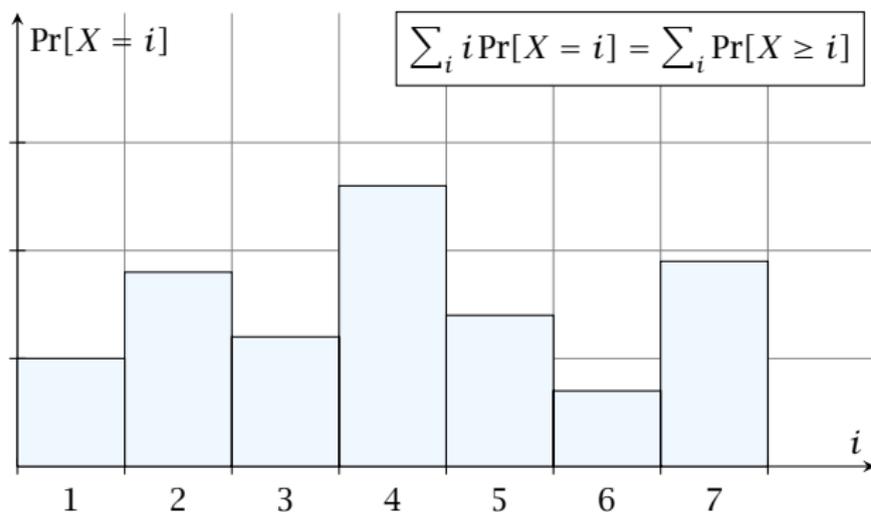
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Analysis of Idealized Open Address Hashing



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The j -th rectangle appears in both sums j times. (j times in the first due to multiplication with j ; and j times in the second for summands $i = 1, 2, \dots, j$)

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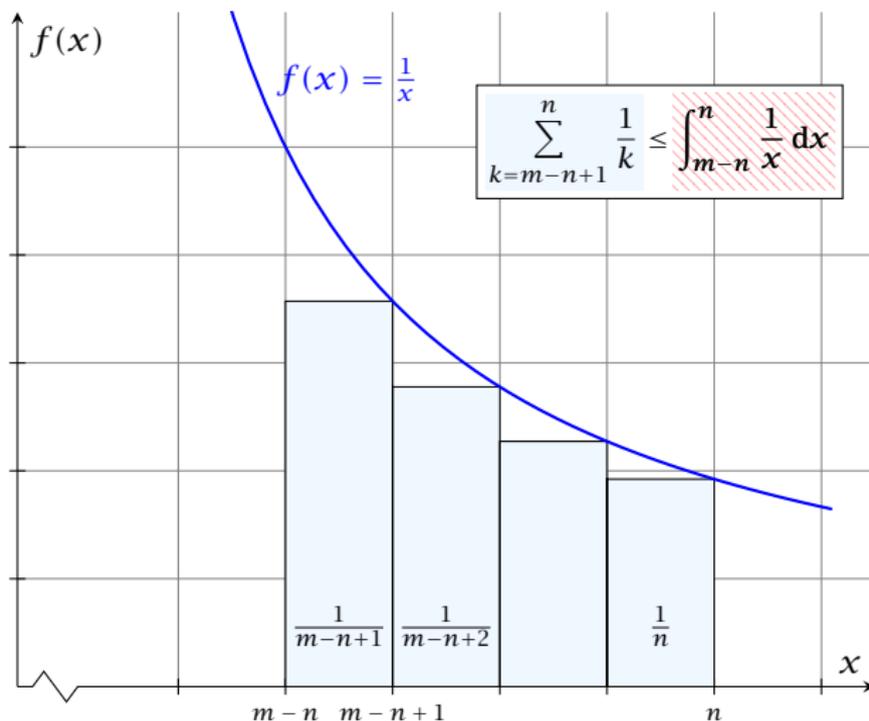
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How do we delete in a hash-table?

- ▶ For hashing with chaining this is not a problem. Simply search for the key, and delete the item in the corresponding list.
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- ▶ Simply removing a key might interrupt the probe sequence of other keys which then cannot be found anymore.
- ▶ One can delete an element by replacing it with a **deleted-marker**.
 - ▶ Deleted markers interrupt the probe sequence of other keys which then cannot be found anymore.
 - ▶ Deleted markers can be ignored during insertions.
 - ▶ Deleted markers can be removed during deletions.
 - ▶ Deleted markers can be converted to empty slots during a rehash.
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- ▶ If a table contains many deleted-markers (linear fraction of the keys) one can rehash the whole table and amortize the cost for this rehash against the cost for the deletions.

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 - ▶ During an insertion if a **deleted**-marker is encountered an element can be inserted there.
 - ▶ During a search a **deleted**-marker must not be used to terminate the probe sequence.
- ▶ The table could fill up with **deleted**-markers leading to bad performance.
- ▶ If a table contains many deleted-markers (linear fraction of the keys) one can rehash the whole table and amortize the cost for this rehash against the cost for the deletions.

Deletions for Linear Probing

- ▶ For Linear Probing one can delete elements without using **deletion**-markers.
- ▶ Upon a deletion elements that are further down in the probe-sequence may be moved to guarantee that they are still found during a search.

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Algorithm 16 delete(p)

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1:  $T[p] \leftarrow \text{null}$ 
2:  $p \leftarrow \text{succ}(p)$ 
3: while  $T[p] \neq \text{null}$  do
4:    $y \leftarrow T[p]$ 
5:    $T[p] \leftarrow \text{null}$ 
6:    $p \leftarrow \text{succ}(p)$ 
7:   insert( $y$ )
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p is the index into the table-cell that contains the object to be deleted.

Pointers into the hash-table become invalid.

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Universal Hashing

Regardless, of the choice of hash-function there is always an input (a set of keys) that has a very poor worst-case behaviour.

Therefore, so far we assumed that the hash-function is random so that regardless of the input the average case behaviour is good.

However, the assumption of uniform hashing that h is chosen randomly from all functions $f: U \rightarrow [0, \dots, n-1]$ is clearly unrealistic as there are $n^{|U|}$ such functions. Even writing down such a function would take $|U| \log n$ bits.

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A class \mathcal{H} of hash-functions from the universe U into the set $\{0, \dots, n-1\}$ is called **universal** if for all $u_1, u_2 \in U$ with $u_1 \neq u_2$

$$\Pr[h(u_1) = h(u_2)] \leq \frac{1}{n} ,$$

where the probability is w. r. t. the choice of a random hash-function from set \mathcal{H} .

Note that this means that the probability of a collision between two arbitrary elements is at most $\frac{1}{n}$.

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A class \mathcal{H} of hash-functions from the universe U into the set $\{0, \dots, n-1\}$ is called **2-independent** (pairwise independent) if the following two conditions hold

- ▶ For any key $u \in U$, and $t \in \{0, \dots, n-1\}$ $\Pr[h(u) = t] = \frac{1}{n}$, i.e., a key is distributed uniformly within the hash-table.
- ▶ For all $u_1, u_2 \in U$ with $u_1 \neq u_2$, and for any two hash-positions t_1, t_2 :

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A class \mathcal{H} of hash-functions from the universe U into the set $\{0, \dots, n-1\}$ is called **k -independent** if for any choice of $\ell \leq k$ distinct keys $u_1, \dots, u_\ell \in U$, and for any set of ℓ not necessarily distinct hash-positions t_1, \dots, t_ℓ :

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where the probability is w. r. t. the choice of a random hash-function from set \mathcal{H} .

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Let $U := \{0, \dots, p-1\}$ for a prime p . Let $\mathbb{Z}_p := \{0, \dots, p-1\}$, and let $\mathbb{Z}_p^* := \{1, \dots, p-1\}$ denote the set of invertible elements in \mathbb{Z}_p .

Define

$$h_{a,b}(x) := (ax + b \bmod p) \bmod n$$

Lemma 20

The class

$$\mathcal{H} = \{h_{a,b} \mid a \in \mathbb{Z}_p^*, b \in \mathbb{Z}_p\}$$

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Proof.

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$$\triangleright ax + b \not\equiv ay + b \pmod{p}$$

if $a \neq 0$, then $ax + b \equiv ay + b \pmod{p}$ is equivalent to $ax \equiv ay \pmod{p}$.

Multiplying with a^{-1} yields $x \equiv y \pmod{p}$.

Since x and y are distinct keys, $x \not\equiv y \pmod{p}$.

Therefore, the probability of a collision is $1/n$.

Since a and b are chosen uniformly at random, the probability of a collision is $1/n$.

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Let $x, y \in U$ be two distinct keys. We have to show that the probability of a collision is only $1/n$.

$$\blacktriangleright ax + b \not\equiv ay + b \pmod{p}$$

If $x \neq y$ then $(x - y) \not\equiv 0 \pmod{p}$.

Multiplying with $a \not\equiv 0 \pmod{p}$ gives

$$a(x - y) \not\equiv 0 \pmod{p}$$

where we use that \mathbb{Z}_p is a field (Körper) and, hence, has no zero divisors (nullteilerfrei).

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- ▶ The hash-function does not generate collisions before the $(\text{mod } n)$ -operation. Furthermore, every choice (a, b) is mapped to a different pair (t_x, t_y) with $t_x := ax + b$ and $t_y := ay + b$.

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$$b \equiv t_y - ay \pmod{p}$$

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There is a one-to-one correspondence between hash-functions (pairs (a, b) , $a \neq 0$) and pairs (t_x, t_y) , $t_x \neq t_y$.

Therefore, we can view the first step (before the mod n -operation) as choosing a pair (t_x, t_y) , $t_x \neq t_y$ uniformly at random.

What happens when we do the mod n operation?

Fix a value t_x . There are $p - 1$ possible values for choosing t_y .

From the range $0, \dots, p - 1$ the values $t_x, t_x + n, t_x + 2n, \dots$ map to t_x after the modulo-operation. These are at most $\lceil p/n \rceil$ values.

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As $t_y \neq t_x$ there are

possibilities for choosing t_y such that the final hash-value creates a collision.

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It is also possible to show that \mathcal{H} is an (almost) pairwise independent class of hash-functions.

$$\Pr_{t_x \neq t_y \in \mathbb{Z}_p^2} \left[\begin{array}{l} t_x \bmod n = h_1 \\ t_y \bmod n = h_2 \end{array} \right]$$

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Note that the middle is the probability that $h(x) = h_1$ and $h(y) = h_2$. The total number of choices for (t_x, t_y) is $p(p-1)$. The number of choices for t_x (t_y) such that $t_x \bmod n = h_1$ ($t_y \bmod n = h_2$) lies between $\lfloor \frac{p}{n} \rfloor$ and $\lceil \frac{p}{n} \rceil$.

Universal Hashing

Definition 21

Let $d \in \mathbb{N}$; $q \geq (d + 1)n$ be a prime; and let $\bar{a} \in \{0, \dots, q - 1\}^{d+1}$. Define for $x \in \{0, \dots, q - 1\}$

$$h_{\bar{a}}(x) := \left(\sum_{i=0}^d a_i x^i \bmod q \right) \bmod n .$$

Let $\mathcal{H}_n^d := \{h_{\bar{a}} \mid \bar{a} \in \{0, \dots, q - 1\}^{d+1}\}$. The class \mathcal{H}_n^d is $(e, d + 1)$ -independent.

Note that in the previous case we had $d = 1$ and chose $a_d \neq 0$.

Universal Hashing

For the coefficients $\bar{a} \in \{0, \dots, q-1\}^{d+1}$ let $f_{\bar{a}}$ denote the polynomial

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Let $A^\ell = \{h_{\bar{a}} \in \mathcal{H} \mid h_{\bar{a}}(x_i) = t_i \text{ for all } i \in \{1, \dots, \ell\}\}$

Then

$$h_{\bar{a}} \in A^\ell \Leftrightarrow h_{\bar{a}} = f_{\bar{a}} \bmod n \text{ and}$$

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In order to obtain the cardinality of A^ℓ we choose our polynomial by fixing $d + 1$ points.

We first fix the values for inputs x_1, \dots, x_ℓ .

We have

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Fix $\ell \leq d + 1$; let $x_1, \dots, x_\ell \in \{0, \dots, q - 1\}$ be keys, and let t_1, \dots, t_ℓ denote the corresponding hash-function values.

Let $A^\ell = \{h_{\bar{a}} \in \mathcal{H} \mid h_{\bar{a}}(x_i) = t_i \text{ for all } i \in \{1, \dots, \ell\}\}$

Then

$$h_{\bar{a}} \in A^\ell \Leftrightarrow h_{\bar{a}} = f_{\bar{a}} \bmod n \text{ and}$$

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In order to obtain the cardinality of A^ℓ we choose our polynomial by fixing $d + 1$ points.

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Now, we choose $d - \ell + 1$ other inputs and choose their value arbitrarily. We have $q^{d-\ell+1}$ possibilities to do this.

Therefore we have

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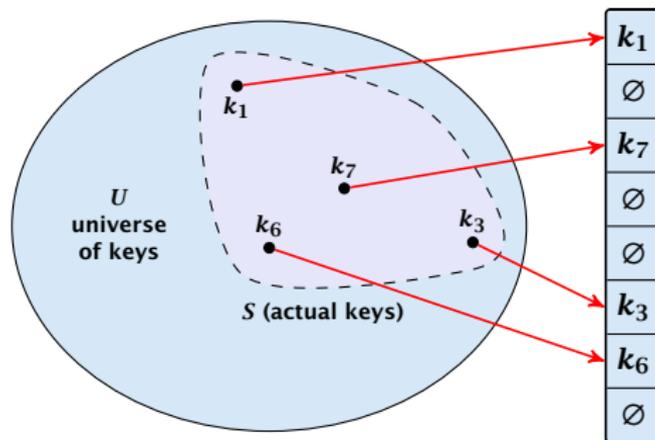
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This shows that the \mathcal{H} is $(e, d+1)$ -universal.

The last step followed from $q \geq (d+1)n$, and $\ell \leq d+1$.

Perfect Hashing

Suppose that we **know** the set S of actual keys (no insert/no delete). Then we may want to design a **simple** hash-function that maps all these keys to different memory locations.



Perfect Hashing

Let $m = |S|$. We could simply choose the hash-table size very large so that we don't get any collisions.

Using a universal hash-function the expected number of collisions is

$$E[\#\text{Collisions}] = \binom{m}{2} \cdot \frac{1}{n}.$$

If we choose $n = m^2$ the expected number of collisions is strictly less than $\frac{1}{2}$.

Can we get an upper bound on the probability of having collisions?

The probability of having 1 or more collisions can be at most $\frac{1}{2}$ as otherwise the expectation would be larger than $\frac{1}{2}$.

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We can find such a hash-function by a few trials.

However, a hash-table size of $n = m^2$ is very very high.

We construct a two-level scheme. We first use a hash-function that maps elements from S to m buckets.

Let m_j denote the number of items that are hashed to the j -th bucket. For each bucket we choose a second hash-function that maps the elements of the bucket into a table of size m_j^2 . The second function can be chosen such that all elements are mapped to different locations.

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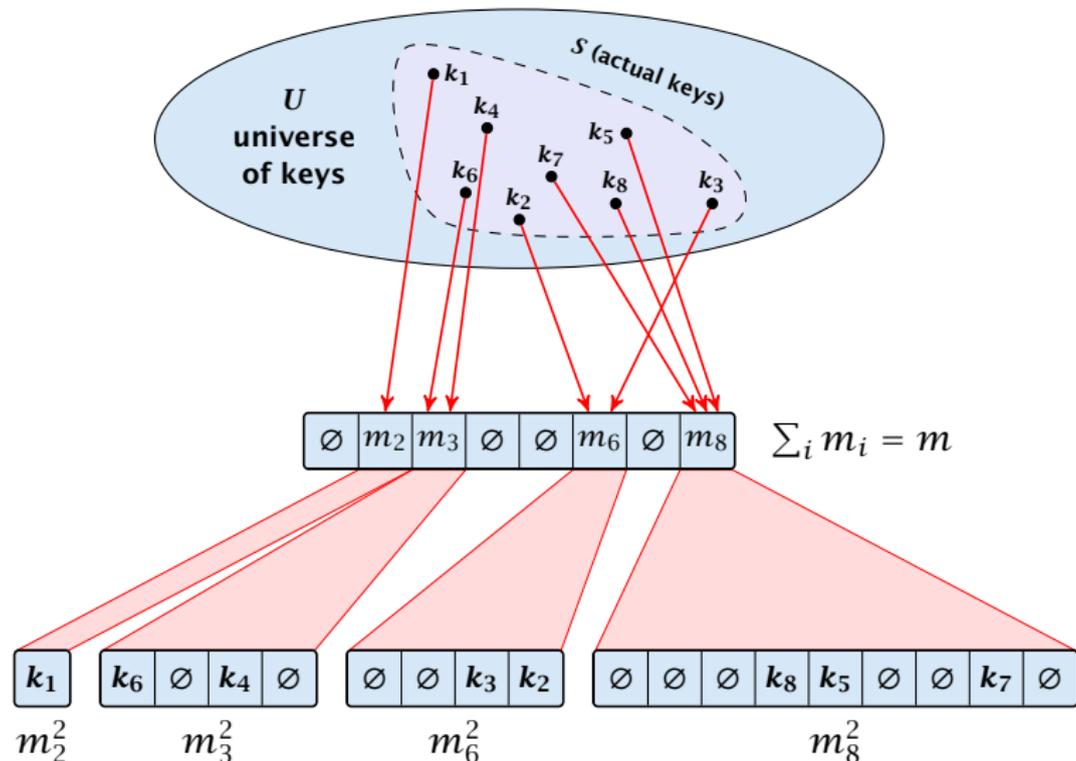
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$$= 2 \binom{m}{2} \frac{1}{m} + m = 2m - 1 .$$

Perfect Hashing

We need only $\mathcal{O}(m)$ time to construct a hash-function h with $\sum_j m_j^2 = \mathcal{O}(4m)$, because with probability at least $1/2$ a random function from a universal family will have this property.

Then we construct a hash-table h_j for every bucket. This takes expected time $\mathcal{O}(m_j)$ for every bucket. A random function h_j is collision-free with probability at least $1/2$. We need $\mathcal{O}(m_j)$ to test this.

We only need that the hash-functions are chosen from a universal family!!!

Cuckoo Hashing

Goal:

Try to generate a hash-table with constant worst-case search time in a dynamic scenario.

Two hash-tables T_1 and T_2 of size m , with hash functions h_1 and h_2 .

An object x is either stored at location $T_1[h_1(x)]$ or $T_2[h_2(x)]$.

Search clearly takes constant time if the above conditions are met.

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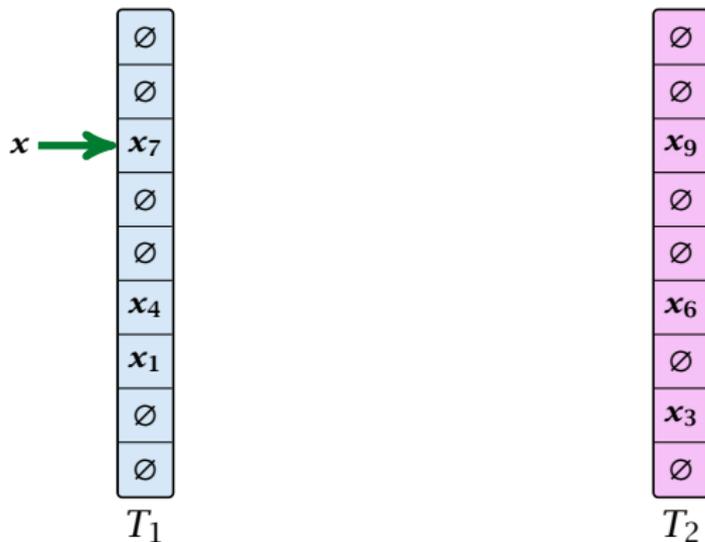
T_1



T_2

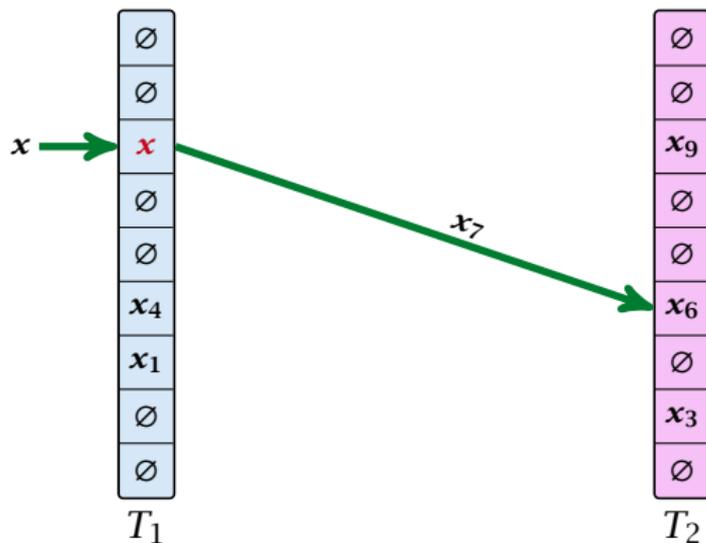
Cuckoo Hashing

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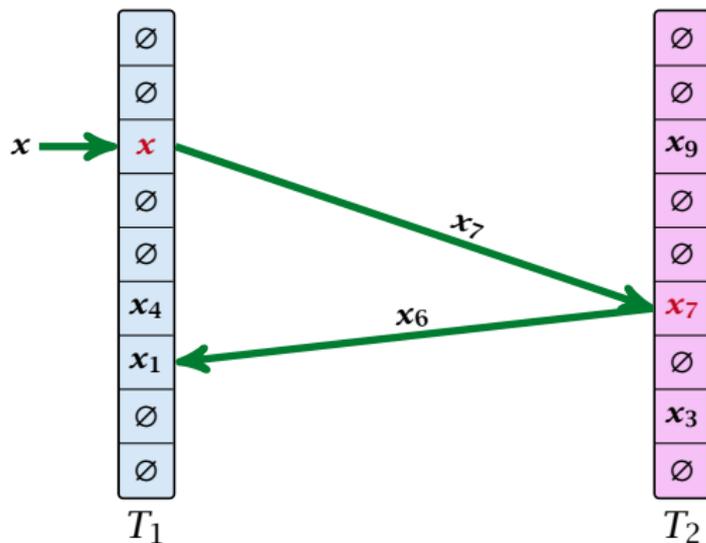
Cuckoo Hashing

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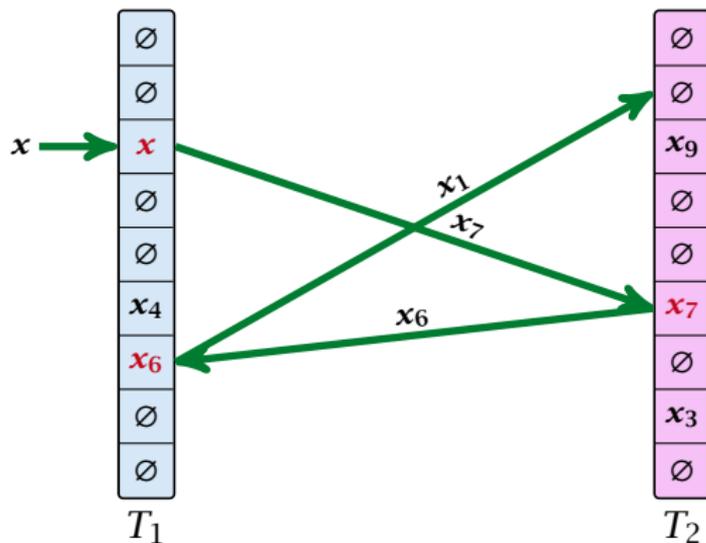
Cuckoo Hashing

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Cuckoo Hashing

Algorithm 17 Cuckoo-Insert(x)

```
1: if  $T_1[h_1(x)] = x \vee T_2[h_2(x)] = x$  then return  
2: steps  $\leftarrow 1$   
3: while steps  $\leq$  maxsteps do  
4:     exchange  $x$  and  $T_1[h_1(x)]$   
5:     if  $x = \text{null}$  then return  
6:     exchange  $x$  and  $T_2[h_2(x)]$   
7:     if  $x = \text{null}$  then return  
8:     steps  $\leftarrow$  steps + 1  
9: rehash() // change hash-functions; rehash everything  
10: Cuckoo-Insert( $x$ )
```

Cuckoo Hashing

- ▶ We call one iteration through the while-loop a **step** of the algorithm.
- ▶ We call a sequence of iterations through the while-loop without the termination condition becoming true a **phase** of the algorithm.
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What is the expected time for an insert-operation?

We first analyze the probability that we end-up in an infinite loop (that is then terminated after `maxsteps` steps).

Formally what is the probability to enter an infinite loop that touches s different keys?

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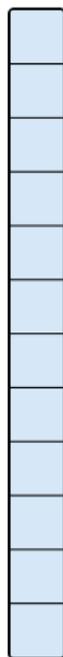
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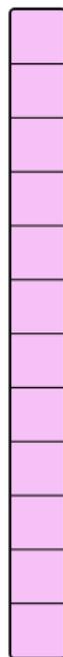
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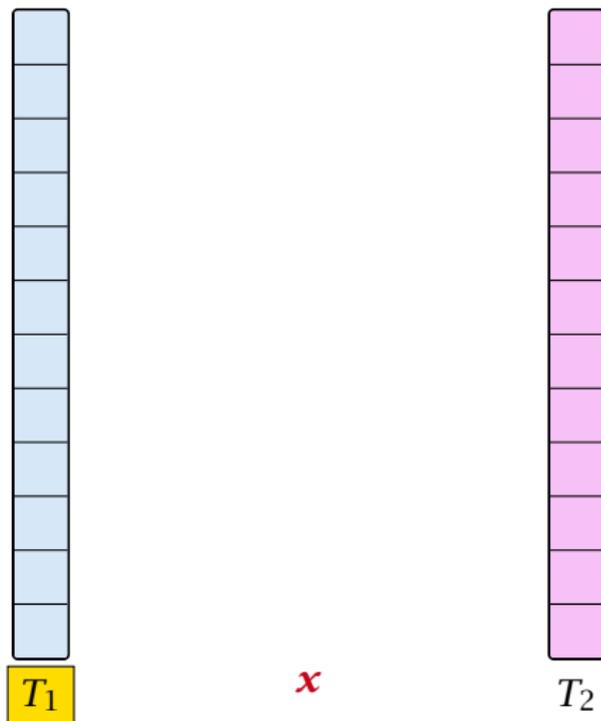


T_1

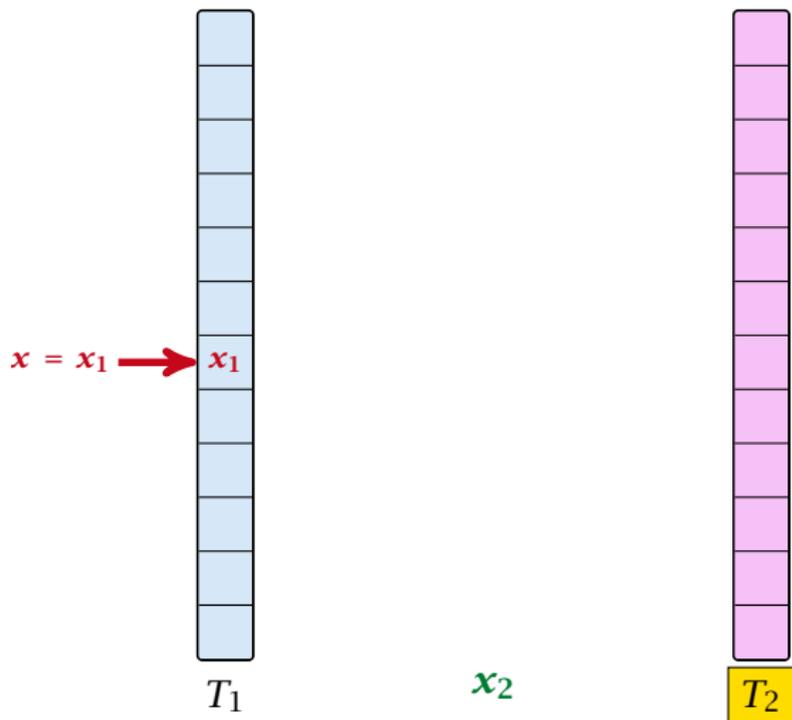


T_2

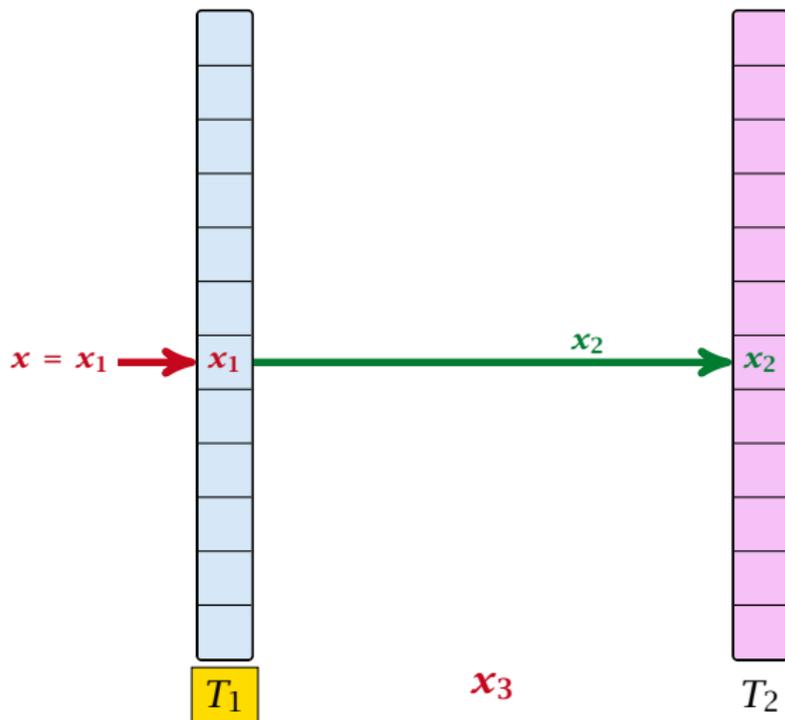
Cuckoo Hashing: Insert



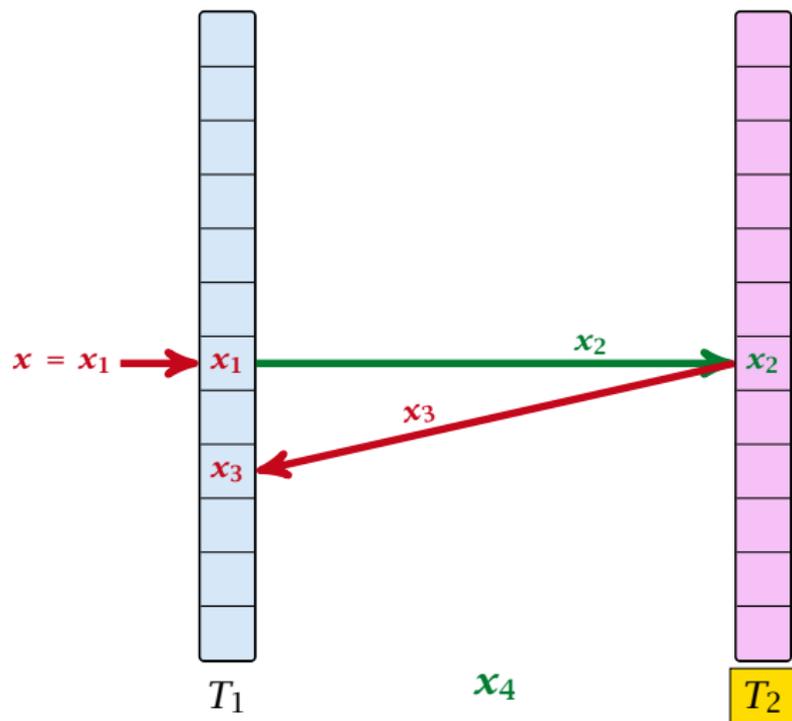
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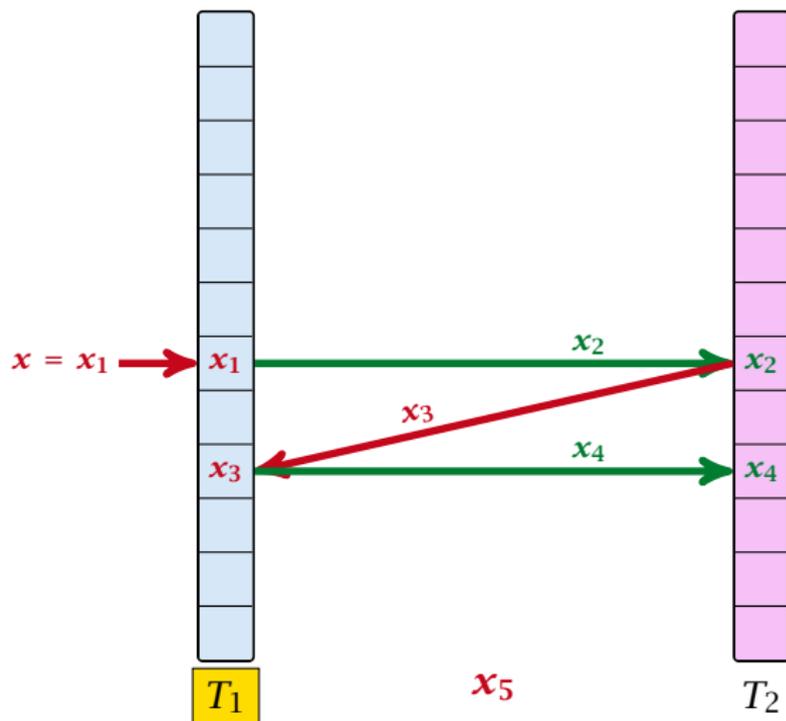
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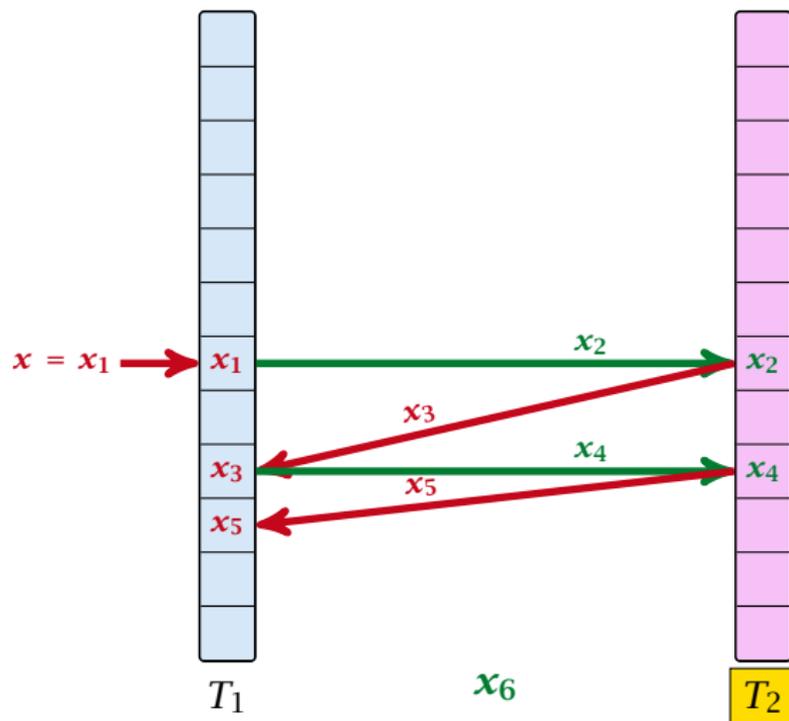
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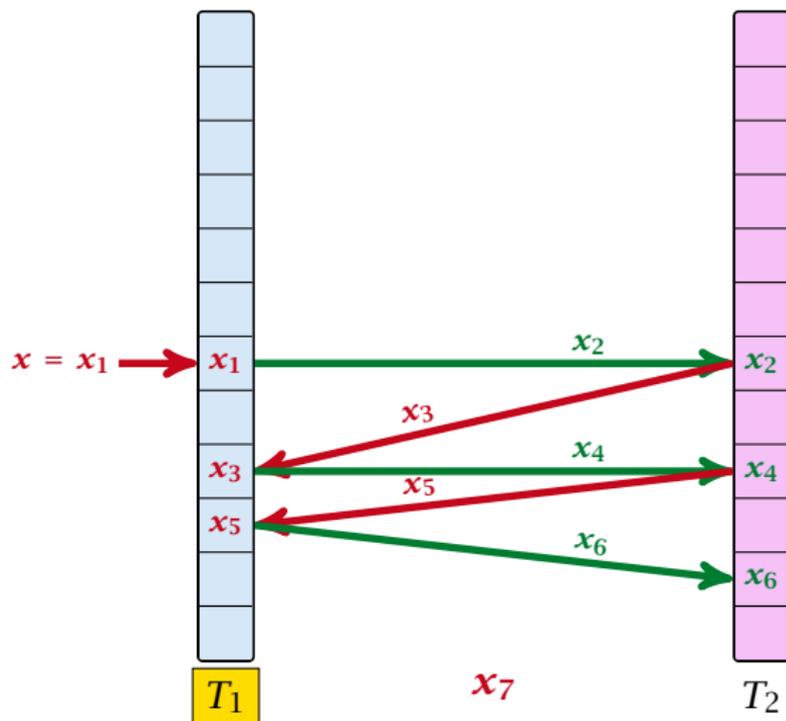
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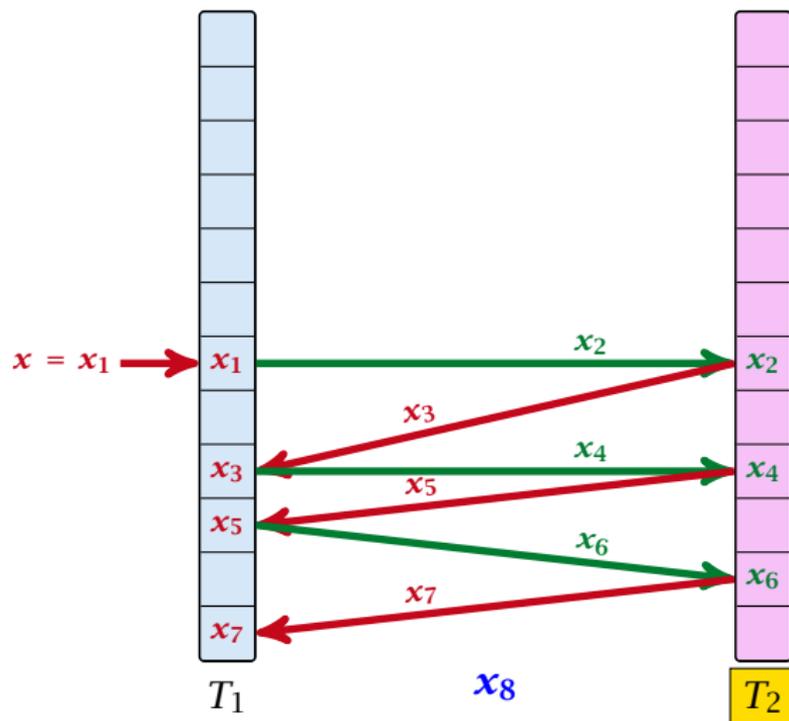
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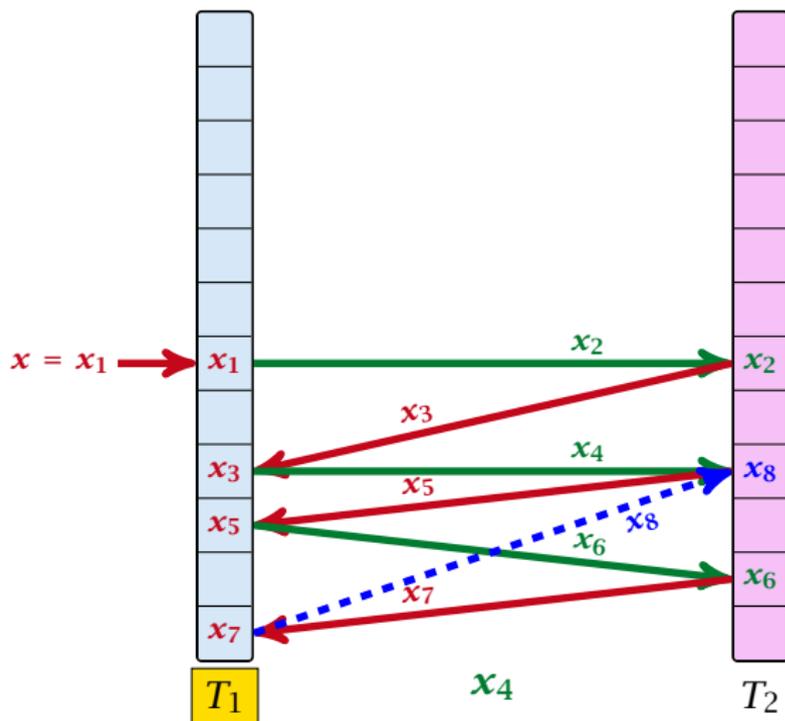
Cuckoo Hashing: Insert



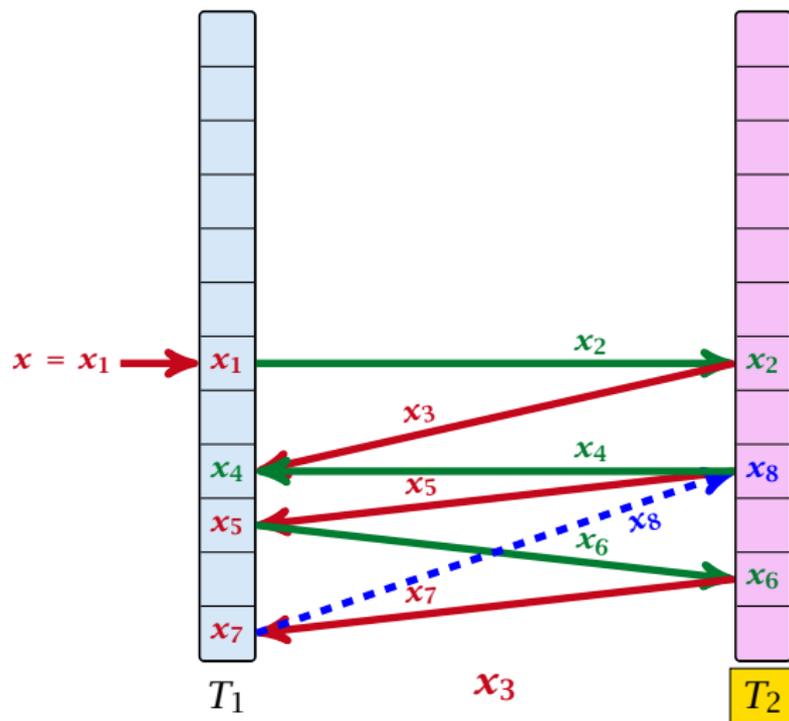
Cuckoo Hashing: Insert



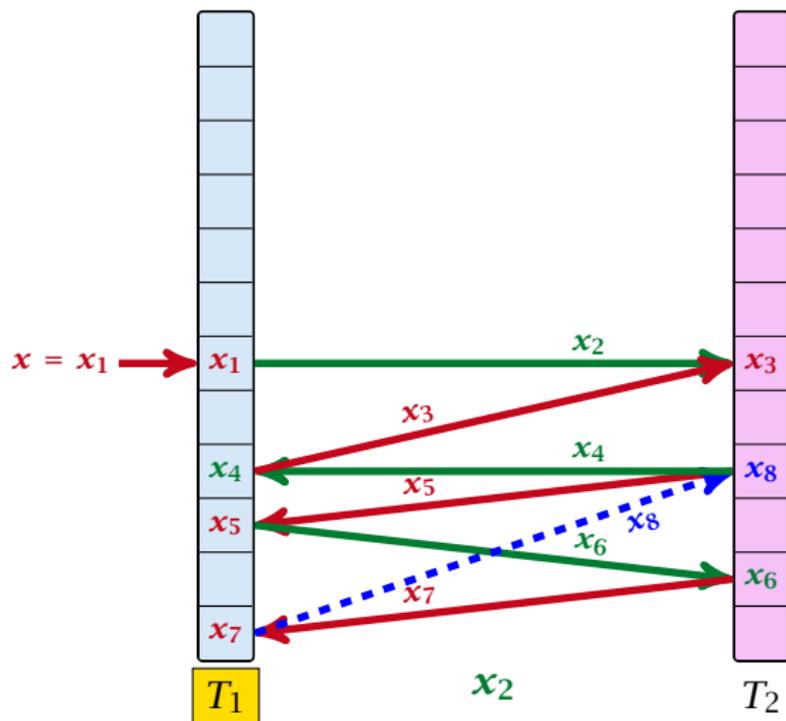
Cuckoo Hashing: Insert



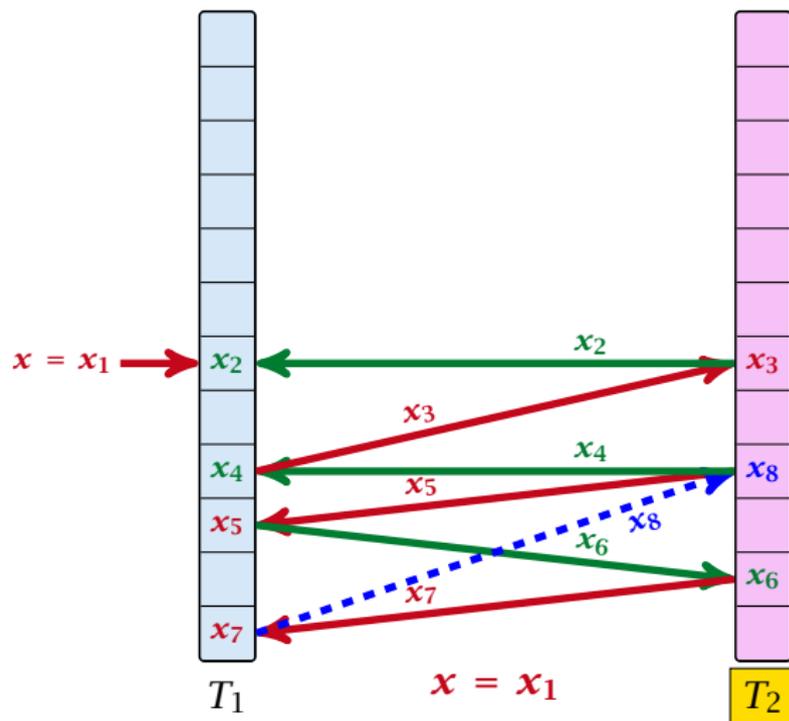
Cuckoo Hashing: Insert



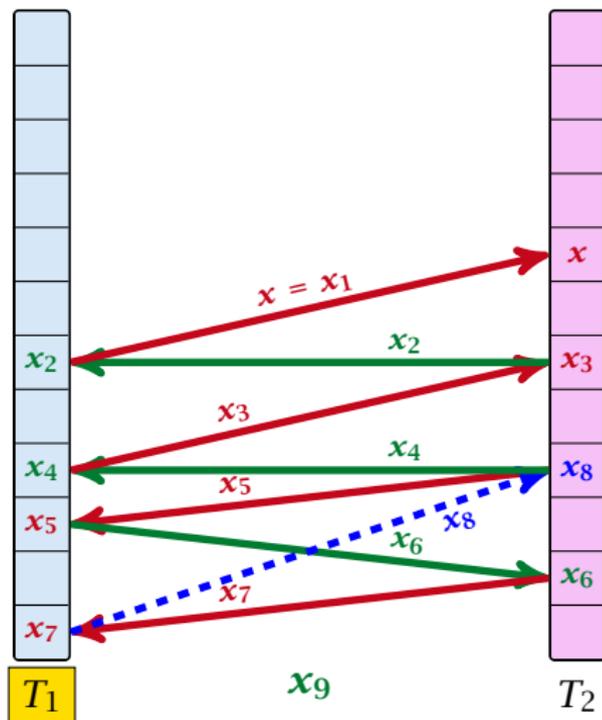
Cuckoo Hashing: Insert



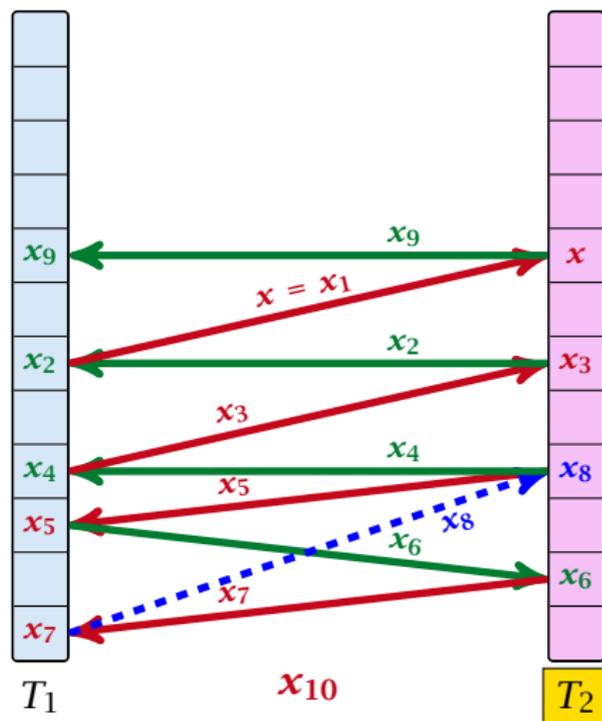
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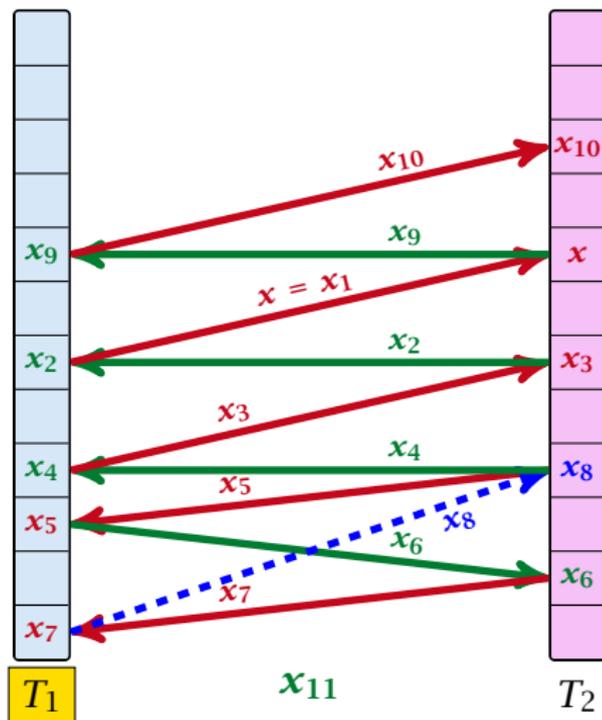
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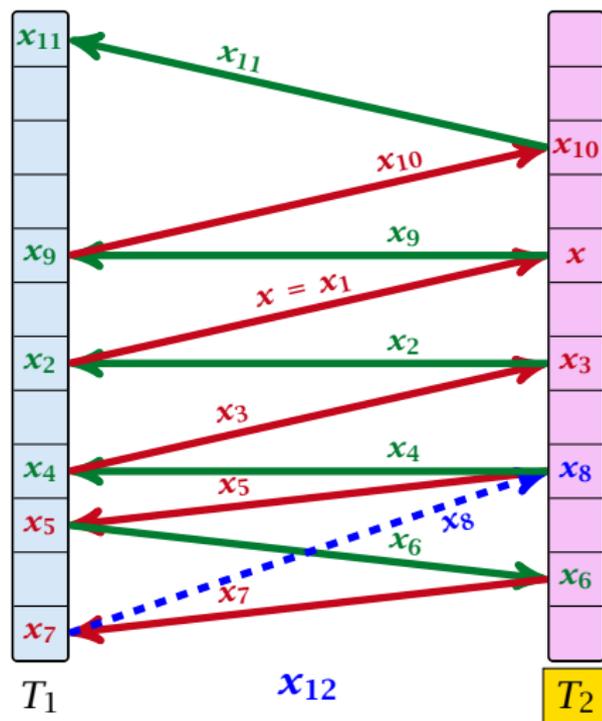
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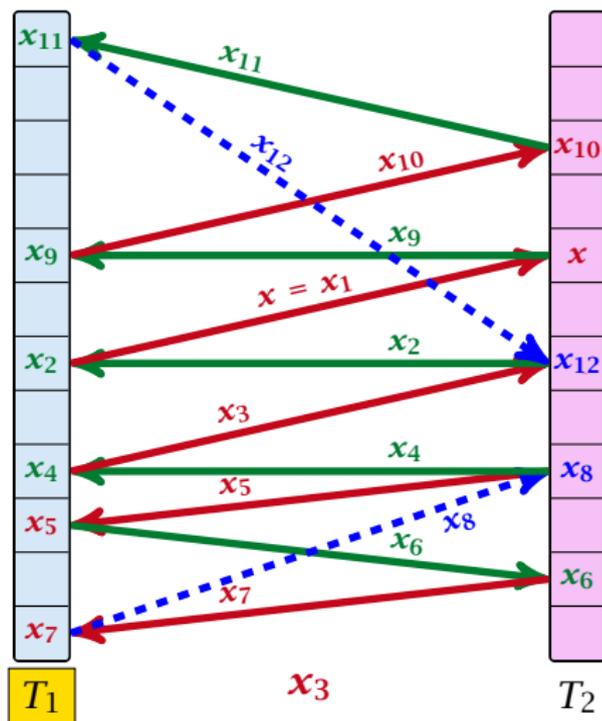
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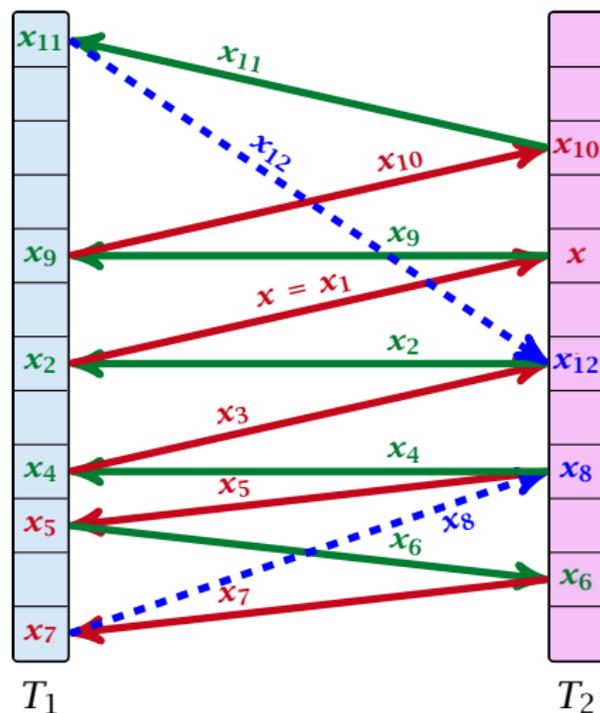
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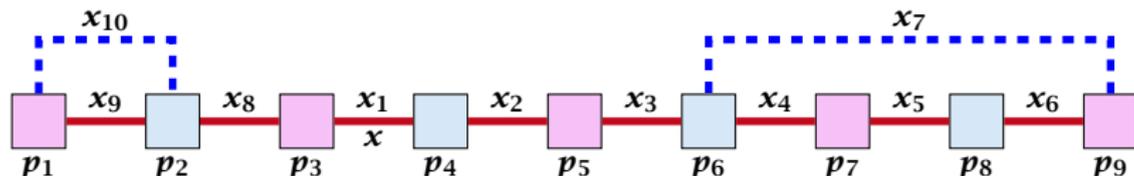
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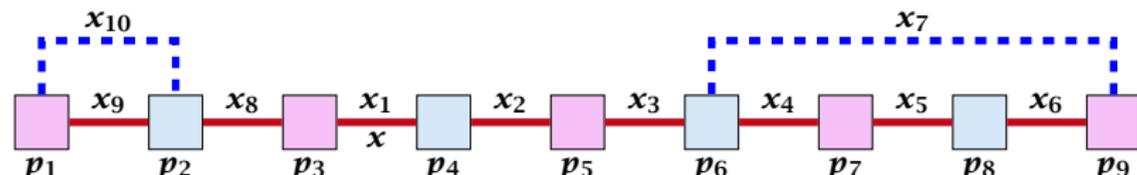


Cuckoo Hashing



A cycle-structure of size s is defined by

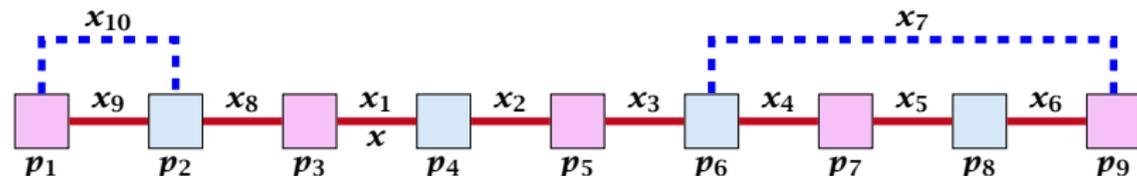
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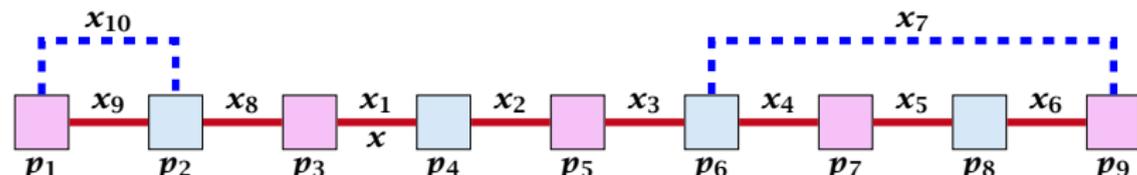
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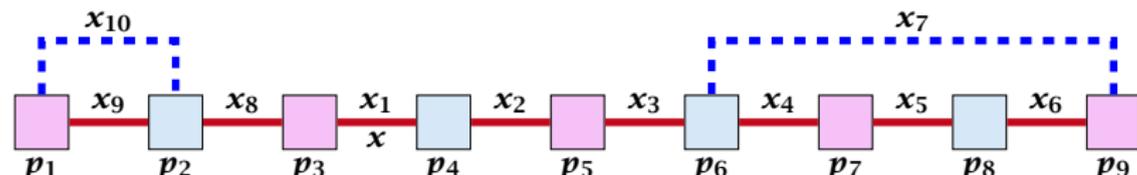
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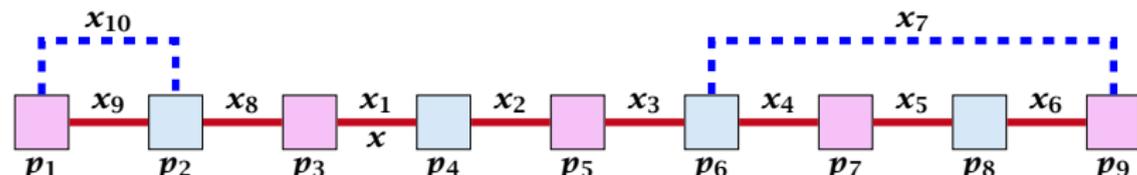
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Cuckoo Hashing

A cycle-structure is **active** if for every key x_ℓ (linking a cell p_i from T_1 and a cell p_j from T_2) we have

$$h_1(x_\ell) = p_i \quad \text{and} \quad h_2(x_\ell) = p_j$$

Observation:

If during a phase the insert-procedure runs into a cycle there must exist an active cycle structure of size $s \geq 3$.

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Observation:

If during a phase the insert-procedure runs into a cycle there must exist an active cycle structure of size $s \geq 3$.

Cuckoo Hashing

What is the probability that all keys in a cycle-structure of size s correctly map into their T_1 -cell?

This probability is at most $\frac{\mu}{n^s}$ since h_1 is a (μ, s) -independent hash-function.

What is the probability that all keys in the cycle-structure of size s correctly map into their T_2 -cell?

This probability is at most $\frac{\mu}{n^s}$ since h_2 is a (μ, s) -independent hash-function.

These events are independent.

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The probability that a given cycle-structure of size s is active is at most $\frac{\mu^2}{n^{2s}}$.

What is the probability that there exists an active cycle structure of size s ?

Cuckoo Hashing

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What is the probability that **there exists** an active cycle structure of size s ?

Cuckoo Hashing

The number of cycle-structures of size s is at most

$$s^3 \cdot n^{s-1} \cdot m^{s-1} .$$

Theorem 11.11 (see [11]) When treated as a directed graph, cuckoo hashing has at most s^3 forward and backward links.

There are at most s possibilities to choose where to place

the s keys. There are at most s possibilities to choose the key that

is moved to the other bucket.

Cuckoo Hashing

The number of cycle-structures of size s is at most

$$s^3 \cdot n^{s-1} \cdot m^{s-1} .$$

- ▶ There are at most s^2 possibilities where to attach the forward and backward links.
- ▶ There are at most s possibilities to choose where to place key x .
- ▶ There are m^{s-1} possibilities to choose the keys apart from x .
- ▶ There are n^{s-1} possibilities to choose the cells.

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Cuckoo Hashing

The probability that there exists an active cycle-structure is therefore at most

$$\sum_{s=3}^{\infty} s^3 \cdot n^{s-1} \cdot m^{s-1} \cdot \frac{\mu^2}{n^{2s}}$$

Cuckoo Hashing

The probability that there exists an active cycle-structure is therefore at most

$$\sum_{s=3}^{\infty} s^3 \cdot n^{s-1} \cdot m^{s-1} \cdot \frac{\mu^2}{n^{2s}} = \frac{\mu^2}{nm} \sum_{s=3}^{\infty} s^3 \left(\frac{m}{n}\right)^s$$

Cuckoo Hashing

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$$\begin{aligned} \sum_{s=3}^{\infty} s^3 \cdot n^{s-1} \cdot m^{s-1} \cdot \frac{\mu^2}{n^{2s}} &= \frac{\mu^2}{nm} \sum_{s=3}^{\infty} s^3 \left(\frac{m}{n}\right)^s \\ &\leq \frac{\mu^2}{m^2} \sum_{s=3}^{\infty} s^3 \left(\frac{1}{1+\epsilon}\right)^s \end{aligned}$$

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Here we used the fact that $(1 + \epsilon)m \leq n$.

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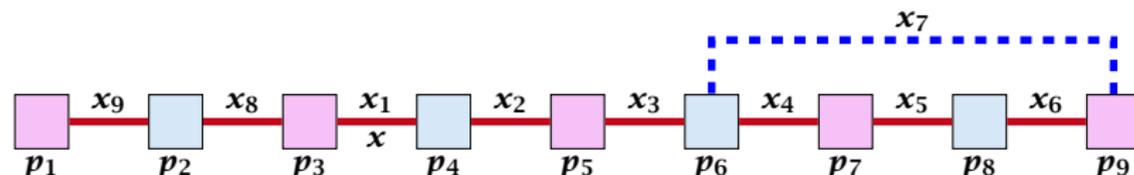
Hence,

$$\Pr[\text{cycle}] = \mathcal{O}\left(\frac{1}{m^2}\right).$$

Cuckoo Hashing

Now, we analyze the probability that a phase is not successful without running into a closed cycle.

Cuckoo Hashing



Sequence of visited keys:

$x = x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_3, x_2, x_1 = x, x_8, x_9, \dots$

Cuckoo Hashing

Consider the sequence of not necessarily distinct keys starting with x in the order that they are visited during the phase.

Lemma 22

If the sequence is of length p then there exists a sub-sequence of at least $\frac{p+2}{3}$ keys starting with x of distinct keys.

Cuckoo Hashing

Consider the sequence of not necessarily distinct keys starting with x in the order that they are visited during the phase.

Lemma 22

*If the sequence is of length p then there exists a sub-sequence of at least $\frac{p+2}{3}$ keys starting with x of *distinct* keys.*

Cuckoo Hashing

Proof.

Let i be the number of keys (including x) that we see before the first repeated key. Let j denote the total number of distinct keys.

The sequence is of the form:

$$x = x_1 \rightarrow x_2 \rightarrow \dots \rightarrow x_i \rightarrow x_r \rightarrow x_{r-1} \rightarrow \dots \rightarrow x_1 \rightarrow x_{i+1} \rightarrow \dots \rightarrow x_j$$

As $r \leq i - 1$ the length p of the sequence is

$$p = i + r + (j - i) \leq i + j - 1 .$$

Either sub-sequence $x_1 \rightarrow x_2 \rightarrow \dots \rightarrow x_i$ or sub-sequence $x_1 \rightarrow x_{i+1} \rightarrow \dots \rightarrow x_j$ has at least $\frac{p+2}{3}$ elements. □

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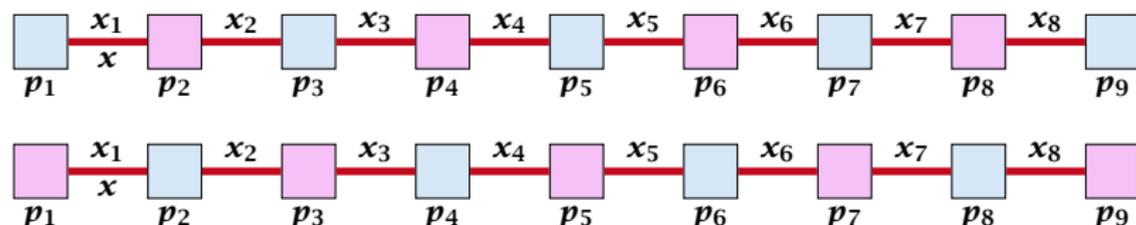
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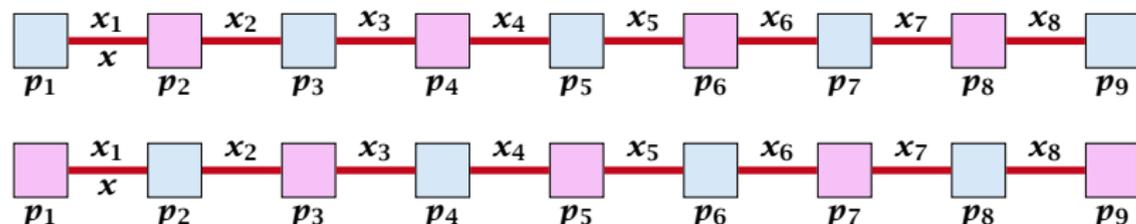
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Cuckoo Hashing



A path-structure of size s is defined by

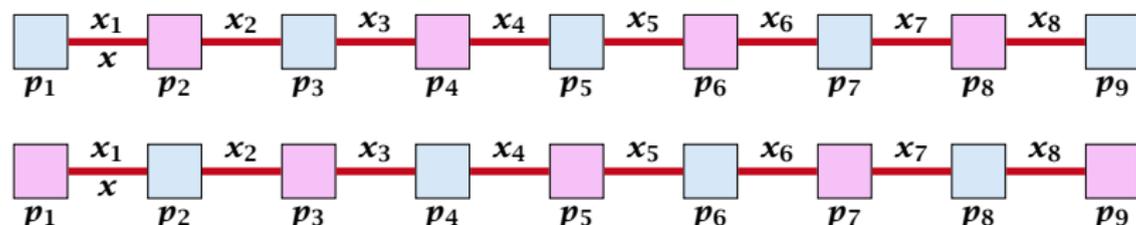
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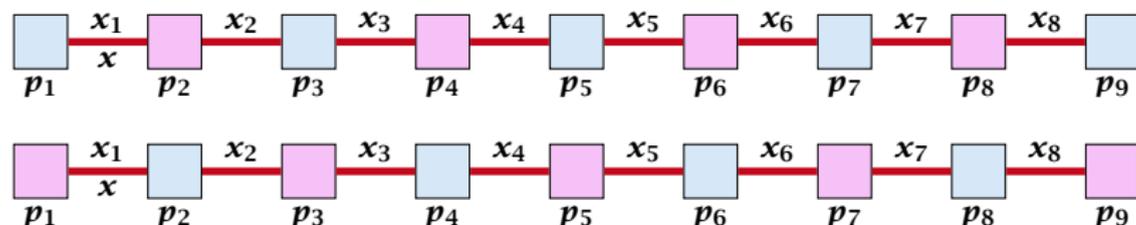
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$$h_1(x_\ell) = p_i \quad \text{and} \quad h_2(x_\ell) = p_j$$

Observation:

If a phase takes at least t steps without running into a cycle there must exist an active path-structure of size $(2t + 2)/3$.

Cuckoo Hashing

The probability that a given path-structure of size s is active is at most $\frac{\mu^2}{n^{2s}}$.

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We choose $\text{maxsteps} \geq 3\ell/2 + 2$. Then the probability that a phase terminates unsuccessfully without running into a cycle is at most

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by choosing $\ell \geq \log \left(\frac{1}{2\mu^2 m^2} \right) / \log \left(\frac{1}{1+\epsilon} \right) = \log (2\mu^2 m^2) / \log (1 + \epsilon)$

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$$\begin{aligned} & \Pr[\text{unsuccessful} \mid \text{no cycle}] \\ & \leq \Pr[\exists \text{ active path-structure of size at least } \frac{2\text{maxsteps}-1}{3}] \\ & \leq \Pr[\exists \text{ active path-structure of size at least } \ell + 1] \\ & \leq \Pr[\exists \text{ active path-structure of size exactly } \ell + 1] \\ & \leq 2\mu^2 \left(\frac{1}{1+\epsilon}\right)^\ell \leq \frac{1}{m^2} \end{aligned}$$

by choosing $\ell \geq \log\left(\frac{1}{2\mu^2 m^2}\right) / \log\left(\frac{1}{1+\epsilon}\right) = \log(2\mu^2 m^2) / \log(1+\epsilon)$

This gives $\text{maxsteps} = \Theta(\log m)$.

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$$\Pr[\text{cycle}] \leq \mathcal{O}\left(\frac{1}{m^2}\right)$$

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for a suitable constant $c > 0$.

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This means the expected cost for a successful phase is constant (even after accounting for the cost of the incomplete step that finishes the phase).

Cuckoo Hashing

A phase that is not successful induces cost for doing a complete rehash (this dominates the cost for the steps in the phase).

The probability that a phase is not successful is $p = \mathcal{O}(1/m^2)$ (probability $\mathcal{O}(1/m^2)$ of running into a cycle and probability $\mathcal{O}(1/m^2)$ of reaching maxsteps without running into a cycle).

A rehash try requires m insertions and takes expected constant time per insertion. It fails with probability $p := \mathcal{O}(1/m)$.

The expected number of unsuccessful rehashes is

$$\sum_{i \geq 1} p^i = \frac{1}{1-p} - 1 = \frac{p}{1-p} = \mathcal{O}(p).$$

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The expected cost for all rehashes is

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What kind of hash-functions do we need?

Since maxsteps is $\Theta(\log m)$ the largest size of a path-structure or cycle-structure contains just $\Theta(\log m)$ different keys.

Therefore, it is sufficient to have $(\mu, \Theta(\log m))$ -independent hash-functions.

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How do we make sure that $n \geq (1 + \epsilon)m$?

- ▶ Let $\alpha := 1/(1 + \epsilon)$.
- ▶ Keep track of the number of elements in the table. When $m \geq \alpha n$ we double n and do a complete re-hash (table-expand).
- ▶ Whenever m drops below $\alpha n/4$ we divide n by 2 and do a rehash (table-shrink).
- ▶ Note that right after a change in table-size we have $m = \alpha n/2$. In order for a table-expand to occur at least $\alpha n/2$ insertions are required. Similar, for a table-shrink at least $\alpha n/4$ deletions must occur.
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Lemma 23

Cuckoo Hashing has an expected constant insert-time and a worst-case constant search-time.

Note that the above lemma only holds if the fill-factor (number of keys/total number of hash-table slots) is at most $\frac{1}{2(1+\epsilon)}$.

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