

## 7 Dictionary

### Dictionary:

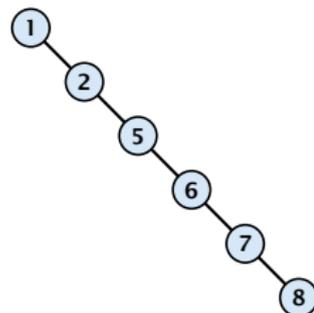
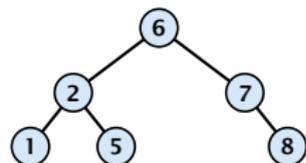
- ▶  **$S$ . insert( $x$ )**: Insert an element  $x$ .
- ▶  **$S$ . delete( $x$ )**: Delete the element pointed to by  $x$ .
- ▶  **$S$ . search( $k$ )**: Return a pointer to an element  $e$  with  $\text{key}[e] = k$  in  $S$  if it exists; otherwise return **null**.

## 7.1 Binary Search Trees

An (**internal**) **binary search tree** stores the elements in a binary tree. Each tree-node corresponds to an element. All elements in the left sub-tree of a node  $v$  have a smaller key-value than  $\text{key}[v]$  and elements in the right sub-tree have a larger-key value. We assume that all key-values are different.

(**External** Search Trees store objects only at leaf-vertices)

Examples:



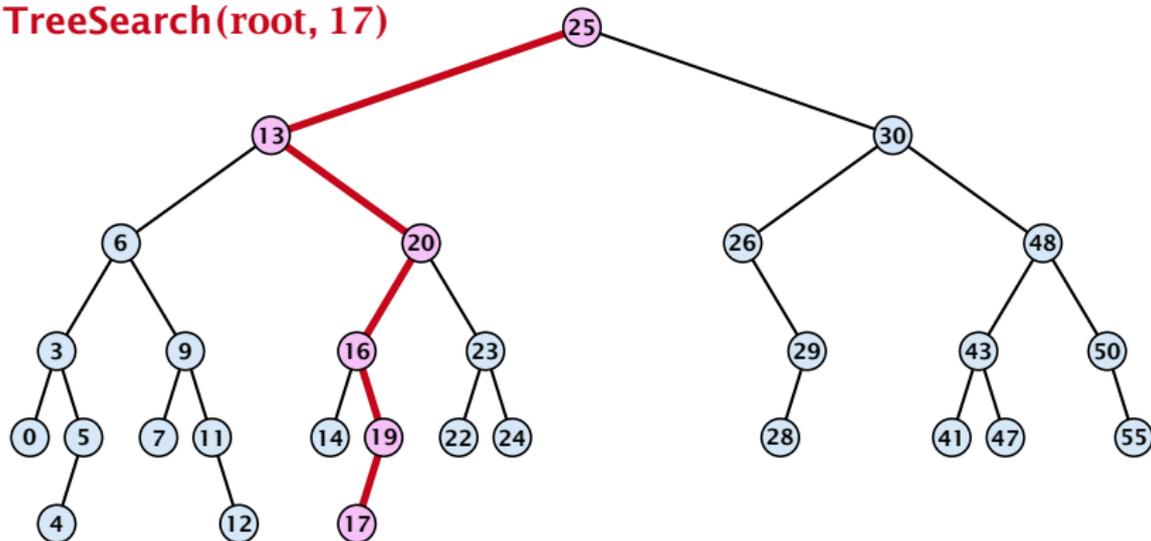
## 7.1 Binary Search Trees

We consider the following operations on binary search trees. Note that this is a super-set of the dictionary-operations.

- ▶  $T.\text{insert}(x)$
- ▶  $T.\text{delete}(x)$
- ▶  $T.\text{search}(k)$
- ▶  $T.\text{successor}(x)$
- ▶  $T.\text{predecessor}(x)$
- ▶  $T.\text{minimum}()$
- ▶  $T.\text{maximum}()$

# Binary Search Trees: Searching

TreeSearch(root, 17)

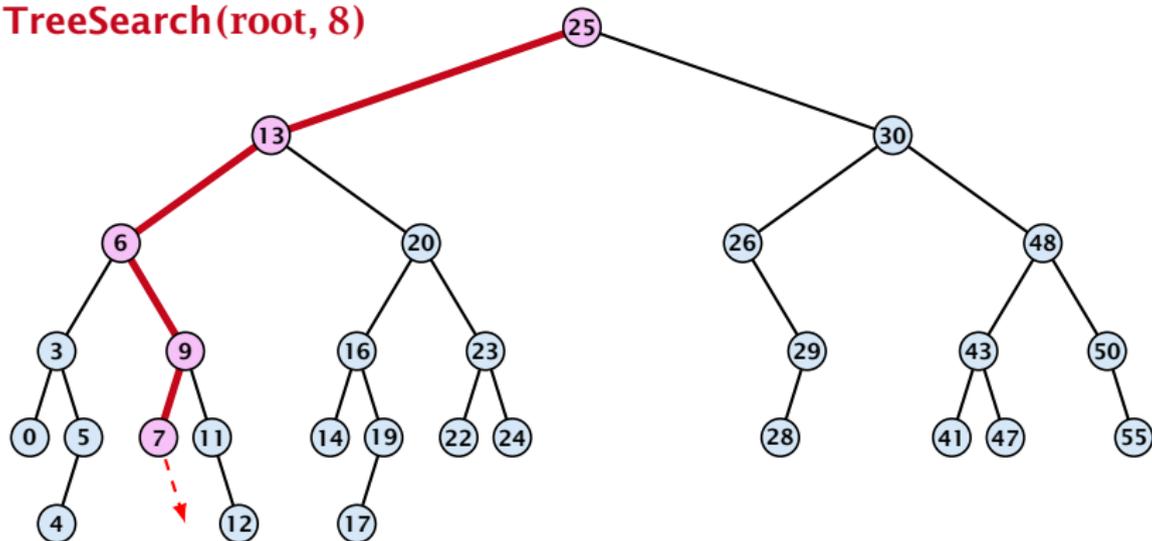


**Algorithm 5** TreeSearch( $x, k$ )

- 1: **if**  $x = \text{null}$  **or**  $k = \text{key}[x]$  **return**  $x$
- 2: **if**  $k < \text{key}[x]$  **return** TreeSearch(left[ $x$ ],  $k$ )
- 3: **else return** TreeSearch(right[ $x$ ],  $k$ )

# Binary Search Trees: Searching

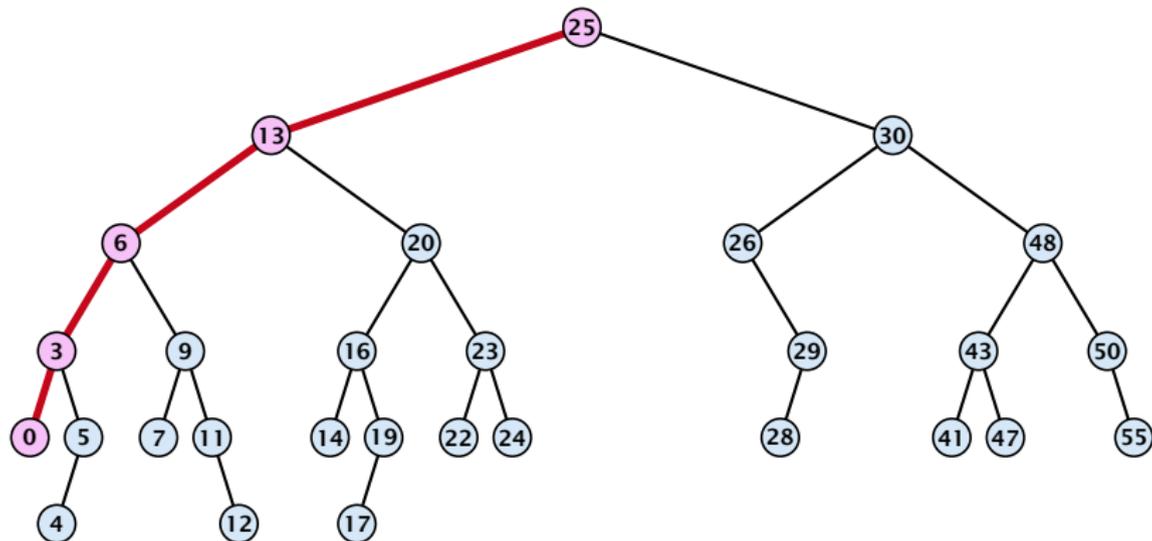
TreeSearch(root, 8)



**Algorithm 5** TreeSearch( $x, k$ )

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- 3: **else return** TreeSearch(right[ $x$ ],  $k$ )

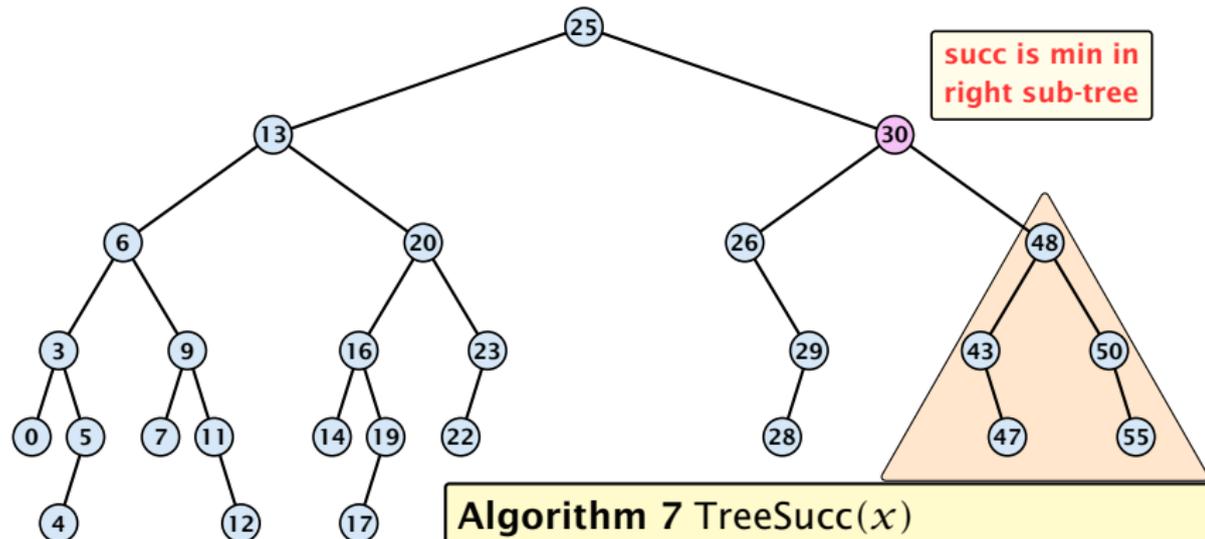
# Binary Search Trees: Minimum



## Algorithm 6 TreeMin( $x$ )

- 1: **if**  $x = \text{null}$  **or**  $\text{left}[x] = \text{null}$  **return**  $x$
- 2: **return** TreeMin(left[ $x$ ])

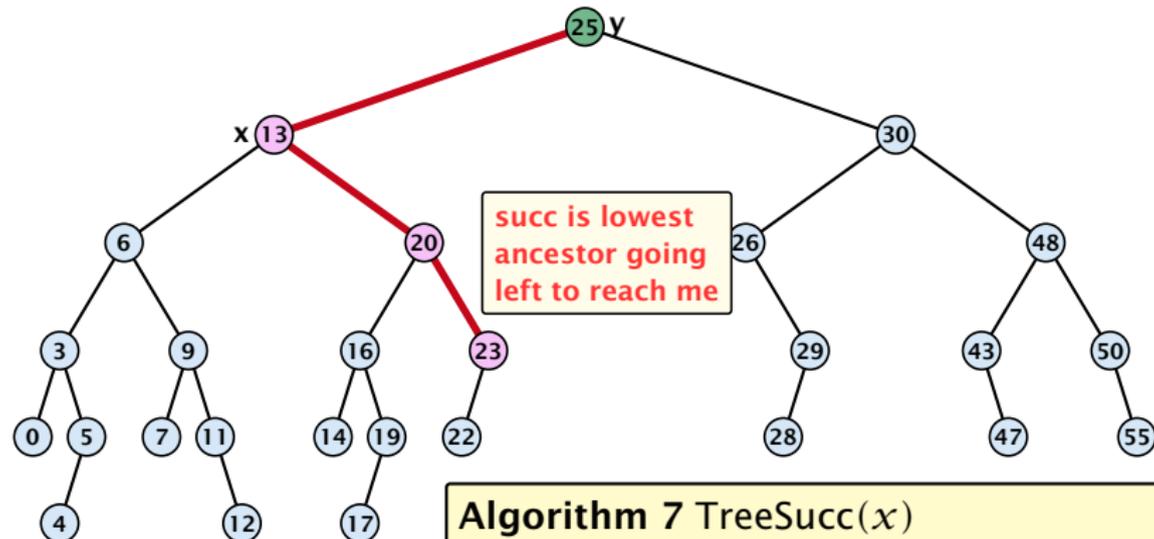
# Binary Search Trees: Successor



## Algorithm 7 TreeSucc( $x$ )

- 1: **if** right[ $x$ ]  $\neq$  null **return** TreeMin(right[ $x$ ])
- 2:  $y \leftarrow$  parent[ $x$ ]
- 3: **while**  $y \neq$  null **and**  $x =$  right[ $y$ ] **do**
- 4:      $x \leftarrow y$ ;  $y \leftarrow$  parent[ $x$ ]
- 5: **return**  $y$ ;

# Binary Search Trees: Successor



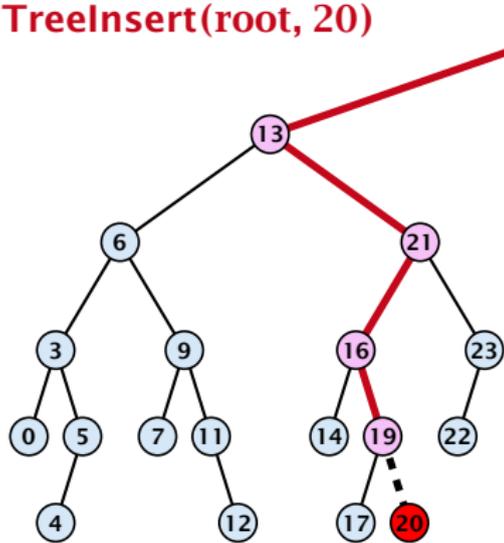
## Algorithm 7 TreeSucc( $x$ )

- 1: **if** right[ $x$ ]  $\neq$  null **return** TreeMin(right[ $x$ ])
- 2:  $y \leftarrow$  parent[ $x$ ]
- 3: **while**  $y \neq$  null **and**  $x =$  right[ $y$ ] **do**
- 4:      $x \leftarrow y$ ;  $y \leftarrow$  parent[ $x$ ]
- 5: **return**  $y$ ;

# Binary Search Trees: Insert

Insert element **not** in the tree.

**TreeInsert**(root, 20)

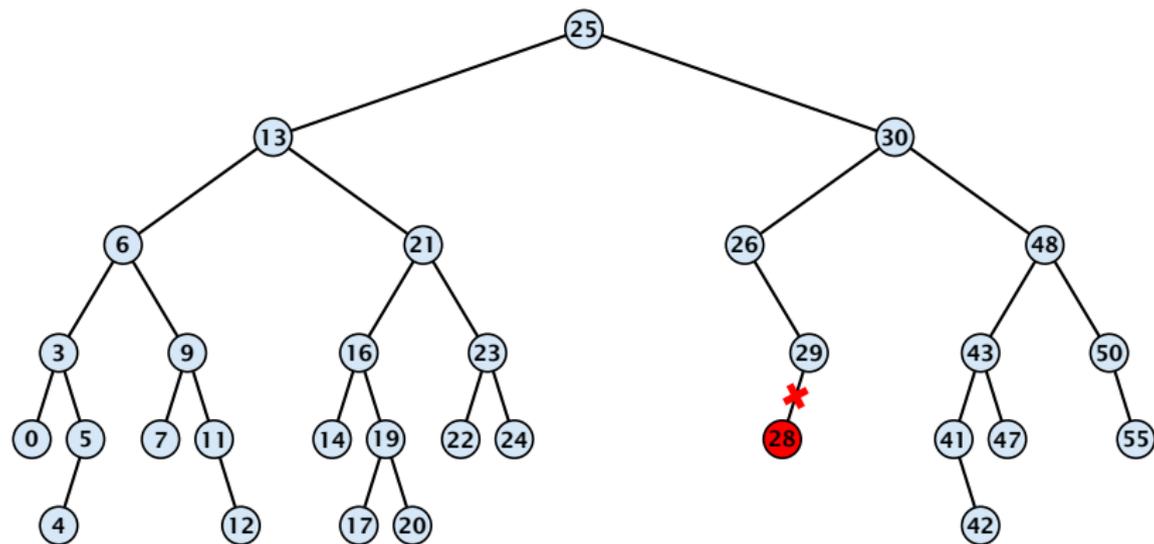


Search for  $z$ . At some point the search stops at a null-pointer. This is the place to insert  $z$ .

## Algorithm 8 TreeInsert( $x, z$ )

- 1: **if**  $x = \text{null}$  **then**
- 2:      $\text{root}[T] \leftarrow z$ ;  $\text{parent}[z] \leftarrow \text{null}$ ;
- 3:     **return**;
- 4: **if**  $\text{key}[x] > \text{key}[z]$  **then**
- 5:     **if**  $\text{left}[x] = \text{null}$  **then**
- 6:          $\text{left}[x] \leftarrow z$ ;  $\text{parent}[z] \leftarrow x$ ;
- 7:     **else** TreeInsert( $\text{left}[x], z$ );
- 8: **else**
- 9:     **if**  $\text{right}[x] = \text{null}$  **then**
- 10:          $\text{right}[x] \leftarrow z$ ;  $\text{parent}[z] \leftarrow x$ ;
- 11:     **else** TreeInsert( $\text{right}[x], z$ );

# Binary Search Trees: Delete

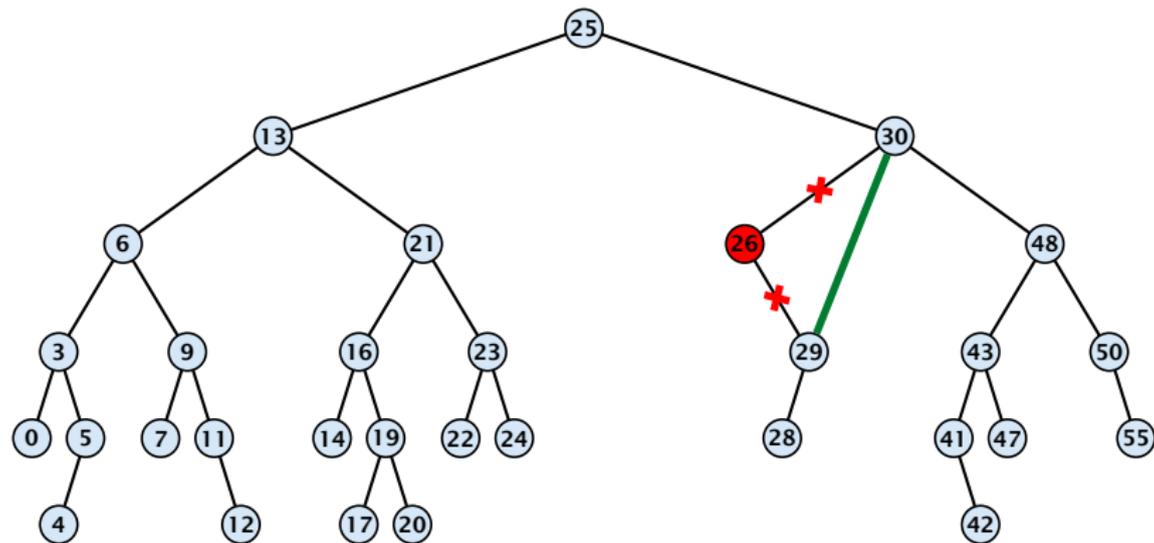


## Case 1:

Element does not have any children

- ▶ Simply go to the parent and set the corresponding pointer to **null**.

# Binary Search Trees: Delete

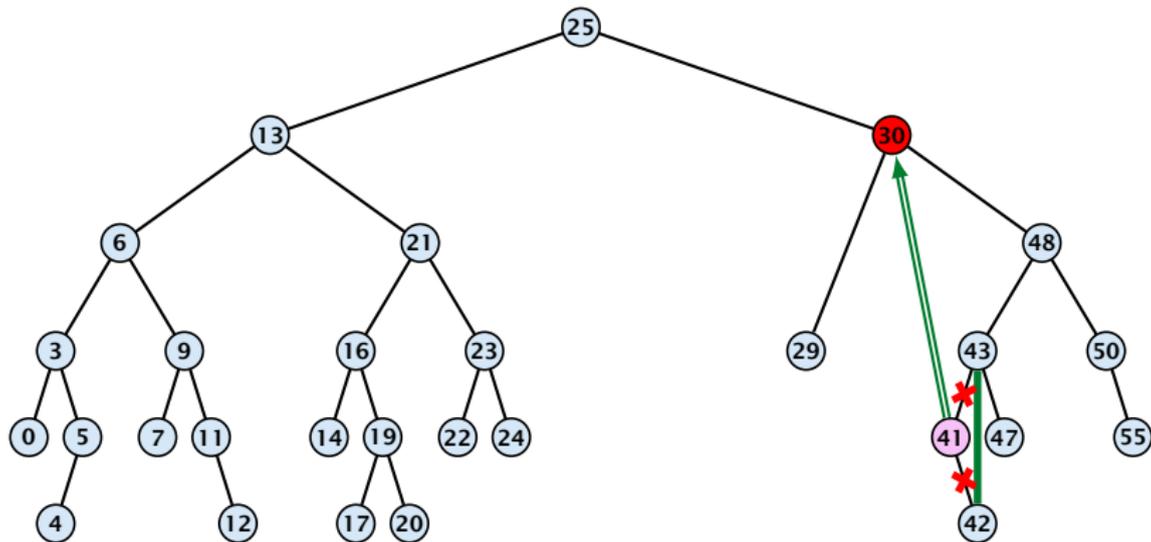


## Case 2:

Element has exactly one child

- ▶ Splice the element out of the tree by connecting its parent to its successor.

# Binary Search Trees: Delete



## Case 3:

Element has two children

- ▶ Find the successor of the element
- ▶ Splice successor out of the tree
- ▶ Replace content of element by content of successor

# Binary Search Trees: Delete

## Algorithm 9 TreeDelete( $z$ )

```
1: if left[ $z$ ] = null or right[ $z$ ] = null
2:   then  $y \leftarrow z$  else  $y \leftarrow \text{TreeSucc}(z)$ ;   select  $y$  to splice out
3: if left[ $y$ ]  $\neq$  null
4:   then  $x \leftarrow \text{left}[y]$  else  $x \leftarrow \text{right}[y]$ ;  $x$  is child of  $y$  (or null)
5: if  $x \neq \text{null}$  then parent[ $x$ ]  $\leftarrow$  parent[ $y$ ];   parent[ $x$ ] is correct
6: if parent[ $y$ ] = null then
7:   root[ $T$ ]  $\leftarrow x$ 
8: else
9:   if  $y = \text{left}[\text{parent}[y]]$  then
10:    left[parent[ $y$ ]]  $\leftarrow x$ 
11:   else
12:    right[parent[ $y$ ]]  $\leftarrow x$ 
13: if  $y \neq z$  then copy  $y$ -data to  $z$ 
```

} fix pointer to  $x$

# Balanced Binary Search Trees

All operations on a binary search tree can be performed in time  $\mathcal{O}(h)$ , where  $h$  denotes the height of the tree.

However the height of the tree may become as large as  $\Theta(n)$ .

## Balanced Binary Search Trees

With each insert- and delete-operation perform **local** adjustments to guarantee a height of  $\mathcal{O}(\log n)$ .

AVL-trees, Red-black trees, Scapegoat trees, 2-3 trees, B-trees, AA trees, Treaps

similar: SPLAY trees.

# Binary Search Trees (BSTs)

## Bibliography

- [MS08] Kurt Mehlhorn, Peter Sanders:  
*Algorithms and Data Structures — The Basic Toolbox*,  
Springer, 2008
- [CLRS90] Thomas H. Cormen, Charles E. Leiserson, Ron L. Rivest, Clifford Stein:  
*Introduction to Algorithms (3rd ed.)*,  
MIT Press and McGraw-Hill, 2009

Binary search trees can be found in every standard text book. For example Chapter 7.1 in [MS08] and Chapter 12 in [CLRS90].

## 7.2 Red Black Trees

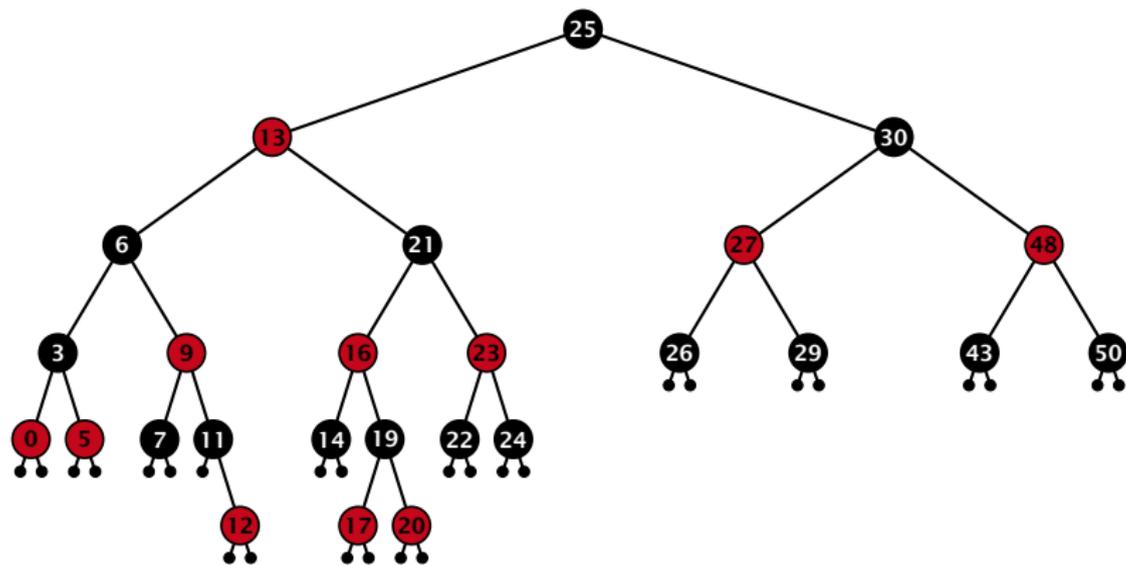
### Definition 1

A red black tree is a balanced binary search tree in which each internal node has two children. Each internal node has a color, such that

1. The root is black.
2. All leaf nodes are black.
3. For each node, all paths to descendant leaves contain the same number of black nodes.
4. If a node is red then both its children are black.

The **null**-pointers in a binary search tree are replaced by pointers to special null-vertices, that do not carry any object-data

# Red Black Trees: Example



## 7.2 Red Black Trees

### Lemma 2

A red-black tree with  $n$  internal nodes has height at most  $\mathcal{O}(\log n)$ .

### Definition 3

The **black height**  $\text{bh}(v)$  of a node  $v$  in a red black tree is the number of black nodes on a path from  $v$  to a leaf vertex (not counting  $v$ ).

We first show:

### Lemma 4

A sub-tree of black height  $\text{bh}(v)$  in a red black tree contains at least  $2^{\text{bh}(v)} - 1$  internal vertices.

## 7.2 Red Black Trees

### Proof of Lemma 4.

#### Induction on the height of $v$ .

##### base case ( $\text{height}(v) = 0$ )

- ▶ If  $\text{height}(v)$  (maximum distance btw.  $v$  and a node in the sub-tree rooted at  $v$ ) is 0 then  $v$  is a leaf.
- ▶ The black height of  $v$  is 0.
- ▶ The sub-tree rooted at  $v$  contains  $0 = 2^{\text{bh}(v)} - 1$  inner vertices.

## 7.2 Red Black Trees

### Proof (cont.)

#### induction step

- ▶ Suppose  $v$  is a node with  $\text{height}(v) > 0$ .
- ▶  $v$  has **two** children with strictly smaller height.
- ▶ These children ( $c_1, c_2$ ) either have  $\text{bh}(c_i) = \text{bh}(v)$  or  $\text{bh}(c_i) = \text{bh}(v) - 1$ .
- ▶ By induction hypothesis both sub-trees contain at least  $2^{\text{bh}(v)-1} - 1$  internal vertices.
- ▶ Then  $T_v$  contains at least  $2(2^{\text{bh}(v)-1} - 1) + 1 \geq 2^{\text{bh}(v)} - 1$  vertices.



## 7.2 Red Black Trees

### Proof of Lemma 2.

Let  $h$  denote the height of the red-black tree, and let  $P$  denote a path from the root to the furthest leaf.

At least half of the nodes on  $P$  must be black, since a red node must be followed by a black node.

Hence, the black height of the root is at least  $h/2$ .

The tree contains at least  $2^{h/2} - 1$  internal vertices. Hence,  
 $2^{h/2} - 1 \leq n$ .

Hence,  $h \leq 2 \log(n + 1) = \mathcal{O}(\log n)$ . □

## 7.2 Red Black Trees

### Definition 1

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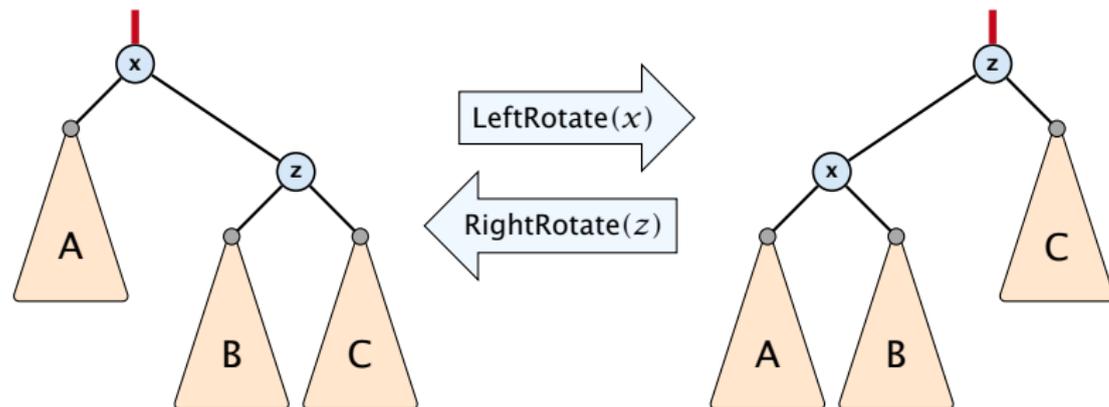
The **null**-pointers in a binary search tree are replaced by pointers to special null-vertices, that do not carry any object-data.

## 7.2 Red Black Trees

We need to adapt the insert and delete operations so that the red black properties are maintained.

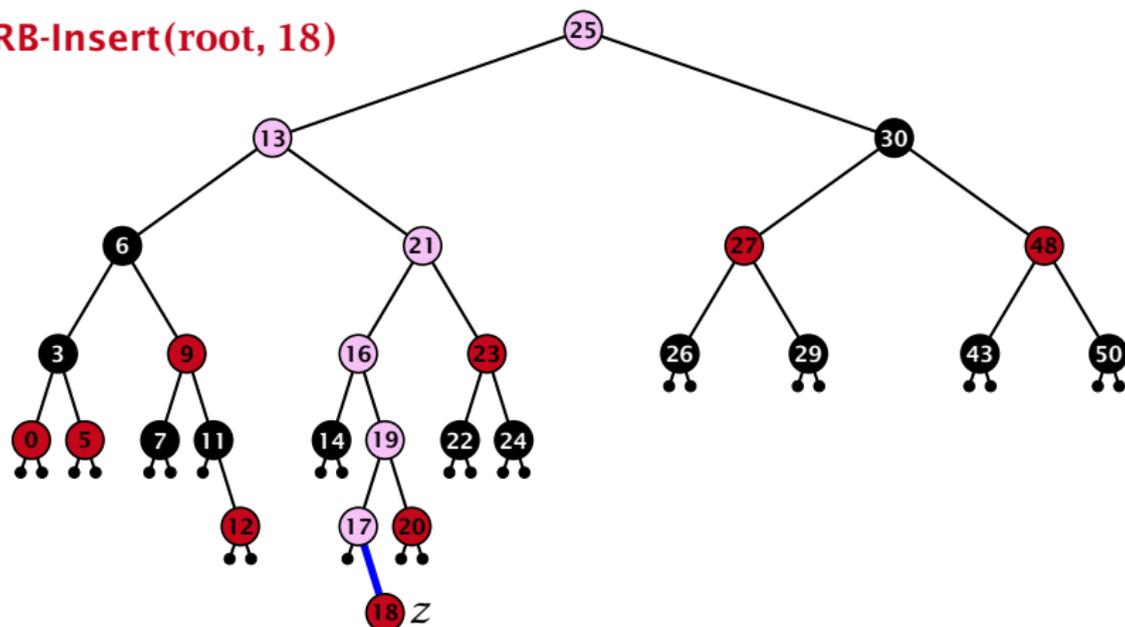
# Rotations

The properties will be maintained through rotations:



# Red Black Trees: Insert

RB-Insert(root, 18)



**Insert:**

- ▶ first make a normal insert into a binary search tree
- ▶ then fix red-black properties

# Red Black Trees: Insert

## Invariant of the fix-up algorithm:

- ▶  $z$  is a red node
- ▶ the black-height property is fulfilled at every node
- ▶ the only violation of red-black properties occurs at  $z$  and  $\text{parent}[z]$ 
  - ▶ either both of them are red  
(most important case)
  - ▶ or the parent does not exist  
(violation since root must be black)

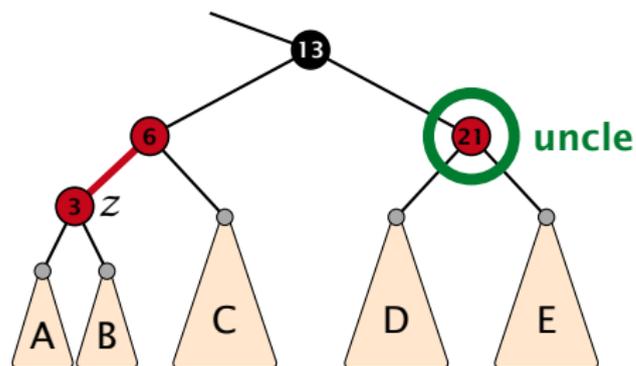
If  $z$  has a parent but no grand-parent we could simply color the parent/root black; however this case never happens.

# Red Black Trees: Insert

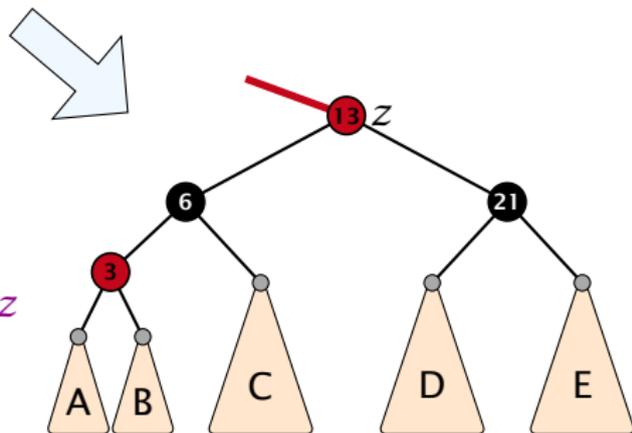
## Algorithm 10 InsertFix( $z$ )

```
1: while parent[ $z$ ]  $\neq$  null and col[parent[ $z$ ]] = red do
2:   if parent[ $z$ ] = left[gp[ $z$ ]] then  $z$  in left subtree of grandparent
3:      $uncle \leftarrow$  right[grandparent[ $z$ ]]
4:     if col[ $uncle$ ] = red then Case 1: uncle red
5:       col[p[ $z$ ]]  $\leftarrow$  black; col[ $u$ ]  $\leftarrow$  black;
6:       col[gp[ $z$ ]]  $\leftarrow$  red;  $z \leftarrow$  grandparent[ $z$ ];
7:     else Case 2: uncle black
8:       if  $z$  = right[parent[ $z$ ]] then 2a:  $z$  right child
9:          $z \leftarrow$  p[ $z$ ]; LeftRotate( $z$ );
10:      col[p[ $z$ ]]  $\leftarrow$  black; col[gp[ $z$ ]]  $\leftarrow$  red; 2b:  $z$  left child
11:      RightRotate(gp[ $z$ ]);
12:     else same as then-clause but right and left exchanged
13: col(root[ $T$ ])  $\leftarrow$  black;
```

## Case 1: Red Uncle

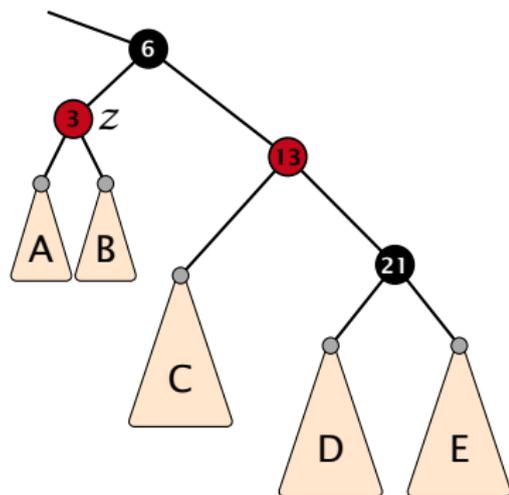
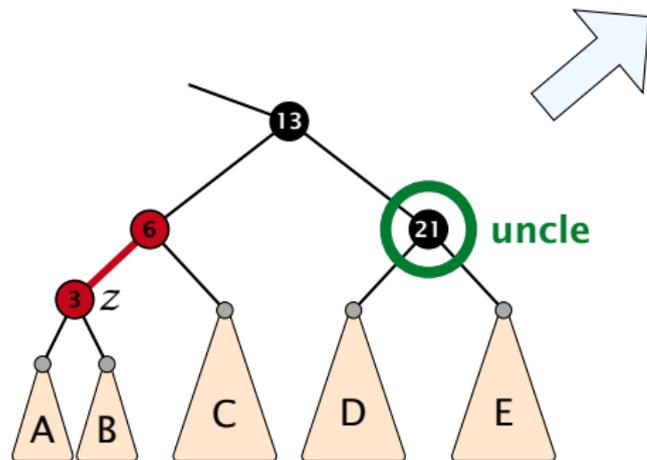


1. recolour
2. move  $z$  to grand-parent
3. invariant is fulfilled for new  $z$
4. you made progress



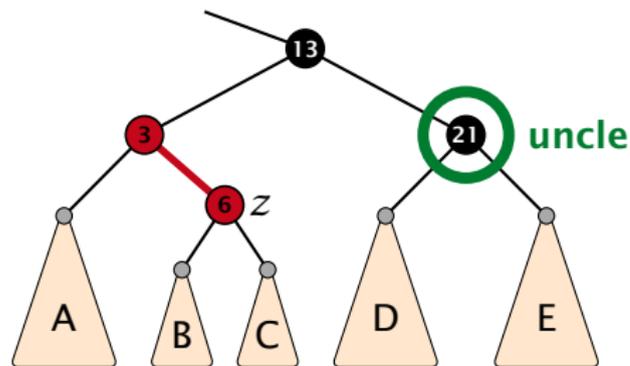
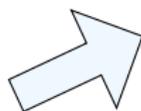
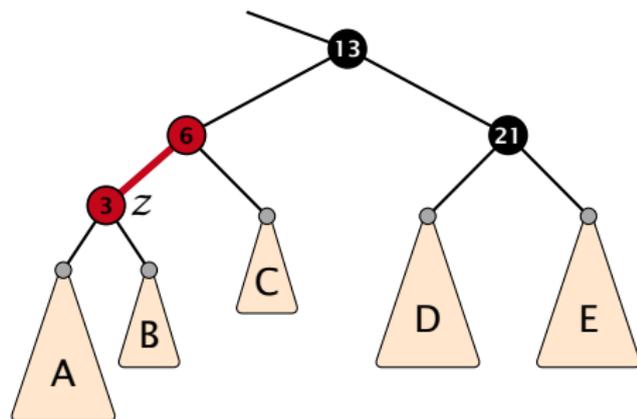
## Case 2b: Black uncle and z is left child

1. rotate around grandparent
2. re-colour to ensure that black height property holds
3. you have a red black tree



## Case 2a: Black uncle and z is right child

1. rotate around parent
2. move  $z$  downwards
3. you have Case 2b.



# Red Black Trees: Insert

## Running time:

- ▶ Only Case 1 may repeat; but only  $h/2$  many steps, where  $h$  is the height of the tree.
- ▶ Case 2a  $\rightarrow$  Case 2b  $\rightarrow$  red-black tree
- ▶ Case 2b  $\rightarrow$  red-black tree

Performing Case 1 at most  $\mathcal{O}(\log n)$  times and every other case at most once, we get a red-black tree. Hence  $\mathcal{O}(\log n)$  re-colorings and at most 2 rotations.

# Red Black Trees: Delete

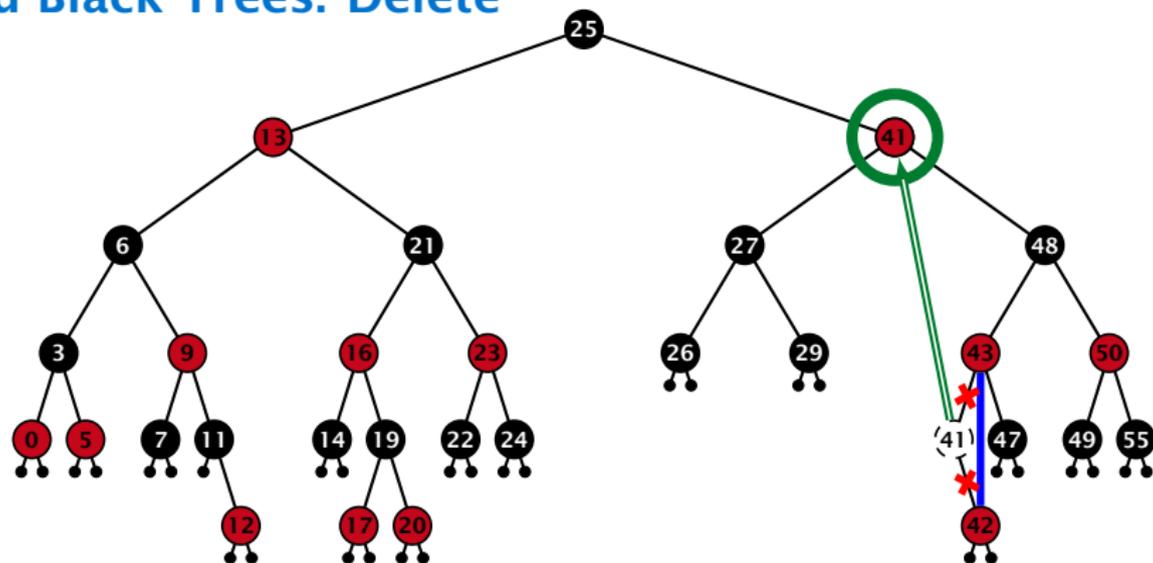
First do a standard delete.

If the spliced out node  $x$  was red everything is fine.

If it was black there may be the following problems.

- ▶ Parent and child of  $x$  were red; two adjacent red vertices.
- ▶ If you delete the root, the root may now be red.
- ▶ Every path from an ancestor of  $x$  to a descendant leaf of  $x$  changes the number of black nodes. Black height property might be violated.

## Red Black Trees: Delete

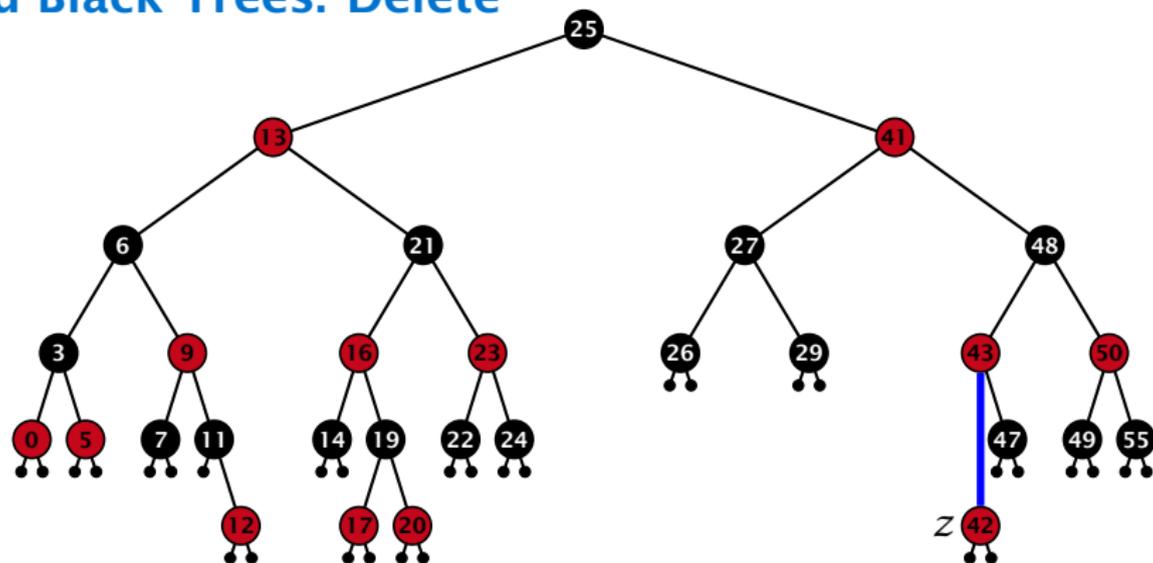


### Case 3:

Element has two children

- ▶ do normal delete
- ▶ when replacing content by content of successor, don't change color of node

## Red Black Trees: Delete



### Delete:

- ▶ deleting black node messes up black-height property
- ▶ if  $z$  is red, we can simply color it black and everything is fine
- ▶ the problem is if  $z$  is black (e.g. a dummy-leaf); we call a fix-up procedure to fix the problem.

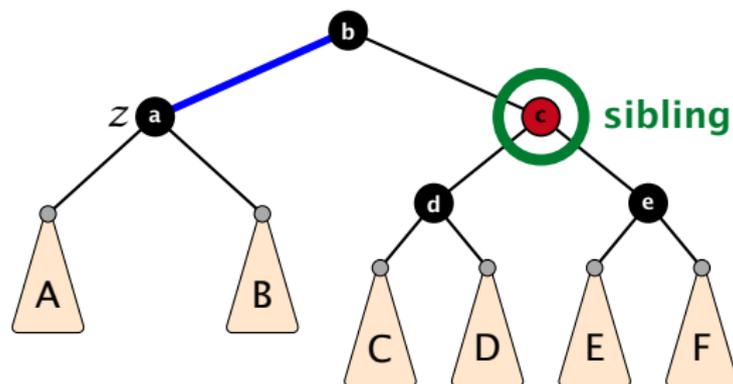
# Red Black Trees: Delete

## Invariant of the fix-up algorithm

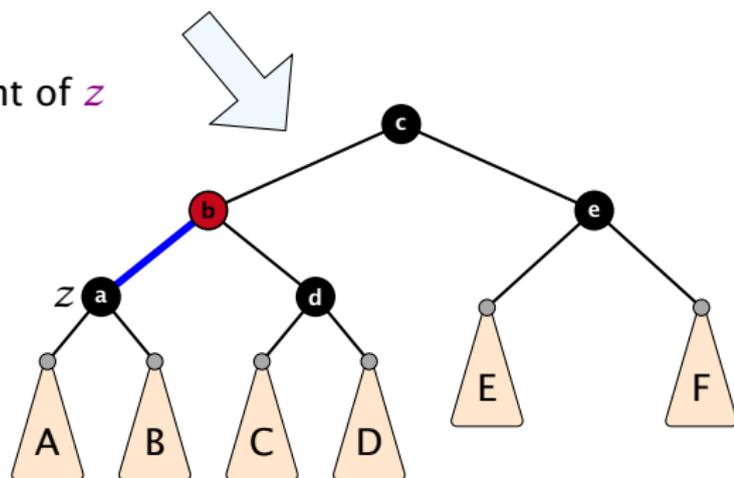
- ▶ the node  $z$  is black
- ▶ if we “assign” a fake black unit to the edge from  $z$  to its parent then the black-height property is fulfilled

**Goal:** make rotations in such a way that you at some point can remove the fake black unit from the edge.

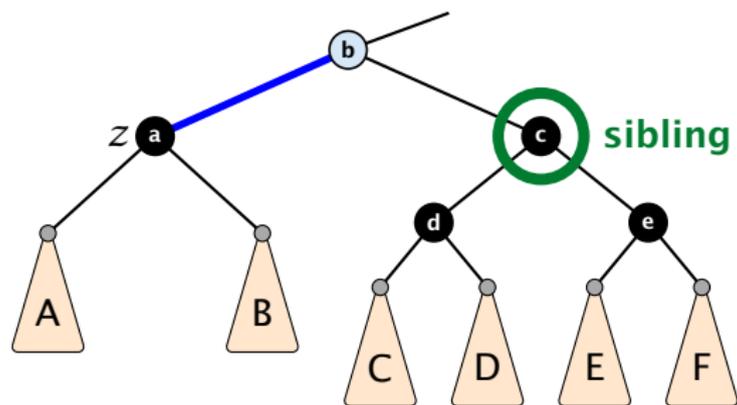
## Case 1: Sibling of $z$ is red



1. left-rotate around parent of  $z$
2. recolor nodes  $b$  and  $c$
3. the new sibling is black (and parent of  $z$  is red)
4. Case 2 (special), or Case 3, or Case 4

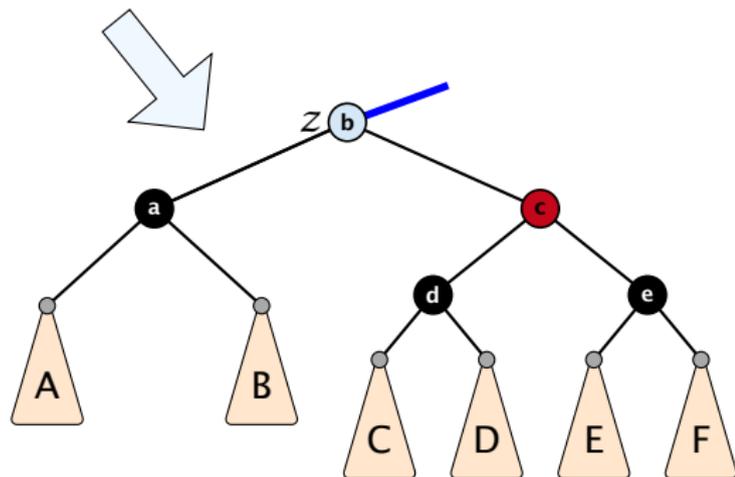


## Case 2: Sibling is black with two black children



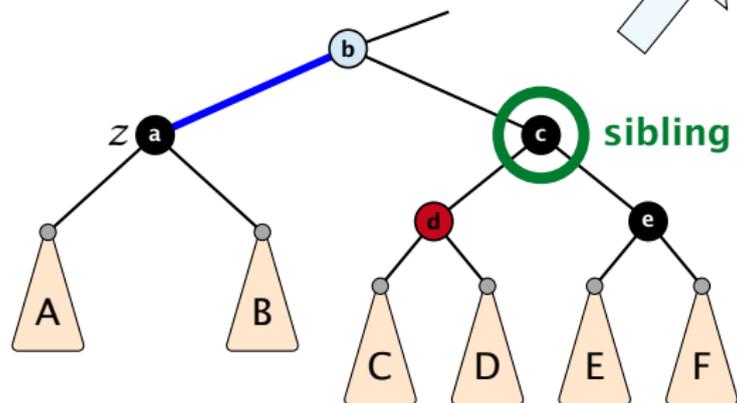
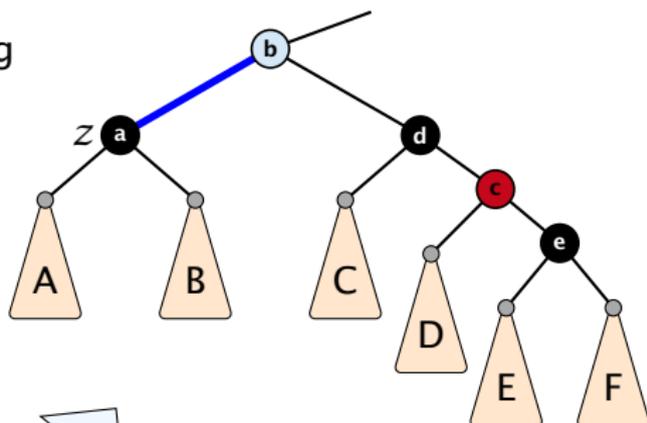
Here  $b$  is either black or red. If it is red we are in a special case that directly leads to a red-black tree.

1. re-color node  $c$
2. move fake black unit upwards
3. move  $z$  upwards
4. we made progress
5. if  $b$  is red we color it black and are done



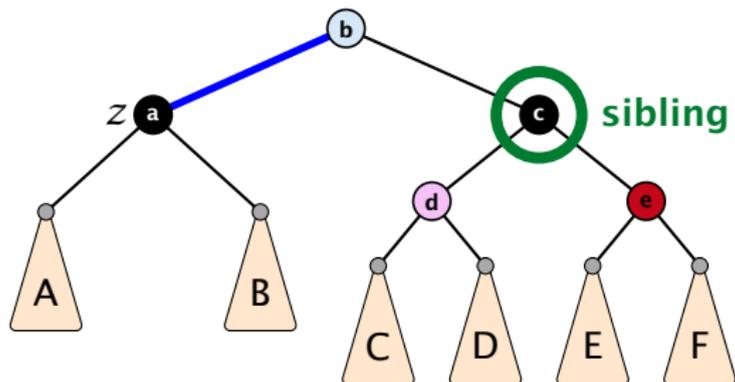
## Case 3: Sibling black with one black child to the right

1. do a right-rotation at sibling
2. recolor  $c$  and  $d$
3. new sibling is black with red right child (Case 4)



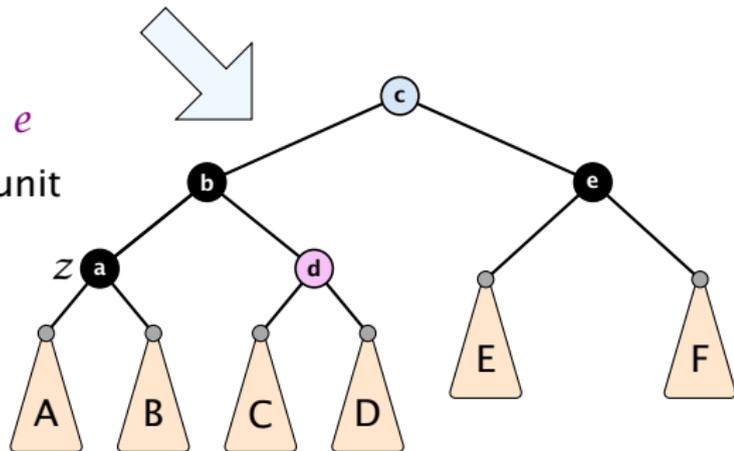
Again the blue color of  $b$  indicates that it can either be black or red.

## Case 4: Sibling is black with red right child



- Here **b** and **d** are either red or black but have possibly different colors.
- We recolor **c** by giving it the color of **b**.

1. left-rotate around **b**
2. recolor nodes **b**, **c**, and **e**
3. remove the fake black unit
4. you have a valid red black tree



## Running time:

- ▶ only Case 2 can repeat; but only  $h$  many steps, where  $h$  is the height of the tree
- ▶ Case 1 → Case 2 (special) → red black tree  
Case 1 → Case 3 → Case 4 → red black tree  
Case 1 → Case 4 → red black tree
- ▶ Case 3 → Case 4 → red black tree
- ▶ Case 4 → red black tree

Performing Case 2 at most  $\mathcal{O}(\log n)$  times and every other step at most once, we get a red black tree. Hence,  $\mathcal{O}(\log n)$  re-colorings and at most 3 rotations.

# Red-Black Trees

## Bibliography

[CLRS90] Thomas H. Cormen, Charles E. Leiserson, Ron L. Rivest, Clifford Stein:  
*Introduction to Algorithms (3rd ed.)*,  
MIT Press and McGraw-Hill, 2009

Red black trees are covered in detail in Chapter 13 of [CLRS90].

## 7.3 AVL-Trees

### Definition 5

AVL-trees are binary search trees that fulfill the following balance condition. For every node  $v$

$$|\text{height}(\text{left sub-tree}(v)) - \text{height}(\text{right sub-tree}(v))| \leq 1 .$$

### Lemma 6

*An AVL-tree of height  $h$  contains at least  $F_{h+2} - 1$  and at most  $2^h - 1$  internal nodes, where  $F_n$  is the  $n$ -th Fibonacci number ( $F_0 = 0$ ,  $F_1 = 1$ ), and the height is the maximal number of edges from the root to an (empty) dummy leaf.*

# AVL trees

## Proof.

The upper bound is clear, as a binary tree of height  $h$  can only contain

$$\sum_{j=0}^{h-1} 2^j = 2^h - 1$$

internal nodes.

# AVL trees

## Proof (cont.)

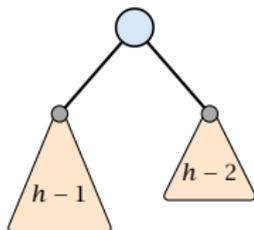
### Induction (base cases):

1. an AVL-tree of height  $h = 1$  contains at least one internal node,  $1 \geq F_3 - 1 = 2 - 1 = 1$ .
2. an AVL tree of height  $h = 2$  contains at least two internal nodes,  $2 \geq F_4 - 1 = 3 - 1 = 2$



### Induction step:

An AVL-tree of height  $h \geq 2$  of minimal size has a root with sub-trees of height  $h - 1$  and  $h - 2$ , respectively. Both, sub-trees have minimal node number.



Let

$g_h := 1 + \text{minimal size of AVL-tree of height } h$  .

Then

$$g_1 = 2 \qquad = F_3$$

$$g_2 = 3 \qquad = F_4$$

$$g_{h-1} = 1 + g_{h-1-1} + g_{h-2-1}, \qquad \text{hence}$$

$$g_h = g_{h-1} + g_{h-2} \qquad = F_{h+2}$$

## 7.3 AVL-Trees

An AVL-tree of height  $h$  contains at least  $F_{h+2} - 1$  internal nodes.

Since

$$n + 1 \geq F_{h+2} = \Omega \left( \left( \frac{1 + \sqrt{5}}{2} \right)^h \right),$$

we get

$$n \geq \Omega \left( \left( \frac{1 + \sqrt{5}}{2} \right)^h \right),$$

and, hence,  $h = \mathcal{O}(\log n)$ .

## 7.3 AVL-Trees

We need to maintain the balance condition through rotations.

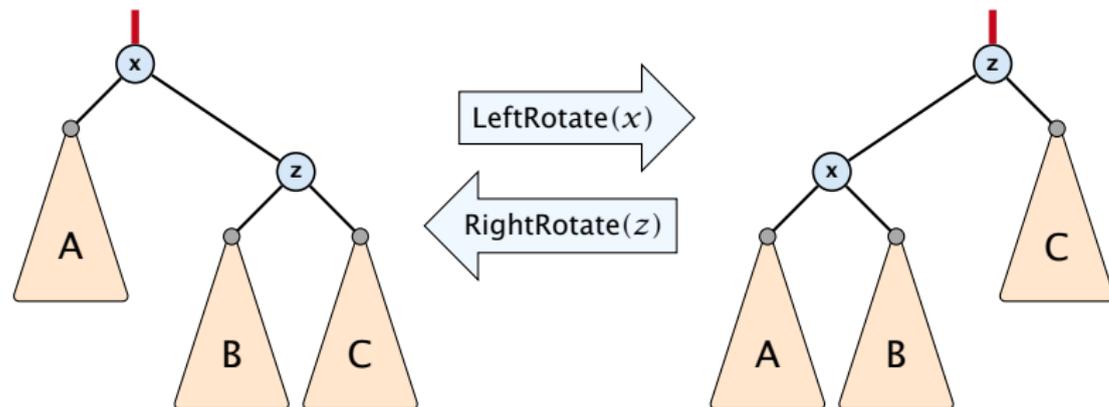
For this we store in every internal tree-node  $v$  the **balance** of the node. Let  $v$  denote a tree node with left child  $c_\ell$  and right child  $c_r$ .

$$\text{balance}[v] := \text{height}(T_{c_\ell}) - \text{height}(T_{c_r}) ,$$

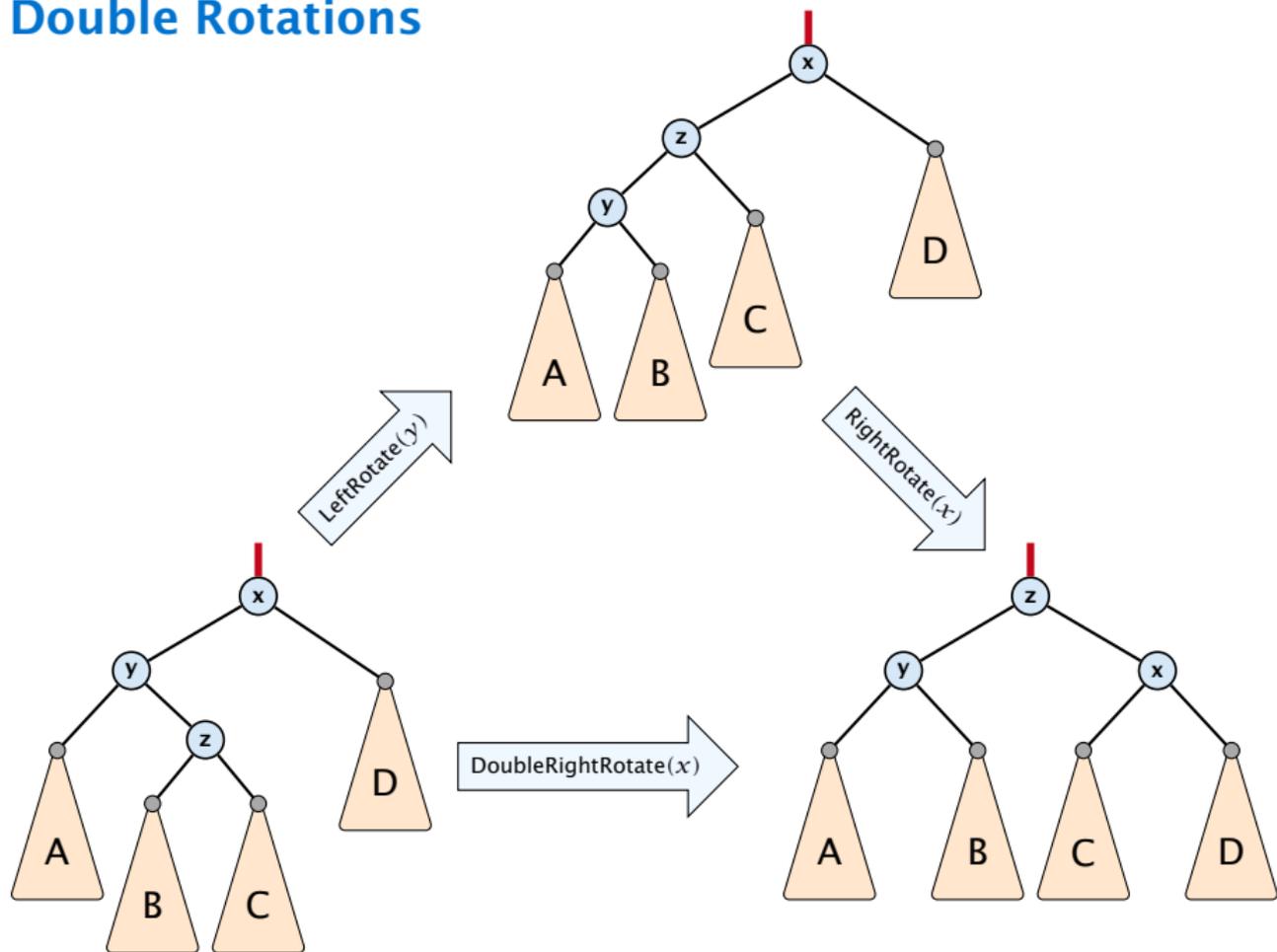
where  $T_{c_\ell}$  and  $T_{c_r}$ , are the sub-trees rooted at  $c_\ell$  and  $c_r$ , respectively.

# Rotations

The properties will be maintained through rotations:



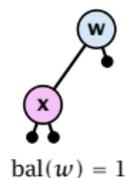
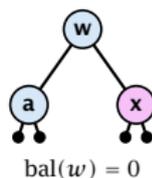
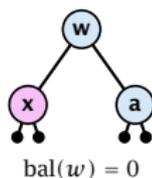
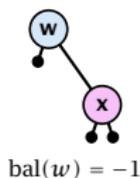
# Double Rotations



# AVL-trees: Insert

Note that before the insertion  $w$  is right above the leaf level, i.e.,  $x$  replaces a child of  $w$  that was a dummy leaf.

- ▶ Insert like in a binary search tree.
- ▶ Let  $w$  denote the parent of the newly inserted node  $x$ .
- ▶ One of the following cases holds:



- ▶ If  $\text{bal}[w] \neq 0$ ,  $T_w$  has changed height; the balance-constraint may be violated at ancestors of  $w$ .
- ▶ Call  $\text{AVL-fix-up-insert}(\text{parent}[w])$  to restore the balance-condition.

# AVL-trees: Insert

## Invariant at the beginning of AVL-fix-up-insert( $v$ ):

1. The balance constraints hold at all descendants of  $v$ .
2. A node has been inserted into  $T_c$ , where  $c$  is either the right or left child of  $v$ .
3.  $T_c$  has increased its height by one (otw. we would already have aborted the fix-up procedure).
4. The balance at node  $c$  fulfills  $\text{balance}[c] \in \{-1, 1\}$ . This holds because if the balance of  $c$  is 0, then  $T_c$  did not change its height, and the whole procedure would have been aborted in the previous step.

Note that these constraints hold for the first call  $\text{AVL-fix-up-insert}(\text{parent}[w])$ .

## AVL-trees: Insert

### Algorithm 11 AVL-fix-up-insert( $v$ )

- 1: **if**  $\text{balance}[v] \in \{-2, 2\}$  **then** DoRotationInsert( $v$ );
- 2: **if**  $\text{balance}[v] \in \{0\}$  **return**;
- 3: AVL-fix-up-insert(parent( $v$ ));

We will show that the above procedure is correct, and that it will do at most one rotation.

## Algorithm 12 DoRotationInsert( $v$ )

```
1: if balance[ $v$ ] = -2 then // insert in right sub-tree
2:     if balance[right[ $v$ ]] = -1 then
3:         LeftRotate( $v$ );
4:     else
5:         DoubleLeftRotate( $v$ );
6: else // insert in left sub-tree
7:     if balance[left[ $v$ ]] = 1 then
8:         RightRotate( $v$ );
9:     else
10:        DoubleRightRotate( $v$ );
```

## AVL-trees: Insert

It is clear that the invariant for the fix-up routine holds as long as no rotations have been done.

We have to show that after doing one rotation **all** balance constraints are fulfilled.

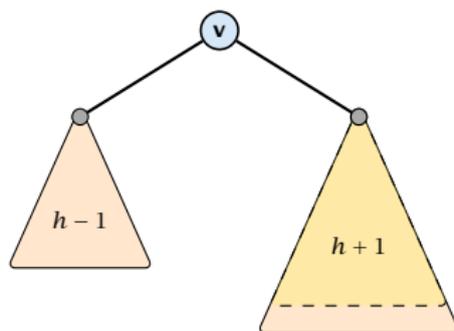
We show that after doing a rotation at  $v$ :

- ▶  $v$  fulfills balance condition.
- ▶ All children of  $v$  still fulfill the balance condition.
- ▶ The height of  $T_v$  is the same as before the insert-operation took place.

We only look at the case where the insert happened into the right sub-tree of  $v$ . The other case is symmetric.

## AVL-trees: Insert

We have the following situation:

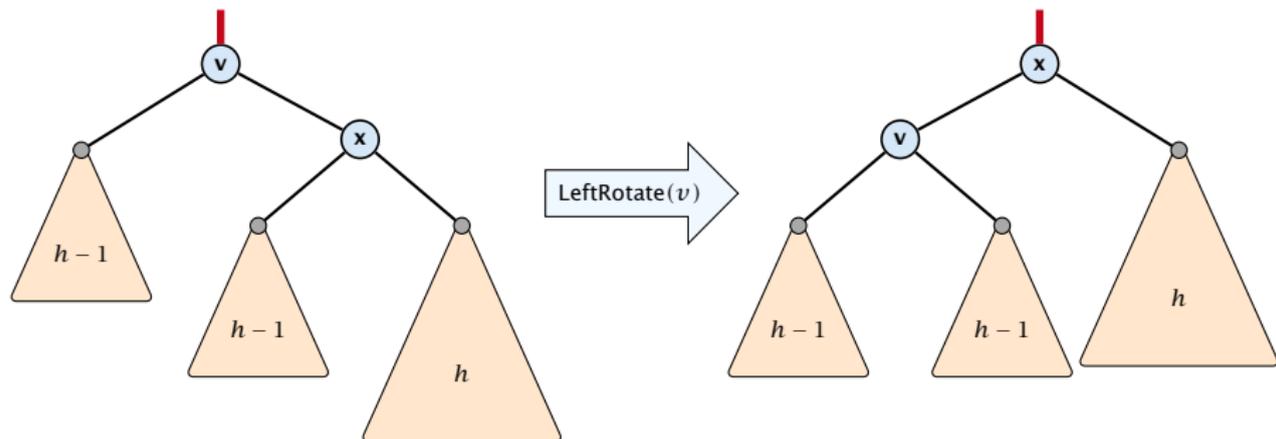


The right sub-tree of  $v$  has increased its height which results in a balance of  $-2$  at  $v$ .

Before the insertion the height of  $T_v$  was  $h+1$ .

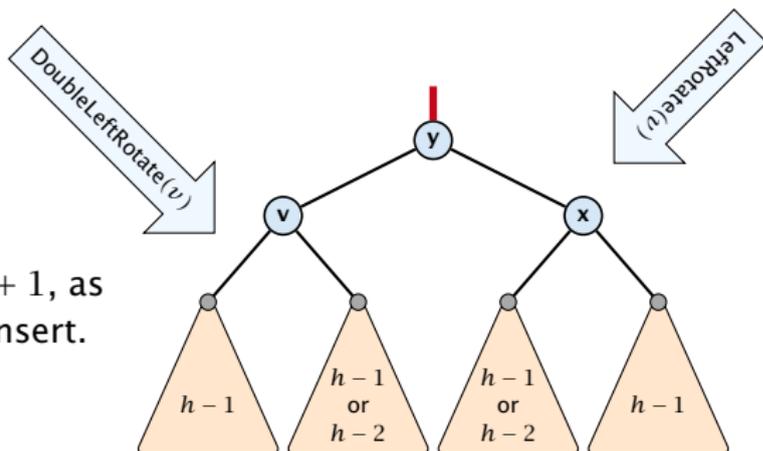
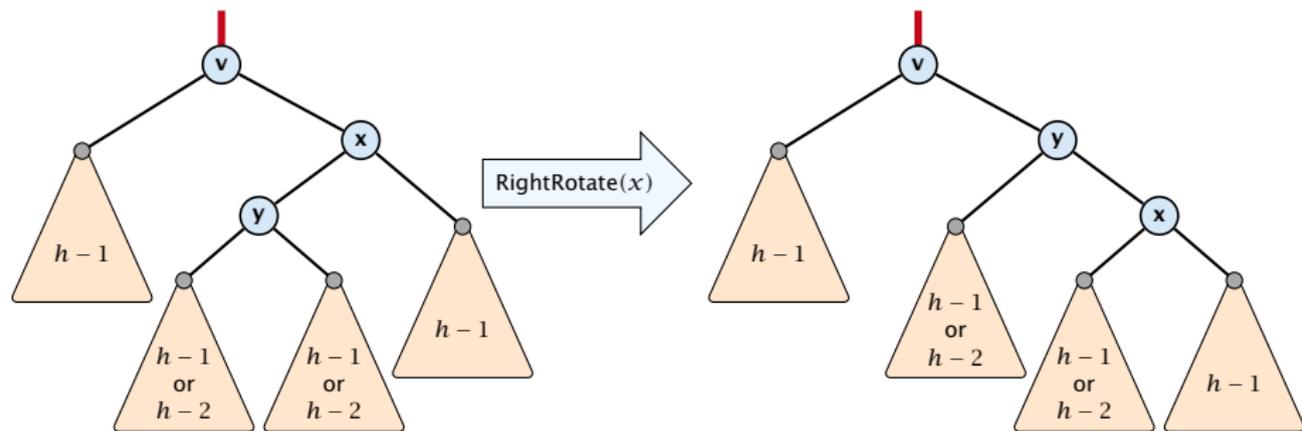
## Case 1: $\text{balance}[\text{right}[v]] = -1$

We do a left rotation at  $v$



Now, the subtree has height  $h + 1$  as before the insertion.  
Hence, we do not need to continue.

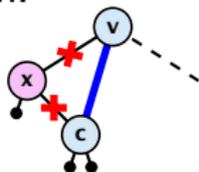
## Case 2: $\text{balance}[\text{right}[v]] = 1$



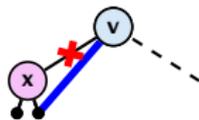
Height is  $h + 1$ , as before the insert.

## AVL-trees: Delete

- ▶ Delete like in a binary search tree.
- ▶ Let  $v$  denote the parent of the node that has been **spliced out**.
- ▶ The balance-constraint may be violated at  $v$ , or at ancestors of  $v$ , as a sub-tree of a child of  $v$  has reduced its height.
- ▶ Initially, the node  $c$ —the new root in the sub-tree that has changed—is either a dummy leaf or a node with two dummy leaves as children.



Case 1



Case 2

In both cases  $\text{bal}[c] = 0$ .

- ▶ Call  $\text{AVL-fix-up-delete}(v)$  to restore the balance-condition.

## Invariant at the beginning AVL-fix-up-delete( $v$ ):

1. The balance constraints holds at all descendants of  $v$ .
2. A node has been deleted from  $T_c$ , where  $c$  is either the right or left child of  $v$ .
3.  $T_c$  has decreased its height by one.
4. The balance at the node  $c$  fulfills  $\text{balance}[c] = 0$ . This holds because if the balance of  $c$  is in  $\{-1, 1\}$ , then  $T_c$  did not change its height, and the whole procedure would have been aborted in the previous step.

## AVL-trees: Delete

### Algorithm 13 AVL-fix-up-delete( $v$ )

- 1: **if**  $\text{balance}[v] \in \{-2, 2\}$  **then** DoRotationDelete( $v$ );
- 2: **if**  $\text{balance}[v] \in \{-1, 1\}$  **return**;
- 3: AVL-fix-up-delete(parent( $v$ ));

We will show that the above procedure is correct. However, for the case of a delete there may be a logarithmic number of rotations.

## Algorithm 14 DoRotationDelete( $v$ )

```
1: if balance[ $v$ ] = -2 then // deletion in left sub-tree
2:     if balance[right[ $v$ ]]  $\in$  {0, -1} then
3:         LeftRotate( $v$ );
4:     else
5:         DoubleLeftRotate( $v$ );
6: else // deletion in right sub-tree
7:     if balance[left[ $v$ ]] = {0, 1} then
8:         RightRotate( $v$ );
9:     else
10:        DoubleRightRotate( $v$ );
```

Note that the case distinction on the second level (bal[right[ $v$ ]] and bal[left[ $v$ ]]) is not done w.r.t. the child  $c$  for which the sub-tree  $T_c$  has changed. This is different to AVL-fix-up-insert.

## AVL-trees: Delete

It is clear that the invariant for the fix-up routine hold as long as no rotations have been done.

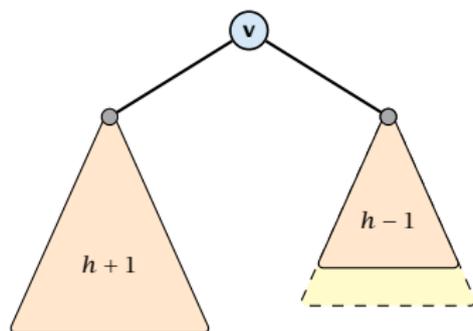
We show that after doing a rotation at  $v$ :

- ▶  $v$  fulfills the balance condition.
- ▶ All children of  $v$  still fulfill the balance condition.
- ▶ If now  $\text{balance}[v] \in \{-1, 1\}$  we can stop as the height of  $T_v$  is the same as before the deletion.

We only look at the case where the deleted node was in the right sub-tree of  $v$ . The other case is symmetric.

## AVL-trees: Delete

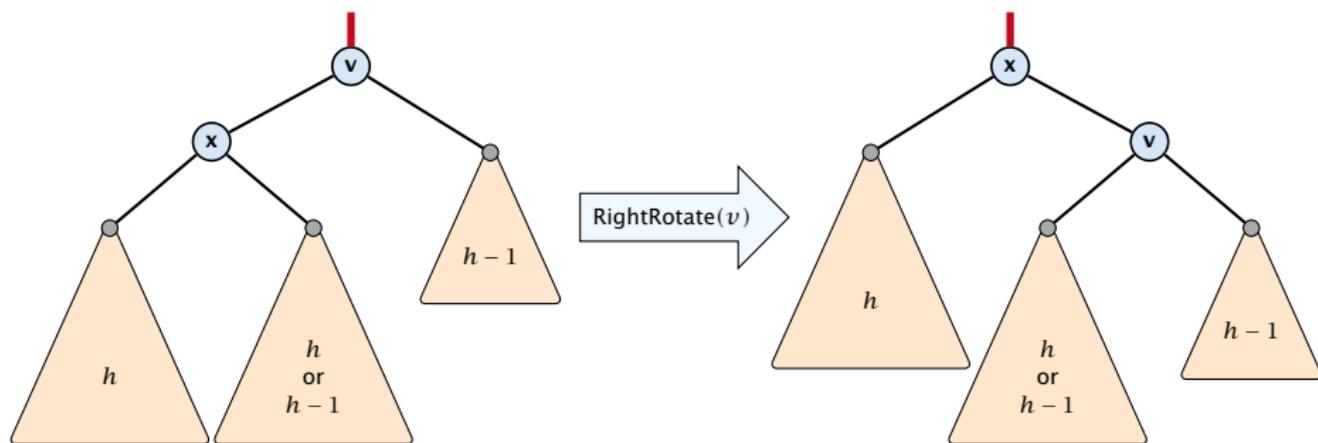
We have the following situation:



The right sub-tree of  $v$  has decreased its height which results in a balance of  $2$  at  $v$ .

Before the deletion the height of  $T_v$  was  $h + 2$ .

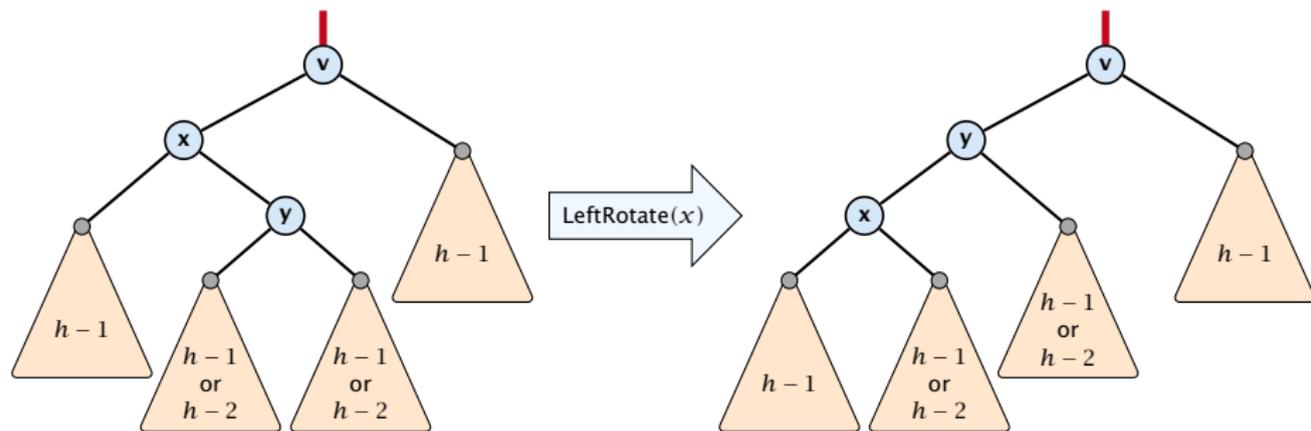
## Case 1: $\text{balance}[\text{left}[v]] \in \{0, 1\}$



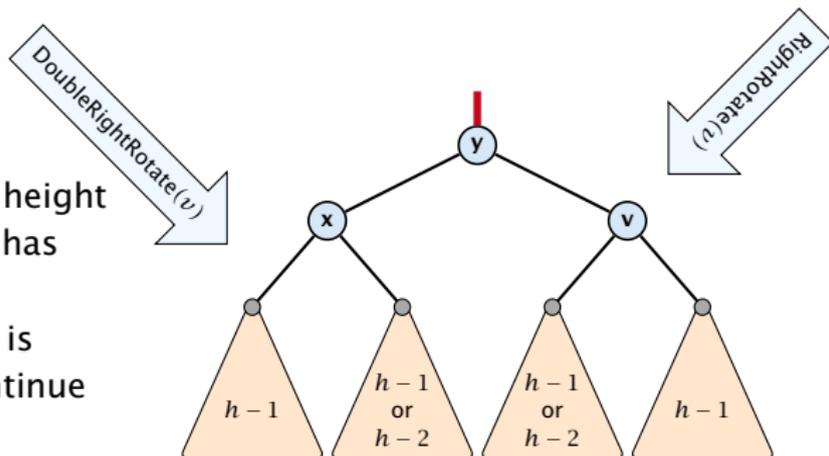
If the middle subtree has height  $h$  the whole tree has height  $h + 2$  as before the deletion. The iteration stops as the balance at the root is non-zero.

If the middle subtree has height  $h - 1$  the whole tree has decreased its height from  $h + 2$  to  $h + 1$ . We do continue the fix-up procedure as the balance at the root is zero.

## Case 2: $\text{balance}[\text{left}[v]] = -1$



Sub-tree has height  $h + 1$ , i.e., it has shrunk. The balance at  $y$  is zero. We continue the iteration.



## Bibliography

- [OW02] Thomas Ottmann, Peter Widmayer:  
*Algorithmen und Datenstrukturen*,  
Spektrum, 4th edition, 2002
- [GT98] Michael T. Goodrich, Roberto Tamassia  
*Data Structures and Algorithms in JAVA*,  
John Wiley, 1998

Chapter 5.2.1 of [OW02] contains a detailed description of AVL-trees, albeit only in German.

AVL-trees are covered in [GT98] in Chapter 7.4. However, the coverage is a lot shorter than in [OW02].

## 7.4 Augmenting Data Structures

Suppose you want to develop a data structure with:

- ▶ **Insert( $x$ )**: insert element  $x$ .
- ▶ **Search( $k$ )**: search for element with key  $k$ .
- ▶ **Delete( $x$ )**: delete element referenced by pointer  $x$ .
- ▶ **find-by-rank( $\ell$ )**: return the  $\ell$ -th element; return “error” if the data-structure contains less than  $\ell$  elements.

**Augment an existing data-structure instead of developing a new one.**

## 7.4 Augmenting Data Structures

### How to augment a data-structure

1. choose an underlying data-structure
  2. determine additional information to be stored in the underlying structure
  3. verify/show how the additional information can be maintained for the basic modifying operations on the underlying structure.
  4. develop the new operations
- Of course, the above steps heavily depend on each other. For example it makes no sense to choose additional information to be stored (Step 2), and later realize that either the information cannot be maintained efficiently (Step 3) or is not sufficient to support the new operations (Step 4).
  - However, the above outline is a good way to describe/document a new data-structure.

## 7.4 Augmenting Data Structures

**Goal: Design a data-structure that supports insert, delete, search, and find-by-rank in time  $\mathcal{O}(\log n)$ .**

1. We choose a red-black tree as the underlying data-structure.
2. We store in each node  $v$  the size of the sub-tree rooted at  $v$ .
3. We need to be able to update the size-field in each node without asymptotically affecting the running time of insert, delete, and search. We come back to this step later...

## 7.4 Augmenting Data Structures

**Goal: Design a data-structure that supports insert, delete, search, and find-by-rank in time  $\mathcal{O}(\log n)$ .**

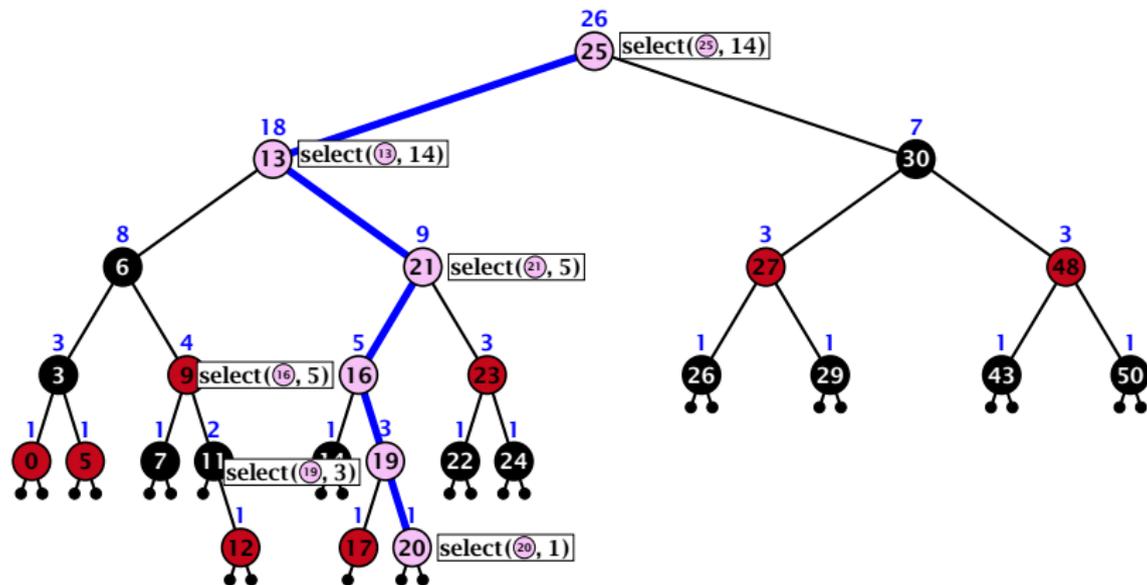
4. How does find-by-rank work?

Find-by-rank( $k$ ) := Select( $\text{root}, k$ ) with

**Algorithm 15** Select( $x, i$ )

```
1: if  $x = \text{null}$  then return error
2: if  $\text{left}[x] \neq \text{null}$  then  $r \leftarrow \text{left}[x].\text{size} + 1$  else  $r \leftarrow 1$ 
3: if  $i = r$  then return  $x$ 
4: if  $i < r$  then
5:     return Select( $\text{left}[x], i$ )
6: else
7:     return Select( $\text{right}[x], i - r$ )
```

## Select( $x, i$ )



### Find-by-rank:

- ▶ decide whether you have to proceed into the left or right sub-tree
- ▶ adjust the rank that you are searching for if you go right

## 7.4 Augmenting Data Structures

**Goal: Design a data-structure that supports insert, delete, search, and find-by-rank in time  $\mathcal{O}(\log n)$ .**

3. How do we maintain information?

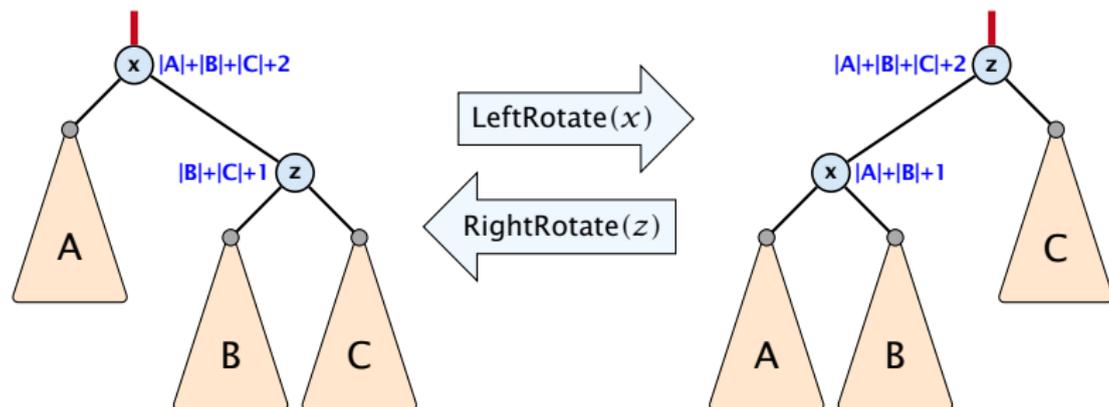
**Search( $k$ ):** Nothing to do.

**Insert( $x$ ):** When going down the search path increase the size field for each visited node. **Maintain the size field during rotations.**

**Delete( $x$ ):** Directly after splicing out a node traverse the path from the spliced out node upwards, and decrease the size counter on every node on this path. **Maintain the size field during rotations.**

## Rotations

The only operation during the fix-up procedure that alters the tree and requires an update of the size-field:



The nodes  $x$  and  $z$  are the only nodes changing their size-fields.

The new size-fields can be computed **locally** from the size-fields of the children.

# Augmenting Data Structures

## Bibliography

- [CLRS90] Thomas H. Cormen, Charles E. Leiserson, Ron L. Rivest, Clifford Stein:  
*Introduction to Algorithms (3rd ed.)*,  
MIT Press and McGraw-Hill, 2009

See Chapter 14 of [CLRS90].

## 7.5 $(a, b)$ -trees

### Definition 7

For  $b \geq 2a - 1$  an  $(a, b)$ -tree is a search tree with the following properties

1. all leaves have the same distance to the root
2. every internal non-root vertex  $v$  has at least  $a$  and at most  $b$  children
3. the root has degree at least 2 if the tree is non-empty
4. the internal vertices do not contain data, but only keys (external search tree)
5. there is a special dummy leaf node with key-value  $\infty$

## 7.5 (a, b)-trees

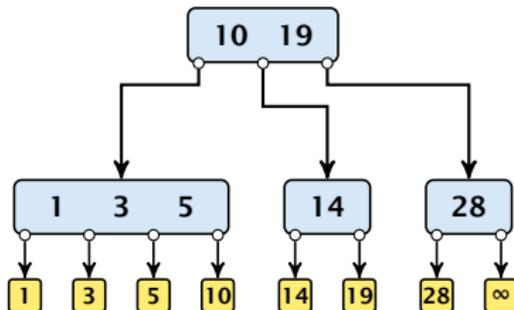
Each internal node  $v$  with  $d(v)$  children stores  $d - 1$  keys  $k_1, \dots, k_{d-1}$ . The  $i$ -th subtree of  $v$  fulfills

$$k_{i-1} < \text{key in } i\text{-th sub-tree} \leq k_i ,$$

where we use  $k_0 = -\infty$  and  $k_d = \infty$ .

## 7.5 (a, b)-trees

### Example 8



## 7.5 ( $a, b$ )-trees

### Variants

- ▶ The dummy leaf element may not exist; it only makes implementation more convenient.
- ▶ Variants in which  $b = 2a$  are commonly referred to as  $B$ -trees.
- ▶ A  $B$ -tree usually refers to the variant in which keys and data are stored at internal nodes.
- ▶ A  $B^+$  tree stores the data only at leaf nodes as in our definition. Sometimes the leaf nodes are also connected in a linear list data structure to speed up the computation of successors and predecessors.
- ▶ A  $B^*$  tree requires that a node is at least  $2/3$ -full as opposed to  $1/2$ -full (the requirement of a  $B$ -tree).

## Lemma 9

Let  $T$  be an  $(a, b)$ -tree for  $n > 0$  elements (i.e.,  $n + 1$  leaf nodes) and height  $h$  (number of edges from root to a leaf vertex). Then

1.  $2a^{h-1} \leq n + 1 \leq b^h$
2.  $\log_b(n + 1) \leq h \leq 1 + \log_a\left(\frac{n+1}{2}\right)$

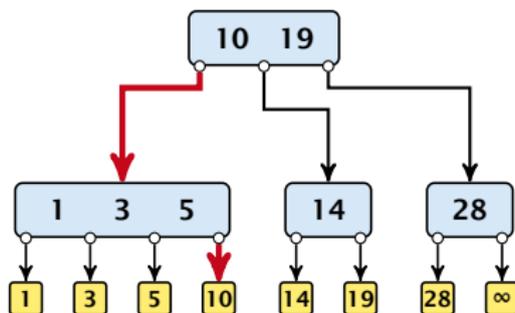
### Proof.

- ▶ If  $n > 0$  the root has degree at least 2 and all other nodes have degree at least  $a$ . This gives that the number of leaf nodes is at least  $2a^{h-1}$ .
- ▶ Analogously, the degree of any node is at most  $b$  and, hence, the number of leaf nodes at most  $b^h$ .



# Search

## Search(8)

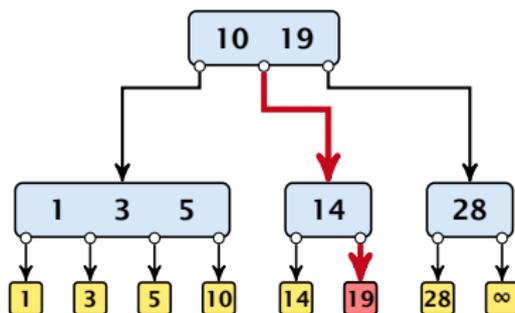


The search is straightforward. It is only important that you need to go all the way to the leaf.

Time:  $\mathcal{O}(b \cdot h) = \mathcal{O}(b \cdot \log n)$ , if the individual nodes are organized as linear lists.

# Search

## Search(19)



The search is straightforward. It is only important that you need to go all the way to the leaf.

Time:  $\mathcal{O}(b \cdot h) = \mathcal{O}(b \cdot \log n)$ , if the individual nodes are organized as linear lists.

# Insert

Insert element  $x$ :

- ▶ Follow the path as if searching for  $\text{key}[x]$ .
- ▶ If this search ends in leaf  $\ell$ , insert  $x$  **before** this leaf.
- ▶ For this add  $\text{key}[x]$  to the key-list of the last internal node  $v$  on the path.
- ▶ If after the insert  $v$  contains  $b$  nodes, do  $\text{Rebalance}(v)$ .

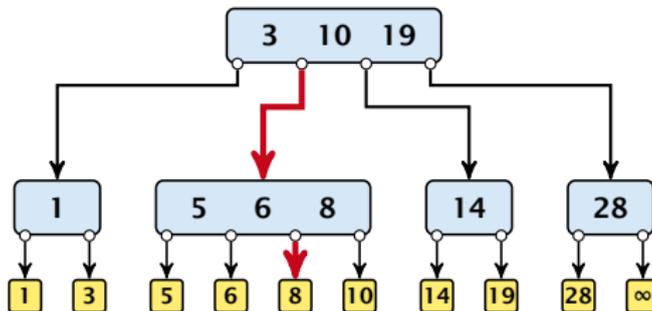
# Insert

Rebalance( $v$ ):

- ▶ Let  $k_i$ ,  $i = 1, \dots, b$  denote the keys stored in  $v$ .
- ▶ Let  $j := \lfloor \frac{b+1}{2} \rfloor$  be the middle element.
- ▶ Create two nodes  $v_1$ , and  $v_2$ .  $v_1$  gets all keys  $k_1, \dots, k_{j-1}$  and  $v_2$  gets keys  $k_{j+1}, \dots, k_b$ .
- ▶ Both nodes get at least  $\lfloor \frac{b-1}{2} \rfloor$  keys, and have therefore degree at least  $\lfloor \frac{b-1}{2} \rfloor + 1 \geq a$  since  $b \geq 2a - 1$ .
- ▶ They get at most  $\lceil \frac{b-1}{2} \rceil$  keys, and have therefore degree at most  $\lceil \frac{b-1}{2} \rceil + 1 \leq b$  (since  $b \geq 2$ ).
- ▶ The key  $k_j$  is promoted to the parent of  $v$ . The current pointer to  $v$  is altered to point to  $v_1$ , and a new pointer (to the right of  $k_j$ ) in the parent is added to point to  $v_2$ .
- ▶ Then, re-balance the parent.

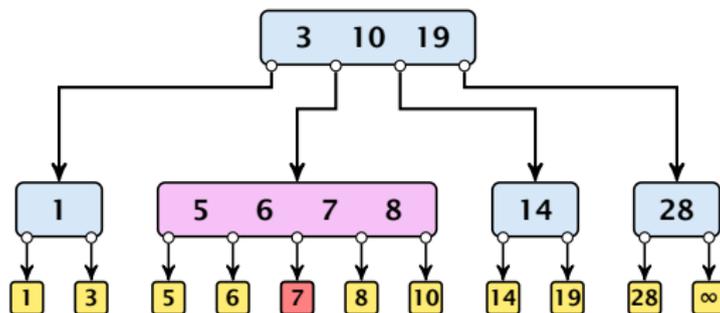
# Insert

## Insert(7)



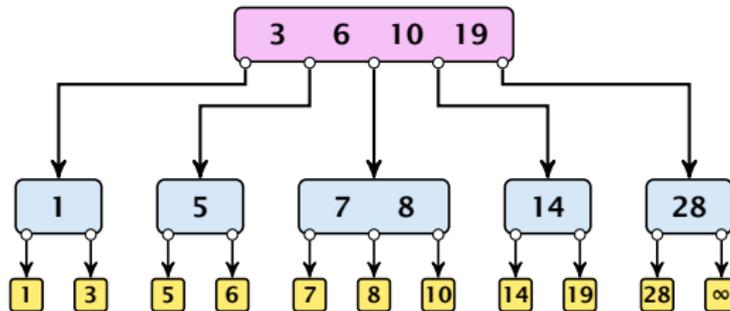
# Insert

## Insert(7)



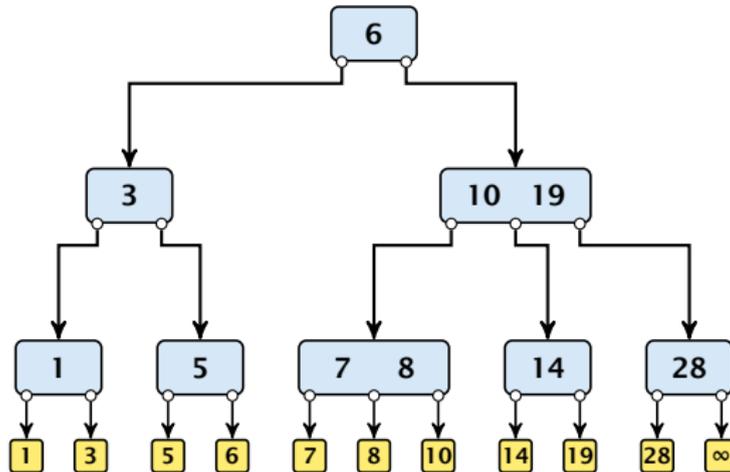
# Insert

## Insert(7)



# Insert

Insert(7)



# Delete

Delete element  $x$  (pointer to leaf vertex):

- ▶ Let  $v$  denote the parent of  $x$ . If  $\text{key}[x]$  is contained in  $v$ , remove the key from  $v$ , and delete the leaf vertex.
- ▶ Otherwise delete the key of the predecessor of  $x$  from  $v$ ; delete the leaf vertex; and replace the occurrence of  $\text{key}[x]$  in internal nodes by the predecessor key. (Note that it appears in exactly one internal vertex).
- ▶ If now the number of keys in  $v$  is below  $a - 1$  perform  $\text{Rebalance}'(v)$ .

# Delete

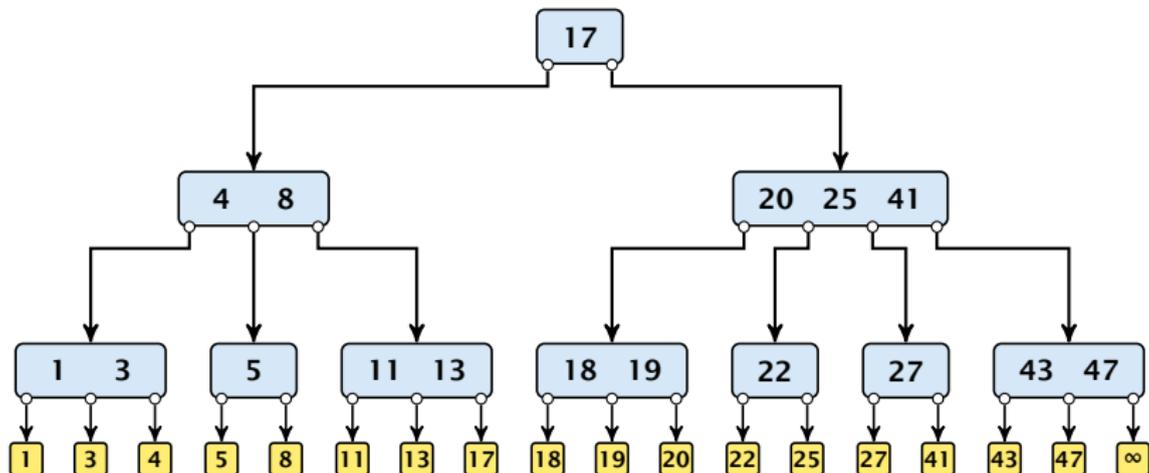
Rebalance' ( $v$ ):

- ▶ If there is a neighbour of  $v$  that has at least  $a$  keys take over the largest (if right neighbor) or smallest (if left neighbour) and the corresponding sub-tree.
- ▶ If not: merge  $v$  with one of its neighbours.
- ▶ The merged node contains at most  $(a - 2) + (a - 1) + 1$  keys, and has therefore at most  $2a - 1 \leq b$  successors.
- ▶ Then rebalance the parent.
- ▶ During this process the root may become empty. In this case the root is deleted and the height of the tree decreases.

Animation for deleting in an  $(a, b)$ -tree is only available in the lecture version of the slides.

## (2, 4)-trees and red black trees

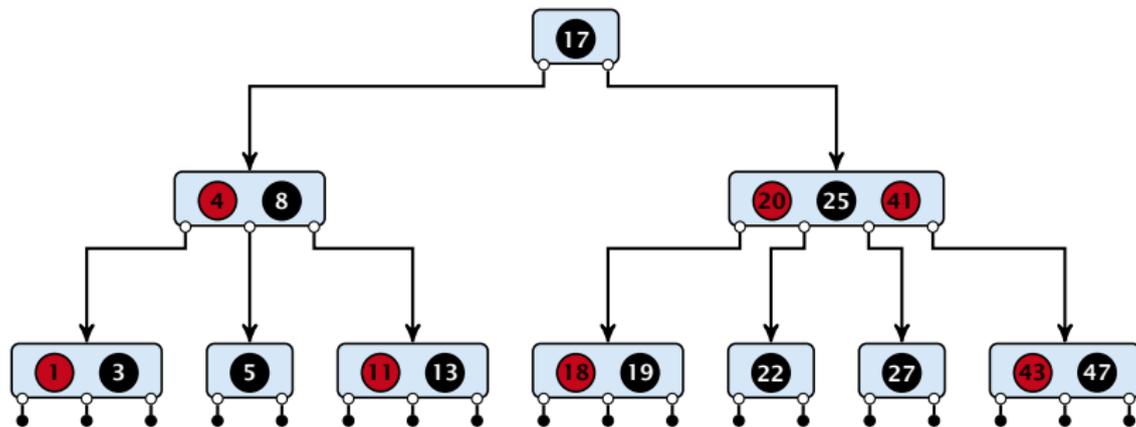
There is a close relation between red-black trees and (2,4)-trees:



First make it into an internal search tree by moving the satellite-data from the leaves to internal nodes. Add dummy leaves.

## (2, 4)-trees and red black trees

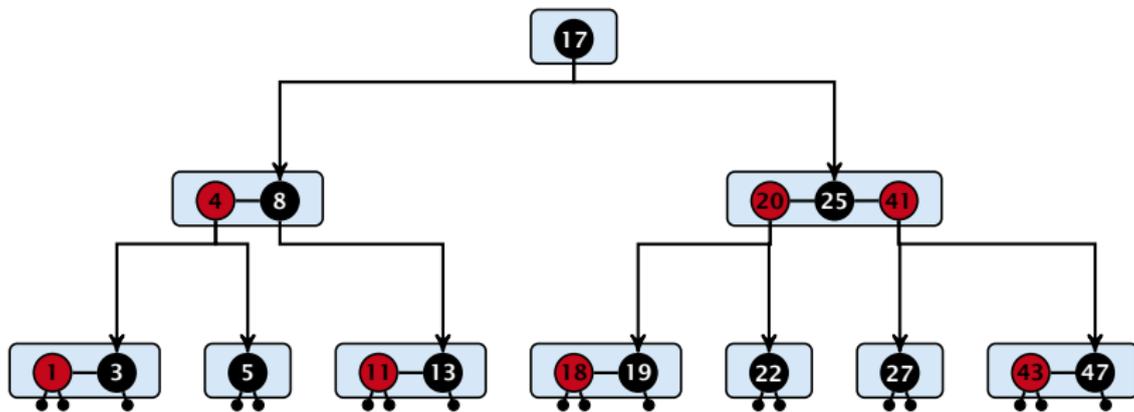
There is a close relation between red-black trees and (2,4)-trees:



Then, color one key in each internal node  $v$  black. If  $v$  contains 3 keys you need to select the middle key otherwise choose a black key arbitrarily. The other keys are colored red.

## (2, 4)-trees and red black trees

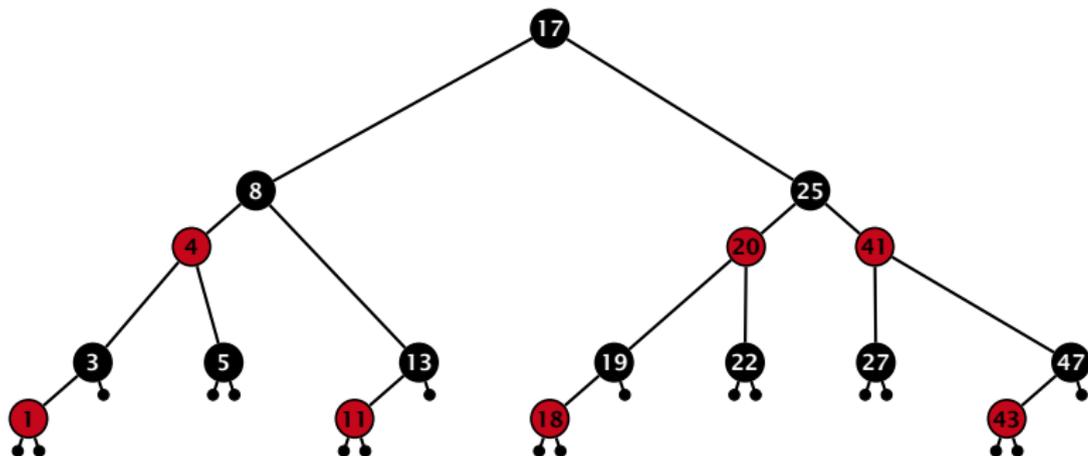
There is a close relation between red-black trees and (2, 4)-trees:



Re-attach the pointers to individual keys. A pointer that is between two keys is attached as a child of the red key. The incoming pointer, points to the black key.

## (2, 4)-trees and red black trees

There is a close relation between red-black trees and (2, 4)-trees:



Note that this correspondence is not unique. In particular, there are different red-black trees that correspond to the same (2, 4)-tree.

# Augmenting Data Structures

## Bibliography

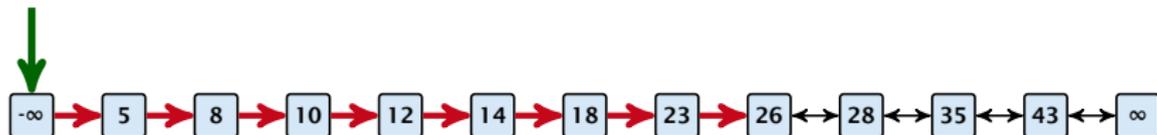
- [MS08] Kurt Mehlhorn, Peter Sanders:  
*Algorithms and Data Structures — The Basic Toolbox*,  
Springer, 2008
- [CLRS90] Thomas H. Cormen, Charles E. Leiserson, Ron L. Rivest, Clifford Stein:  
*Introduction to algorithms (3rd ed.)*,  
MIT Press and McGraw-Hill, 2009

A description of B-trees (a specific variant of  $(a, b)$ -trees) can be found in Chapter 18 of [CLRS90]. Chapter 7.2 of [MS08] discusses  $(a, b)$ -trees as discussed in the lecture.

## 7.6 Skip Lists

Why do we not use a list for implementing the ADT Dynamic Set?

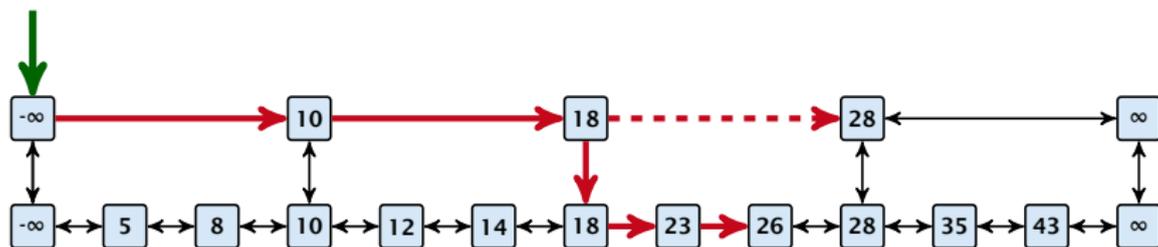
- ▶ time for search  $\Theta(n)$
- ▶ time for insert  $\Theta(n)$  (dominated by searching the item)
- ▶ time for delete  $\Theta(1)$  if we are given a handle to the object, otw.  $\Theta(n)$



## 7.6 Skip Lists

How can we improve the search-operation?

**Add an express lane:**



Let  $|L_1|$  denote the number of elements in the “express lane”, and  $|L_0| = n$  the number of all elements (ignoring dummy elements).

Worst case search time:  $|L_1| + \frac{|L_0|}{|L_1|}$  (ignoring additive constants)

Choose  $|L_1| = \sqrt{n}$ . Then search time  $\Theta(\sqrt{n})$ .

## 7.6 Skip Lists

Add more express lanes. Lane  $L_i$  contains roughly every  $\frac{L_{i-1}}{L_i}$ -th item from list  $L_{i-1}$ .

### Search(x) ( $k + 1$ lists $L_0, \dots, L_k$ )

- ▶ Find the largest item in list  $L_k$  that is smaller than  $x$ . At most  $|L_k| + 2$  steps.
- ▶ Find the largest item in list  $L_{k-1}$  that is smaller than  $x$ . At most  $\lceil \frac{|L_{k-1}|}{|L_k|+1} \rceil + 2$  steps.
- ▶ Find the largest item in list  $L_{k-2}$  that is smaller than  $x$ . At most  $\lceil \frac{|L_{k-2}|}{|L_{k-1}|+1} \rceil + 2$  steps.
- ▶ ...
- ▶ At most  $|L_k| + \sum_{i=1}^k \frac{L_{i-1}}{L_i} + 3(k + 1)$  steps.

## 7.6 Skip Lists

Choose ratios between list-lengths evenly, i.e.,  $\frac{|L_{i-1}|}{|L_i|} = r$ , and, hence,  $L_k \approx r^{-k}n$ .

Worst case running time is:  $\mathcal{O}(r^{-k}n + kr)$ .

Choose  $r = n^{\frac{1}{k+1}}$ . Then

$$\begin{aligned}r^{-k}n + kr &= \left(n^{\frac{1}{k+1}}\right)^{-k}n + kn^{\frac{1}{k+1}} \\ &= n^{1-\frac{k}{k+1}} + kn^{\frac{1}{k+1}} \\ &= (k+1)n^{\frac{1}{k+1}}.\end{aligned}$$

Choosing  $k = \Theta(\log n)$  gives a logarithmic running time.

## 7.6 Skip Lists

### How to do insert and delete?

- ▶ If we want that in  $L_i$  we always skip over roughly the same number of elements in  $L_{i-1}$  an insert or delete may require a lot of re-organisation.

**Use randomization instead!**

## 7.6 Skip Lists

### Insert:

- ▶ A search operation gives you the insert position for element  $x$  in every list.
- ▶ Flip a coin until it shows head, and record the number  $t \in \{1, 2, \dots\}$  of trials needed.
- ▶ Insert  $x$  into lists  $L_0, \dots, L_{t-1}$ .

### Delete:

- ▶ You get all predecessors via backward pointers.
- ▶ Delete  $x$  in all lists it actually appears in.

**The time for both operations is dominated by the search time.**



# High Probability

## Definition 10 (High Probability)

We say a **randomized** algorithm has running time  $\mathcal{O}(\log n)$  with **high probability** if for any constant  $\alpha$  the running time is at most  $\mathcal{O}(\log n)$  with probability at least  $1 - \frac{1}{n^\alpha}$ .

Here the  $\mathcal{O}$ -notation hides a constant that may depend on  $\alpha$ .

# High Probability

Suppose there are a **polynomially** many events  $E_1, E_2, \dots, E_\ell$ ,  $\ell = n^c$  each holding with high probability (e.g.  $E_i$  may be the event that the  $i$ -th search in a skip list takes time at most  $\mathcal{O}(\log n)$ ).

Then the probability that all  $E_i$  hold is at least

$$\begin{aligned}\Pr[E_1 \wedge \dots \wedge E_\ell] &= 1 - \Pr[\bar{E}_1 \vee \dots \vee \bar{E}_\ell] \\ &\geq 1 - n^c \cdot n^{-\alpha} \\ &= 1 - n^{c-\alpha} .\end{aligned}$$

This means  $\Pr[E_1 \wedge \dots \wedge E_\ell]$  holds with high probability.

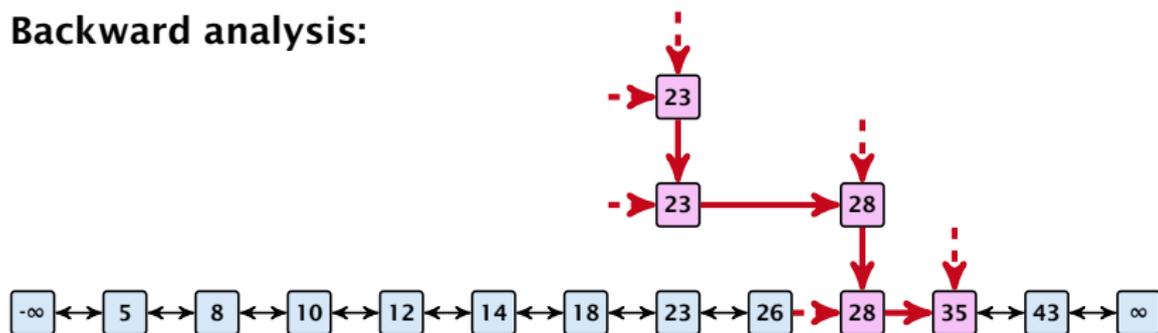
## 7.6 Skip Lists

### Lemma 11

*A search (and, hence, also insert and delete) in a skip list with  $n$  elements takes time  $\mathcal{O}(\log n)$  with high probability (w. h. p.).*

## 7.6 Skip Lists

Backward analysis:



At each point the path goes up with probability  $1/2$  and left with probability  $1/2$ .

We show that w.h.p.:

- ▶ A “long” search path must also go very high.
- ▶ There are no elements in high lists.

From this it follows that w.h.p. there are no long paths.

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{en}{k}\right)^k$$

$$\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!} = \frac{n \cdot \dots \cdot (n-k+1)}{k \cdot \dots \cdot 1} \geq \left(\frac{n}{k}\right)^k$$

$$\begin{aligned} \binom{n}{k} &= \frac{n \cdot \dots \cdot (n-k+1)}{k!} \leq \frac{n^k}{k!} = \frac{n^k \cdot k^k}{k^k \cdot k!} \\ &= \left(\frac{n}{k}\right)^k \cdot \frac{k^k}{k!} \leq \left(\frac{en}{k}\right)^k \end{aligned}$$

## 7.6 Skip Lists

Let  $E_{z,k}$  denote the event that a search path is of length  $z$  (number of edges) but does not visit a list above  $L_k$ .

In particular, this means that during the construction in the backward analysis we see at most  $k$  heads (i.e., coin flips that tell you to go up) in  $z$  trials.

## 7.6 Skip Lists

$$\Pr[E_{z,k}] \leq \Pr[\text{at most } k \text{ heads in } z \text{ trials}]$$

$$\leq \binom{z}{k} 2^{-(z-k)} \leq \left(\frac{ez}{k}\right)^k 2^{-(z-k)} \leq \left(\frac{2ez}{k}\right)^k 2^{-z}$$

choosing  $k = \gamma \log n$  with  $\gamma \geq 1$  and  $z = (\beta + \alpha)\gamma \log n$

$$\begin{aligned} &\leq \left(\frac{2ez}{k}\right)^k 2^{-\beta k} \cdot n^{-\gamma\alpha} \leq \left(\frac{2ez}{2^\beta k}\right)^k \cdot n^{-\alpha} \\ &\leq \left(\frac{2e(\beta + \alpha)}{2^\beta}\right)^k n^{-\alpha} \end{aligned}$$

now choosing  $\beta = 6\alpha$  gives

$$\leq \left(\frac{42\alpha}{64\alpha}\right)^k n^{-\alpha} \leq n^{-\alpha}$$

for  $\alpha \geq 1$ .

## 7.6 Skip Lists

So far we fixed  $k = \gamma \log n$ ,  $\gamma \geq 1$ , and  $z = 7\alpha\gamma \log n$ ,  $\alpha \geq 1$ .

This means that a search path of length  $\Omega(\log n)$  visits a list on a level  $\Omega(\log n)$ , w.h.p.

Let  $A_{k+1}$  denote the event that the list  $L_{k+1}$  is non-empty. Then

$$\Pr[A_{k+1}] \leq n2^{-(k+1)} \leq n^{-(\gamma-1)} .$$

For the search to take at least  $z = 7\alpha\gamma \log n$  steps either the event  $E_{z,k}$  or the event  $A_{k+1}$  must hold.

Hence,

$$\begin{aligned} \Pr[\text{search requires } z \text{ steps}] &\leq \Pr[E_{z,k}] + \Pr[A_{k+1}] \\ &\leq n^{-\alpha} + n^{-(\gamma-1)} \end{aligned}$$

This means, the search requires at most  $z$  steps, w. h. p.

# Skip Lists

## Bibliography

[GT98] Michael T. Goodrich, Roberto Tamassia  
*Data Structures and Algorithms in JAVA*,  
John Wiley, 1998

Skip lists are covered in Chapter 7.5 of [GT98].

## 7.7 Hashing

### Dictionary:

- ▶  **$S.insert(x)$** : Insert an element  $x$ .
- ▶  **$S.delete(x)$** : Delete the element pointed to by  $x$ .
- ▶  **$S.search(k)$** : Return a pointer to an element  $e$  with  $key[e] = k$  in  $S$  if it exists; otherwise return **null**.

So far we have implemented the search for a key by carefully choosing split-elements.

Then the memory location of an object  $x$  with key  $k$  is determined by successively comparing  $k$  to split-elements.

**Hashing** tries to **directly** compute the memory location from the given key. The goal is to have constant search time.

## 7.7 Hashing

### Definitions:

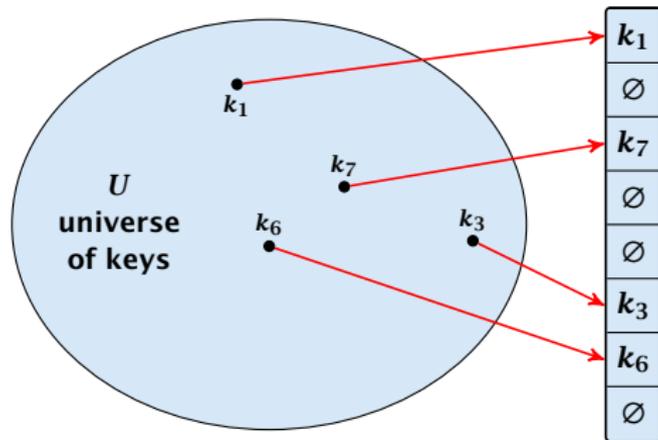
- ▶ Universe  $U$  of keys, e.g.,  $U \subseteq \mathbb{N}_0$ .  $U$  very large.
- ▶ Set  $S \subseteq U$  of keys,  $|S| = m \leq |U|$ .
- ▶ Array  $T[0, \dots, n - 1]$  hash-table.
- ▶ Hash function  $h : U \rightarrow [0, \dots, n - 1]$ .

### The hash-function $h$ should fulfill:

- ▶ Fast to evaluate.
- ▶ Small storage requirement.
- ▶ Good distribution of elements over the whole table.

## Direct Addressing

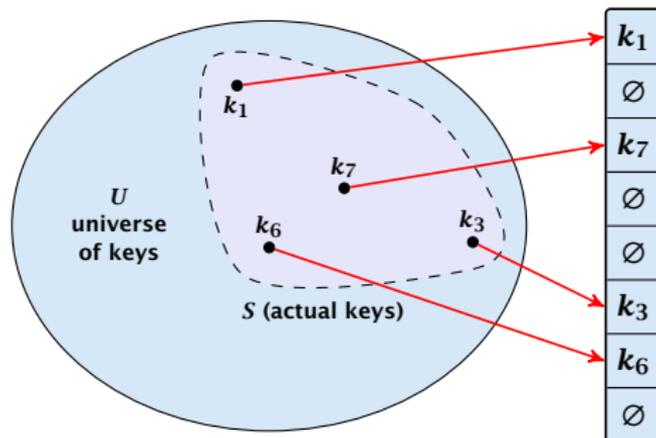
Ideally the hash function maps **all** keys to different memory locations.



This special case is known as **Direct Addressing**. It is usually very unrealistic as the universe of keys typically is quite large, and in particular larger than the available memory.

# Perfect Hashing

Suppose that we **know** the set  $S$  of actual keys (no insert/no delete). Then we may want to design a **simple** hash-function that maps all these keys to different memory locations.



Such a hash function  $h$  is called a **perfect hash function** for set  $S$ .

# Collisions

If we do not know the keys in advance, the best we can hope for is that the hash function distributes keys evenly across the table.

## Problem: Collisions

Usually the universe  $U$  is much larger than the table-size  $n$ .

Hence, there may be two elements  $k_1, k_2$  from the set  $S$  that map to the same memory location (i.e.,  $h(k_1) = h(k_2)$ ). This is called a **collision**.

# Collisions

Typically, collisions do not appear once the size of the set  $S$  of actual keys gets close to  $n$ , but already when  $|S| \geq \omega(\sqrt{n})$ .

## Lemma 12

*The probability of having a collision when hashing  $m$  elements into a table of size  $n$  under uniform hashing is at least*

$$1 - e^{-\frac{m(m-1)}{2n}} \approx 1 - e^{-\frac{m^2}{2n}} .$$

### Uniform hashing:

Choose a hash function uniformly at random from all functions  $f : U \rightarrow [0, \dots, n-1]$ .

# Collisions

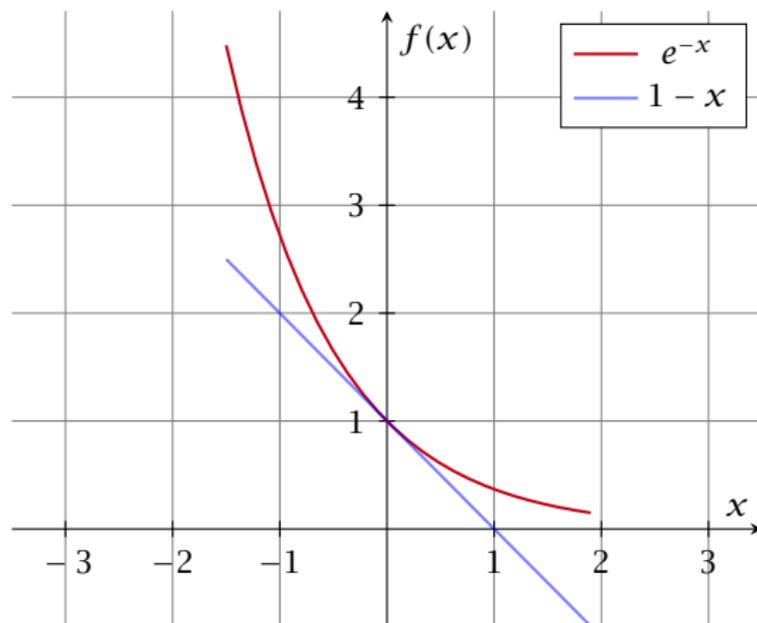
## Proof.

Let  $A_{m,n}$  denote the event that inserting  $m$  keys into a table of size  $n$  does **not** generate a collision. Then

$$\begin{aligned}\Pr[A_{m,n}] &= \prod_{\ell=1}^m \frac{n - \ell + 1}{n} = \prod_{j=0}^{m-1} \left(1 - \frac{j}{n}\right) \\ &\leq \prod_{j=0}^{m-1} e^{-j/n} = e^{-\sum_{j=0}^{m-1} \frac{j}{n}} = e^{-\frac{m(m-1)}{2n}}.\end{aligned}$$

Here the first equality follows since the  $\ell$ -th element that is hashed has a probability of  $\frac{n-\ell+1}{n}$  to not generate a collision under the condition that the previous elements did not induce collisions. □

# Collisions



The inequality  $1 - x \leq e^{-x}$  is derived by stopping the Taylor-expansion of  $e^{-x}$  after the second term.

# Resolving Collisions

The methods for dealing with collisions can be classified into the two main types

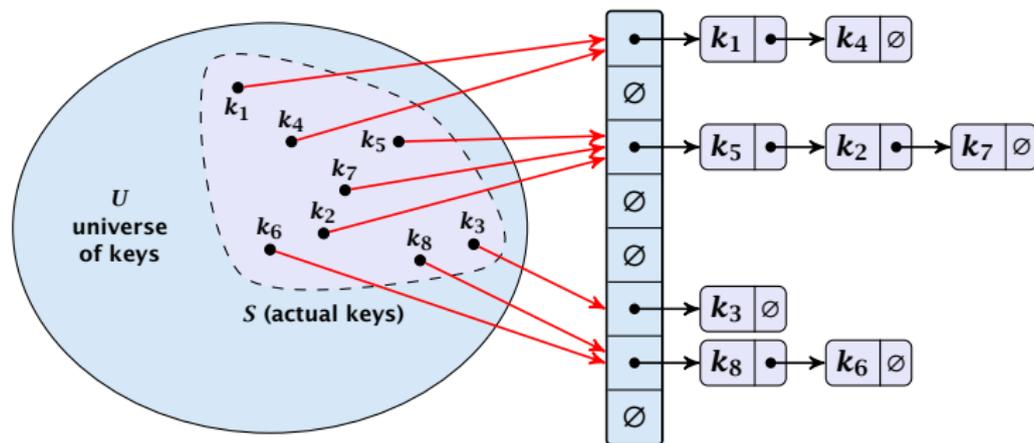
- ▶ **open addressing**, aka. closed hashing
- ▶ **hashing with chaining**, aka. closed addressing, open hashing.

There are applications e.g. computer chess where you do not resolve collisions at all.

# Hashing with Chaining

Arrange elements that map to the same position in a linear list.

- ▶ Access: compute  $h(x)$  and search list for  $\text{key}[x]$ .
- ▶ Insert: insert at the front of the list.



# Hashing with Chaining

Let  $A$  denote a strategy for resolving collisions. We use the following notation:

- ▶  $A^+$  denotes the average time for a **successful** search when using  $A$ ;
- ▶  $A^-$  denotes the average time for an **unsuccessful** search when using  $A$ ;
- ▶ We parameterize the complexity results in terms of  $\alpha := \frac{m}{n}$ , the so-called **fill factor** of the hash-table.

We assume **uniform hashing** for the following analysis.

# Hashing with Chaining

The time required for an unsuccessful search is 1 plus the length of the list that is examined. The average length of a list is  $\alpha = \frac{m}{n}$ . Hence, if  $A$  is the collision resolving strategy “Hashing with Chaining” we have

$$A^- = 1 + \alpha .$$

## Hashing with Chaining

For a successful search observe that we do **not** choose a list at random, but we consider a random key  $k$  in the hash-table and ask for the search-time for  $k$ .

This is 1 plus the number of elements that lie before  $k$  in  $k$ 's list.

Let  $k_\ell$  denote the  $\ell$ -th key inserted into the table.

Let for two keys  $k_i$  and  $k_j$ ,  $X_{ij}$  denote the indicator variable for the event that  $k_i$  and  $k_j$  hash to the same position. Clearly,  $\Pr[X_{ij} = 1] = 1/n$  for uniform hashing.

The expected successful search cost is

$$\mathbb{E} \left[ \frac{1}{m} \sum_{i=1}^m \left( 1 + \sum_{\substack{j=i+1 \\ \text{keys before } k_i}}^m X_{ij} \right) \right]$$

cost for key  $k_i$

## Hashing with Chaining

$$\begin{aligned} E \left[ \frac{1}{m} \sum_{i=1}^m \left( 1 + \sum_{j=i+1}^m X_{ij} \right) \right] &= \frac{1}{m} \sum_{i=1}^m \left( 1 + \sum_{j=i+1}^m E[X_{ij}] \right) \\ &= \frac{1}{m} \sum_{i=1}^m \left( 1 + \sum_{j=i+1}^m \frac{1}{n} \right) \\ &= 1 + \frac{1}{mn} \sum_{i=1}^m (m - i) \\ &= 1 + \frac{1}{mn} \left( m^2 - \frac{m(m+1)}{2} \right) \\ &= 1 + \frac{m-1}{2n} = 1 + \frac{\alpha}{2} - \frac{\alpha}{2m} . \end{aligned}$$

Hence, the expected cost for a successful search is  $A^+ \leq 1 + \frac{\alpha}{2}$ .

# Hashing with Chaining

## Disadvantages:

- ▶ pointers increase memory requirements
- ▶ pointers may lead to bad cache efficiency

## Advantages:

- ▶ no à priori limit on the number of elements
- ▶ deletion can be implemented efficiently
- ▶ by using balanced trees instead of linked list one can also obtain worst-case guarantees.

# Open Addressing

All objects are stored in the table itself.

Define a function  $h(k, j)$  that determines the table-position to be examined in the  $j$ -th step. The values  $h(k, 0), \dots, h(k, n - 1)$  must form a permutation of  $0, \dots, n - 1$ .

**Search( $k$ ):** Try position  $h(k, 0)$ ; if it is empty your search fails; otw. continue with  $h(k, 1), h(k, 2), \dots$

**Insert( $x$ ):** Search until you find an empty slot; insert your element there. If your search reaches  $h(k, n - 1)$ , and this slot is non-empty then your table is full.

# Open Addressing

Choices for  $h(k, j)$ :

- ▶ **Linear probing:**

$$h(k, i) = h(k) + i \pmod n$$

(sometimes:  $h(k, i) = h(k) + ci \pmod n$ ).

- ▶ **Quadratic probing:**

$$h(k, i) = h(k) + c_1i + c_2i^2 \pmod n.$$

- ▶ **Double hashing:**

$$h(k, i) = h_1(k) + ih_2(k) \pmod n.$$

For quadratic probing and double hashing one has to ensure that the search covers all positions in the table (i.e., for double hashing  $h_2(k)$  must be relatively prime to  $n$  (**teilerfremd**); for quadratic probing  $c_1$  and  $c_2$  have to be chosen carefully).

## Linear Probing

- ▶ Advantage: **Cache-efficiency**. The new probe position is very likely to be in the cache.
- ▶ Disadvantage: **Primary clustering**. Long sequences of occupied table-positions get longer as they have a larger probability to be hit. Furthermore, they can merge forming larger sequences.

### Lemma 13

Let  $L$  be the method of linear probing for resolving collisions:

$$L^+ \approx \frac{1}{2} \left( 1 + \frac{1}{1 - \alpha} \right)$$

$$L^- \approx \frac{1}{2} \left( 1 + \frac{1}{(1 - \alpha)^2} \right)$$

## Quadratic Probing

- ▶ Not as cache-efficient as Linear Probing.
- ▶ **Secondary clustering**: caused by the fact that all keys mapped to the same position have the same probe sequence.

### Lemma 14

Let  $Q$  be the method of quadratic probing for resolving collisions:

$$Q^+ \approx 1 + \ln\left(\frac{1}{1-\alpha}\right) - \frac{\alpha}{2}$$

$$Q^- \approx \frac{1}{1-\alpha} + \ln\left(\frac{1}{1-\alpha}\right) - \alpha$$

# Double Hashing

- ▶ Any probe into the hash-table usually creates a cache-miss.

## Lemma 15

Let  $A$  be the method of double hashing for resolving collisions:

$$D^+ \approx \frac{1}{\alpha} \ln \left( \frac{1}{1 - \alpha} \right)$$

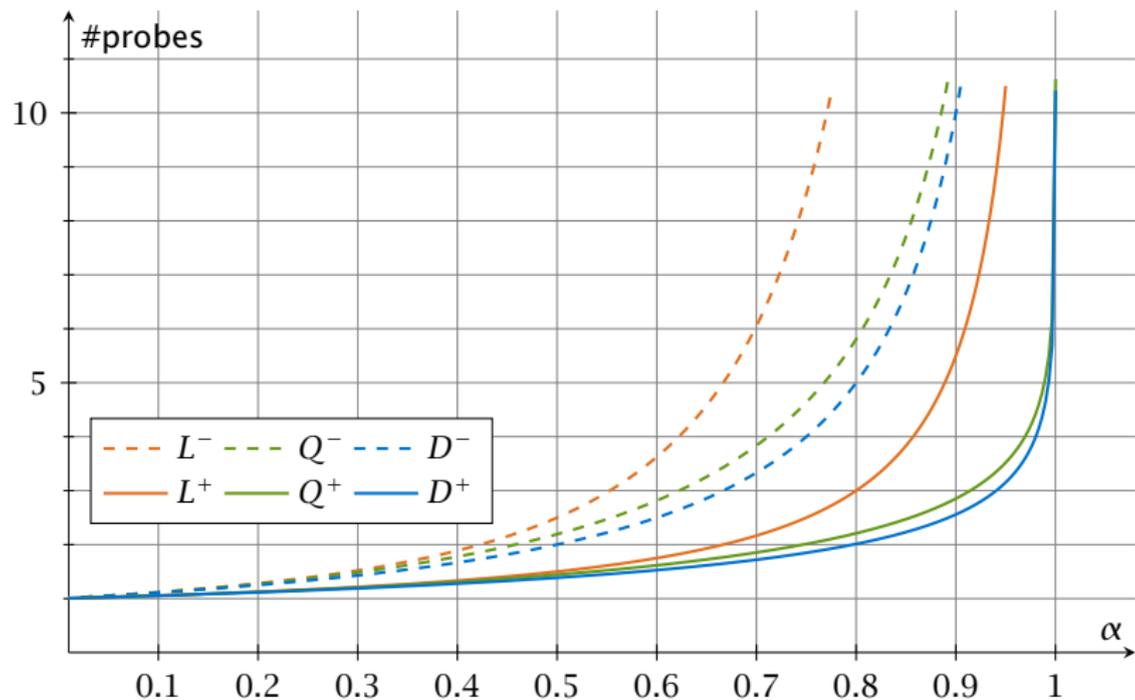
$$D^- \approx \frac{1}{1 - \alpha}$$

# Open Addressing

Some values:

$\alpha$	<i>Linear Probing</i>		<i>Quadratic Probing</i>		<i>Double Hashing</i>	
	$L^+$	$L^-$	$Q^+$	$Q^-$	$D^+$	$D^-$
0.5	1.5	2.5	1.44	2.19	1.39	2
0.9	5.5	50.5	2.85	11.40	2.55	10
0.95	10.5	200.5	3.52	22.05	3.15	20

# Open Addressing



# Analysis of Idealized Open Address Hashing

We analyze the time for a search in a very idealized Open Addressing scheme.

- ▶ The probe sequence  $h(k, 0), h(k, 1), h(k, 2), \dots$  is equally likely to be any permutation of  $\langle 0, 1, \dots, n - 1 \rangle$ .

## Analysis of Idealized Open Address Hashing

Let  $X$  denote a random variable describing the number of probes in an **unsuccessful** search.

Let  $A_i$  denote the event that the  $i$ -th probe **occurs** and is to a non-empty slot.

$$\begin{aligned}\Pr[A_1 \cap A_2 \cap \dots \cap A_{i-1}] \\ &= \Pr[A_1] \cdot \Pr[A_2 \mid A_1] \cdot \Pr[A_3 \mid A_1 \cap A_2] \cdot \\ &\quad \dots \cdot \Pr[A_{i-1} \mid A_1 \cap \dots \cap A_{i-2}]\end{aligned}$$

$$\begin{aligned}\Pr[X \geq i] &= \frac{m}{n} \cdot \frac{m-1}{n-1} \cdot \frac{m-2}{n-2} \cdot \dots \cdot \frac{m-i+2}{n-i+2} \\ &\leq \left(\frac{m}{n}\right)^{i-1} = \alpha^{i-1} .\end{aligned}$$

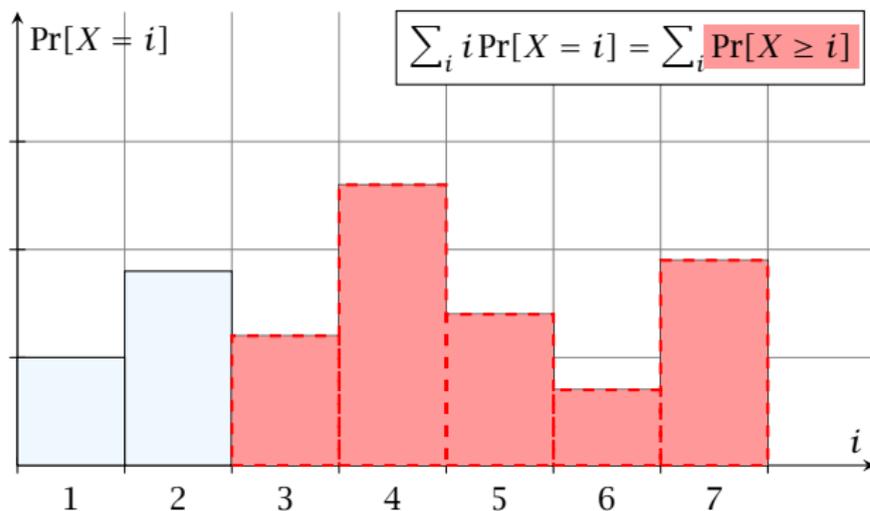
# Analysis of Idealized Open Address Hashing

$$E[X] = \sum_{i=1}^{\infty} \Pr[X \geq i] \leq \sum_{i=1}^{\infty} \alpha^{i-1} = \sum_{i=0}^{\infty} \alpha^i = \frac{1}{1-\alpha} .$$

$$\frac{1}{1-\alpha} = 1 + \alpha + \alpha^2 + \alpha^3 + \dots$$

# Analysis of Idealized Open Address Hashing

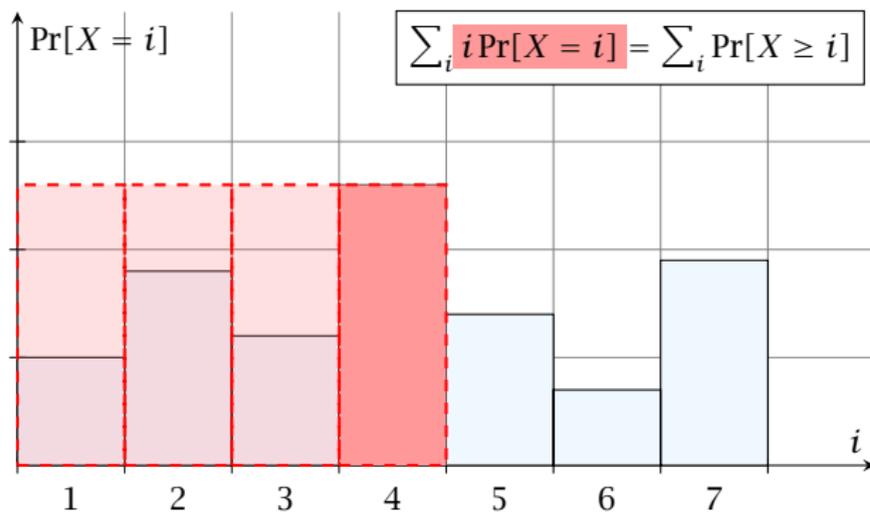
$$i = 3$$



The  $j$ -th rectangle appears in both sums  $j$  times. ( $j$  times in the first due to multiplication with  $j$ ; and  $j$  times in the second for summands  $i = 1, 2, \dots, j$ )

# Analysis of Idealized Open Address Hashing

$$i = 4$$



The  $j$ -th rectangle appears in both sums  $j$  times. ( $j$  times in the first due to multiplication with  $j$ ; and  $j$  times in the second for summands  $i = 1, 2, \dots, j$ )

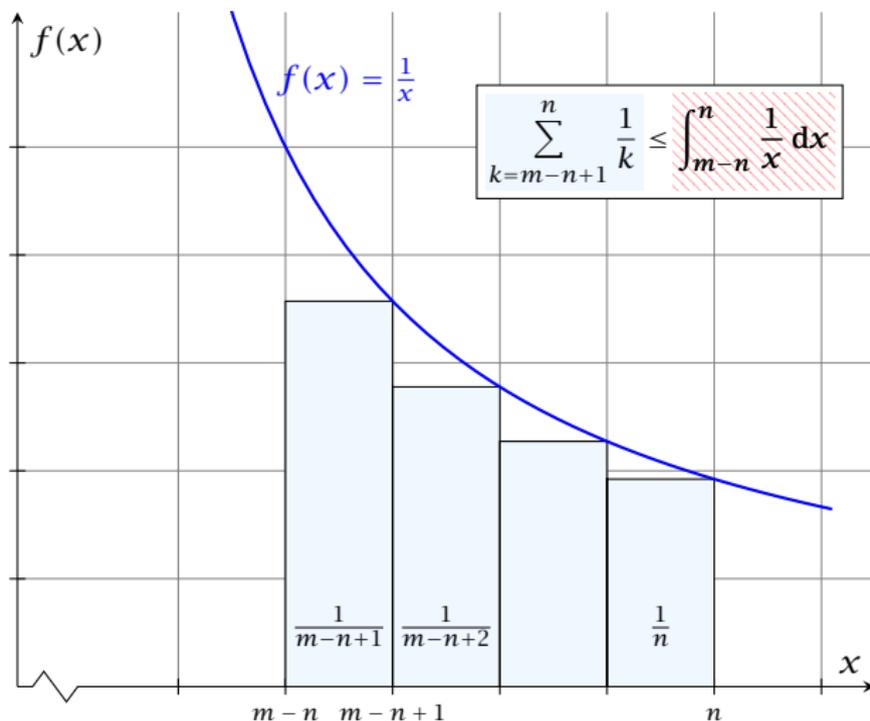
# Analysis of Idealized Open Address Hashing

The number of probes in a **successful** search for  $k$  is equal to the number of probes made in an unsuccessful search for  $k$  at the time that  $k$  is inserted.

Let  $k$  be the  $i + 1$ -st element. The expected time for a search for  $k$  is at most  $\frac{1}{1-i/n} = \frac{n}{n-i}$ .

$$\begin{aligned} \frac{1}{m} \sum_{i=0}^{m-1} \frac{n}{n-i} &= \frac{n}{m} \sum_{i=0}^{m-1} \frac{1}{n-i} = \frac{1}{\alpha} \sum_{k=n-m+1}^n \frac{1}{k} \\ &\leq \frac{1}{\alpha} \int_{n-m}^n \frac{1}{x} dx = \frac{1}{\alpha} \ln \frac{n}{n-m} = \frac{1}{\alpha} \ln \frac{1}{1-\alpha} . \end{aligned}$$

# Analysis of Idealized Open Address Hashing



# Deletions in Hashtables

## How do we delete in a hash-table?

- ▶ For hashing with chaining this is not a problem. Simply search for the key, and delete the item in the corresponding list.
- ▶ For open addressing this is difficult.

## Deletions in Hashtables

- ▶ Simply removing a key might interrupt the probe sequence of other keys which then cannot be found anymore.
- ▶ One can delete an element by replacing it with a **deleted**-marker.
  - ▶ During an insertion if a **deleted**-marker is encountered an element can be inserted there.
  - ▶ During a search a **deleted**-marker must not be used to terminate the probe sequence.
- ▶ The table could fill up with **deleted**-markers leading to bad performance.
- ▶ If a table contains many deleted-markers (linear fraction of the keys) one can rehash the whole table and amortize the cost for this rehash against the cost for the deletions.

# Deletions for Linear Probing

- ▶ For Linear Probing one can delete elements without using **deletion**-markers.
- ▶ Upon a deletion elements that are further down in the probe-sequence may be moved to guarantee that they are still found during a search.

## Deletions for Linear Probing

### Algorithm 16 delete( $p$ )

```
1:  $T[p] \leftarrow \text{null}$ 
2:  $p \leftarrow \text{succ}(p)$ 
3: while  $T[p] \neq \text{null}$  do
4:    $y \leftarrow T[p]$ 
5:    $T[p] \leftarrow \text{null}$ 
6:    $p \leftarrow \text{succ}(p)$ 
7:   insert( $y$ )
```

$p$  is the index into the table-cell that contains the object to be deleted.

Pointers into the hash-table become invalid.

# Universal Hashing

Regardless, of the choice of hash-function there is always an input (a set of keys) that has a very poor worst-case behaviour.

Therefore, so far we assumed that the hash-function is random so that regardless of the input the average case behaviour is good.

However, the assumption of uniform hashing that  $h$  is chosen randomly from all functions  $f : U \rightarrow [0, \dots, n - 1]$  is clearly unrealistic as there are  $n^{|U|}$  such functions. Even writing down such a function would take  $|U| \log n$  bits.

Universal hashing tries to define a set  $\mathcal{H}$  of functions that is much smaller but still leads to good average case behaviour when selecting a hash-function uniformly at random from  $\mathcal{H}$ .

# Universal Hashing

## Definition 16

A class  $\mathcal{H}$  of hash-functions from the universe  $U$  into the set  $\{0, \dots, n-1\}$  is called **universal** if for all  $u_1, u_2 \in U$  with  $u_1 \neq u_2$

$$\Pr[h(u_1) = h(u_2)] \leq \frac{1}{n} ,$$

where the probability is w. r. t. the choice of a random hash-function from set  $\mathcal{H}$ .

Note that this means that the probability of a collision between two arbitrary elements is at most  $\frac{1}{n}$ .

# Universal Hashing

## Definition 17

A class  $\mathcal{H}$  of hash-functions from the universe  $U$  into the set  $\{0, \dots, n-1\}$  is called **2-independent** (pairwise independent) if the following two conditions hold

- ▶ For any key  $u \in U$ , and  $t \in \{0, \dots, n-1\}$   $\Pr[h(u) = t] = \frac{1}{n}$ , i.e., a key is distributed uniformly within the hash-table.
- ▶ For all  $u_1, u_2 \in U$  with  $u_1 \neq u_2$ , and for any two hash-positions  $t_1, t_2$ :

$$\Pr[h(u_1) = t_1 \wedge h(u_2) = t_2] \leq \frac{1}{n^2} .$$

This requirement clearly implies a universal hash-function.

# Universal Hashing

## Definition 18

A class  $\mathcal{H}$  of hash-functions from the universe  $U$  into the set  $\{0, \dots, n-1\}$  is called  **$k$ -independent** if for any choice of  $\ell \leq k$  distinct keys  $u_1, \dots, u_\ell \in U$ , and for any set of  $\ell$  not necessarily distinct hash-positions  $t_1, \dots, t_\ell$ :

$$\Pr[h(u_1) = t_1 \wedge \dots \wedge h(u_\ell) = t_\ell] \leq \frac{1}{n^\ell} ,$$

where the probability is w. r. t. the choice of a random hash-function from set  $\mathcal{H}$ .

# Universal Hashing

## Definition 19

A class  $\mathcal{H}$  of hash-functions from the universe  $U$  into the set  $\{0, \dots, n-1\}$  is called  $(\mu, k)$ -independent if for any choice of  $\ell \leq k$  distinct keys  $u_1, \dots, u_\ell \in U$ , and for any set of  $\ell$  not necessarily distinct hash-positions  $t_1, \dots, t_\ell$ :

$$\Pr[h(u_1) = t_1 \wedge \dots \wedge h(u_\ell) = t_\ell] \leq \frac{\mu}{n^\ell},$$

where the probability is w. r. t. the choice of a random hash-function from set  $\mathcal{H}$ .

# Universal Hashing

Let  $U := \{0, \dots, p-1\}$  for a prime  $p$ . Let  $\mathbb{Z}_p := \{0, \dots, p-1\}$ , and let  $\mathbb{Z}_p^* := \{1, \dots, p-1\}$  denote the set of invertible elements in  $\mathbb{Z}_p$ .

Define

$$h_{a,b}(x) := (ax + b \bmod p) \bmod n$$

## Lemma 20

*The class*

$$\mathcal{H} = \{h_{a,b} \mid a \in \mathbb{Z}_p^*, b \in \mathbb{Z}_p\}$$

*is a universal class of hash-functions from  $U$  to  $\{0, \dots, n-1\}$ .*

# Universal Hashing

## Proof.

Let  $x, y \in U$  be two distinct keys. We have to show that the probability of a collision is only  $1/n$ .

$$\blacktriangleright ax + b \not\equiv ay + b \pmod{p}$$

If  $x \neq y$  then  $(x - y) \not\equiv 0 \pmod{p}$ .

Multiplying with  $a \not\equiv 0 \pmod{p}$  gives

$$a(x - y) \not\equiv 0 \pmod{p}$$

where we use that  $\mathbb{Z}_p$  is a field (**Körper**) and, hence, has no zero divisors (**nullteilerfrei**).

# Universal Hashing

- ▶ The hash-function does not generate collisions before the  $(\text{mod } n)$ -operation. Furthermore, every choice  $(a, b)$  is mapped to a different pair  $(t_x, t_y)$  with  $t_x := ax + b$  and  $t_y := ay + b$ .

This holds because we can compute  $a$  and  $b$  when given  $t_x$  and  $t_y$ :

$$t_x \equiv ax + b \pmod{p}$$

$$t_y \equiv ay + b \pmod{p}$$

$$t_x - t_y \equiv a(x - y) \pmod{p}$$

$$t_y \equiv ay + b \pmod{p}$$

$$a \equiv (t_x - t_y)(x - y)^{-1} \pmod{p}$$

$$b \equiv t_y - ay \pmod{p}$$

# Universal Hashing

There is a one-to-one correspondence between hash-functions (pairs  $(a, b)$ ,  $a \neq 0$ ) and pairs  $(t_x, t_y)$ ,  $t_x \neq t_y$ .

Therefore, we can view the first step (before the  $\text{mod } n$ -operation) as choosing a pair  $(t_x, t_y)$ ,  $t_x \neq t_y$  uniformly at random.

What happens when we do the  $\text{mod } n$  operation?

Fix a value  $t_x$ . There are  $p - 1$  possible values for choosing  $t_y$ .

From the range  $0, \dots, p - 1$  the values  $t_x, t_x + n, t_x + 2n, \dots$  map to  $t_x$  after the modulo-operation. These are at most  $\lceil p/n \rceil$  values.

# Universal Hashing

As  $t_y \neq t_x$  there are

$$\left\lceil \frac{p}{n} \right\rceil - 1 \leq \frac{p}{n} + \frac{n-1}{n} - 1 \leq \frac{p-1}{n}$$

possibilities for choosing  $t_y$  such that the final hash-value creates a collision.

This happens with probability at most  $\frac{1}{n}$ .

# Universal Hashing

It is also possible to show that  $\mathcal{H}$  is an (almost) pairwise independent class of hash-functions.

$$\frac{\lfloor \frac{p}{n} \rfloor^2}{p(p-1)} \leq \Pr_{t_x \neq t_y \in \mathbb{Z}_p^2} \left[ \begin{array}{l} t_x \bmod n = h_1 \\ t_y \bmod n = h_2 \end{array} \right] \leq \frac{\lceil \frac{p}{n} \rceil^2}{p(p-1)}$$

Note that the middle is the probability that  $h(x) = h_1$  and  $h(y) = h_2$ . The total number of choices for  $(t_x, t_y)$  is  $p(p-1)$ . The number of choices for  $t_x$  ( $t_y$ ) such that  $t_x \bmod n = h_1$  ( $t_y \bmod n = h_2$ ) lies between  $\lfloor \frac{p}{n} \rfloor$  and  $\lceil \frac{p}{n} \rceil$ .

# Universal Hashing

## Definition 21

Let  $d \in \mathbb{N}$ ;  $q \geq (d + 1)n$  be a prime; and let  $\bar{a} \in \{0, \dots, q - 1\}^{d+1}$ . Define for  $x \in \{0, \dots, q - 1\}$

$$h_{\bar{a}}(x) := \left( \sum_{i=0}^d a_i x^i \bmod q \right) \bmod n .$$

Let  $\mathcal{H}_n^d := \{h_{\bar{a}} \mid \bar{a} \in \{0, \dots, q - 1\}^{d+1}\}$ . The class  $\mathcal{H}_n^d$  is  $(e, d + 1)$ -independent.

Note that in the previous case we had  $d = 1$  and chose  $a_d \neq 0$ .

# Universal Hashing

For the coefficients  $\bar{a} \in \{0, \dots, q-1\}^{d+1}$  let  $f_{\bar{a}}$  denote the polynomial

$$f_{\bar{a}}(x) = \left( \sum_{i=0}^d a_i x^i \right) \bmod q$$

The polynomial is defined by  $d+1$  distinct points.

# Universal Hashing

Fix  $\ell \leq d + 1$ ; let  $x_1, \dots, x_\ell \in \{0, \dots, q - 1\}$  be keys, and let  $t_1, \dots, t_\ell$  denote the corresponding hash-function values.

Let  $A^\ell = \{h_{\bar{a}} \in \mathcal{H} \mid h_{\bar{a}}(x_i) = t_i \text{ for all } i \in \{1, \dots, \ell\}\}$

Then

$$h_{\bar{a}} \in A^\ell \Leftrightarrow h_{\bar{a}} = f_{\bar{a}} \bmod n \text{ and}$$

$$f_{\bar{a}}(x_i) \in \underbrace{\{t_i + \alpha \cdot n \mid \alpha \in \{0, \dots, \lceil \frac{q}{n} \rceil - 1\}\}}_{=: B_i}$$

In order to obtain the cardinality of  $A^\ell$  we choose our polynomial by fixing  $d + 1$  points.

We first fix the values for inputs  $x_1, \dots, x_\ell$ .

We have

$$|B_1| \cdot \dots \cdot |B_\ell|$$

possibilities to do this (so that  $h_{\bar{a}}(x_i) = t_i$ ).

- $A^\ell$  denotes the set of hash-functions such that every  $x_i$  hits its pre-defined position  $t_i$ .
- $B_i$  is the set of positions that  $f_{\bar{a}}$  can hit so that  $h_{\bar{a}}$  still hits  $t_i$ .

# Universal Hashing

Now, we choose  $d - \ell + 1$  other inputs and choose their value arbitrarily. We have  $q^{d-\ell+1}$  possibilities to do this.

Therefore we have

$$|B_1| \cdot \dots \cdot |B_\ell| \cdot q^{d-\ell+1} \leq \left\lceil \frac{q}{n} \right\rceil^\ell \cdot q^{d-\ell+1}$$

possibilities to choose  $\bar{a}$  such that  $h_{\bar{a}} \in A_\ell$ .

# Universal Hashing

Therefore the probability of choosing  $h_{\bar{a}}$  from  $A_\ell$  is only

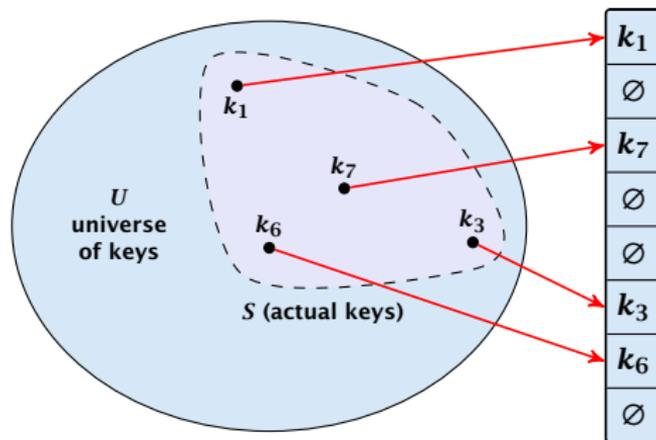
$$\begin{aligned}\frac{\lceil \frac{q}{n} \rceil^\ell \cdot q^{d-\ell+1}}{q^{d+1}} &\leq \frac{(\frac{q+n}{n})^\ell}{q^\ell} \leq \left(\frac{q+n}{q}\right)^\ell \cdot \frac{1}{n^\ell} \\ &\leq \left(1 + \frac{1}{\ell}\right)^\ell \cdot \frac{1}{n^\ell} \leq \frac{e}{n^\ell}.\end{aligned}$$

This shows that the  $\mathcal{H}$  is  $(e, d+1)$ -universal.

The last step followed from  $q \geq (d+1)n$ , and  $\ell \leq d+1$ .

# Perfect Hashing

Suppose that we **know** the set  $S$  of actual keys (no insert/no delete). Then we may want to design a **simple** hash-function that maps all these keys to different memory locations.



# Perfect Hashing

Let  $m = |S|$ . We could simply choose the hash-table size very large so that we don't get any collisions.

Using a universal hash-function the expected number of collisions is

$$E[\text{\#Collisions}] = \binom{m}{2} \cdot \frac{1}{n}.$$

If we choose  $n = m^2$  the **expected number** of collisions is strictly less than  $\frac{1}{2}$ .

Can we get an upper bound on the **probability of having collisions**?

The probability of having **1** or more collisions can be at most  $\frac{1}{2}$  as otherwise the expectation would be larger than  $\frac{1}{2}$ .

# Perfect Hashing

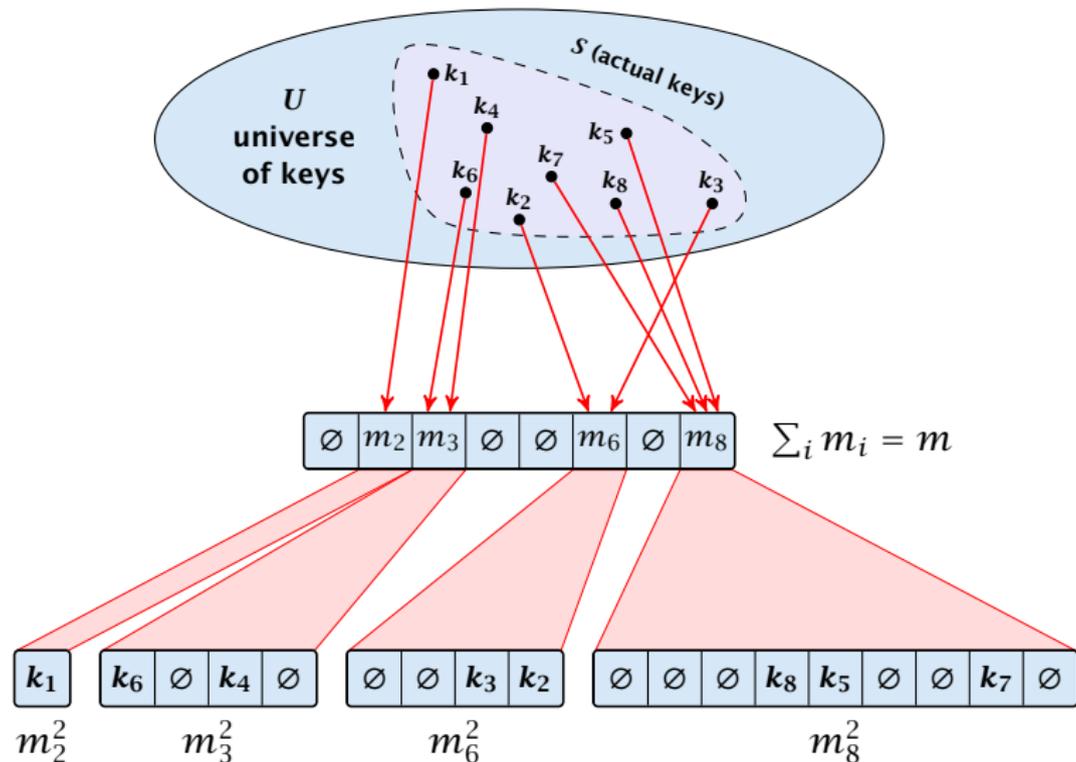
We can find such a hash-function by a few trials.

However, a hash-table size of  $n = m^2$  is very very high.

We construct a two-level scheme. We first use a hash-function that maps elements from  $S$  to  $m$  buckets.

Let  $m_j$  denote the number of items that are hashed to the  $j$ -th bucket. For each bucket we choose a second hash-function that maps the elements of the bucket into a table of size  $m_j^2$ . The second function can be chosen such that all elements are mapped to different locations.

# Perfect Hashing



# Perfect Hashing

The total memory that is required by all hash-tables is  $\mathcal{O}(\sum_j m_j^2)$ . Note that  $m_j$  is a random variable.

$$\begin{aligned} \mathbb{E} \left[ \sum_j m_j^2 \right] &= \mathbb{E} \left[ 2 \sum_j \binom{m_j}{2} + \sum_j m_j \right] \\ &= 2 \mathbb{E} \left[ \sum_j \binom{m_j}{2} \right] + \mathbb{E} \left[ \sum_j m_j \right] \end{aligned}$$

The first expectation is simply the expected number of collisions, for the first level. Since we use universal hashing we have

$$= 2 \binom{m}{2} \frac{1}{m} + m = 2m - 1 .$$

# Perfect Hashing

We need only  $\mathcal{O}(m)$  time to construct a hash-function  $h$  with  $\sum_j m_j^2 = \mathcal{O}(4m)$ , because with probability at least  $1/2$  a random function from a universal family will have this property.

Then we construct a hash-table  $h_j$  for every bucket. This takes expected time  $\mathcal{O}(m_j)$  for every bucket. A random function  $h_j$  is collision-free with probability at least  $1/2$ . We need  $\mathcal{O}(m_j)$  to test this.

We only need that the hash-functions are chosen from a universal family!!!

# Cuckoo Hashing

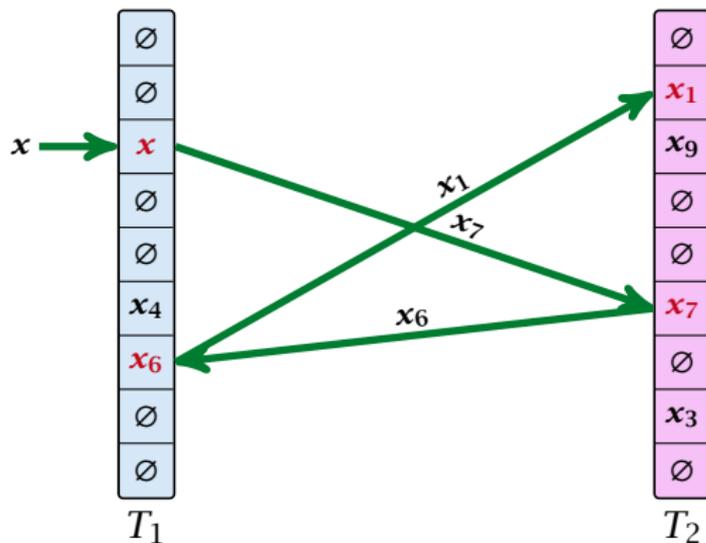
## Goal:

Try to generate a hash-table with constant worst-case search time in a dynamic scenario.

- ▶ Two hash-tables  $T_1[0, \dots, n - 1]$  and  $T_2[0, \dots, n - 1]$ , with hash-functions  $h_1$ , and  $h_2$ .
- ▶ An object  $x$  is either stored at location  $T_1[h_1(x)]$  or  $T_2[h_2(x)]$ .
- ▶ A search clearly takes constant time if the above constraint is met.

# Cuckoo Hashing

Insert:



## Algorithm 17 Cuckoo-Insert( $x$ )

```
1: if  $T_1[h_1(x)] = x \vee T_2[h_2(x)] = x$  then return  
2: steps  $\leftarrow 1$   
3: while steps  $\leq$  maxsteps do  
4:   exchange  $x$  and  $T_1[h_1(x)]$   
5:   if  $x = \text{null}$  then return  
6:   exchange  $x$  and  $T_2[h_2(x)]$   
7:   if  $x = \text{null}$  then return  
8:   steps  $\leftarrow$  steps + 1  
9: rehash() // change hash-functions; rehash everything  
10: Cuckoo-Insert( $x$ )
```

# Cuckoo Hashing

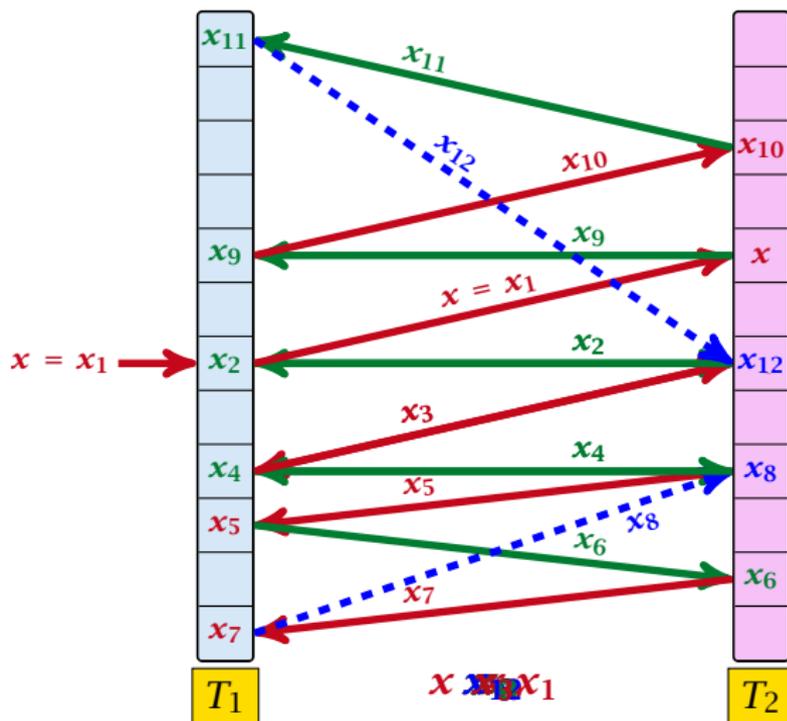
- ▶ We call one iteration through the while-loop a **step** of the algorithm.
- ▶ We call a sequence of iterations through the while-loop without the termination condition becoming true a **phase** of the algorithm.
- ▶ We say a phase is **successful** if it is not terminated by the **maxstep**-condition, but the while loop is left because  $x = \text{null}$ .

## What is the expected time for an insert-operation?

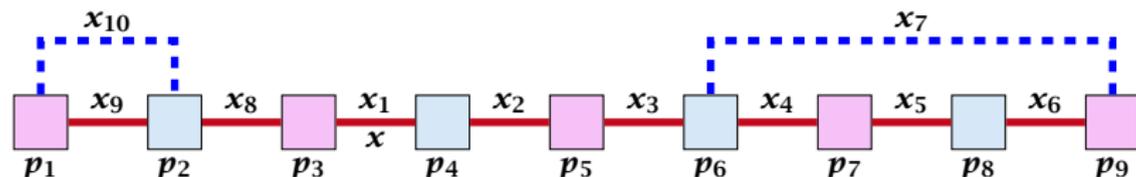
We first analyze the probability that we end-up in an infinite loop (that is then terminated after  $\text{maxsteps}$  steps).

Formally what is the probability to enter an infinite loop that touches  $s$  different keys?

# Cuckoo Hashing: Insert



# Cuckoo Hashing



A cycle-structure of size  $s$  is defined by

- ▶  $s - 1$  different cells (alternating btw. cells from  $T_1$  and  $T_2$ ).
- ▶  $s$  distinct keys  $x = x_1, x_2, \dots, x_s$ , linking the cells.
- ▶ The leftmost cell is “linked forward” to some cell on the right.
- ▶ The rightmost cell is “linked backward” to a cell on the left.
- ▶ One link represents key  $x$ ; this is where the counting starts.

# Cuckoo Hashing

A cycle-structure is **active** if for every key  $x_\ell$  (linking a cell  $p_i$  from  $T_1$  and a cell  $p_j$  from  $T_2$ ) we have

$$h_1(x_\ell) = p_i \quad \text{and} \quad h_2(x_\ell) = p_j$$

## Observation:

If during a phase the insert-procedure runs into a cycle there must exist an active cycle structure of size  $s \geq 3$ .

# Cuckoo Hashing

What is the probability that all keys in a cycle-structure of size  $s$  correctly map into their  $T_1$ -cell?

This probability is at most  $\frac{\mu}{n^s}$  since  $h_1$  is a  $(\mu, s)$ -independent hash-function.

What is the probability that all keys in the cycle-structure of size  $s$  correctly map into their  $T_2$ -cell?

This probability is at most  $\frac{\mu}{n^s}$  since  $h_2$  is a  $(\mu, s)$ -independent hash-function.

These events are independent.

# Cuckoo Hashing

The probability that a given cycle-structure of size  $s$  is active is at most  $\frac{\mu^2}{n^{2s}}$ .

What is the probability that **there exists** an active cycle structure of size  $s$ ?

# Cuckoo Hashing

The number of cycle-structures of size  $s$  is at most

$$s^3 \cdot n^{s-1} \cdot m^{s-1} .$$

- ▶ There are at most  $s^2$  possibilities where to attach the forward and backward links.
- ▶ There are at most  $s$  possibilities to choose where to place key  $x$ .
- ▶ There are  $m^{s-1}$  possibilities to choose the keys apart from  $x$ .
- ▶ There are  $n^{s-1}$  possibilities to choose the cells.

# Cuckoo Hashing

The probability that there exists an active cycle-structure is therefore at most

$$\begin{aligned} \sum_{s=3}^{\infty} s^3 \cdot n^{s-1} \cdot m^{s-1} \cdot \frac{\mu^2}{n^{2s}} &= \frac{\mu^2}{nm} \sum_{s=3}^{\infty} s^3 \left(\frac{m}{n}\right)^s \\ &\leq \frac{\mu^2}{m^2} \sum_{s=3}^{\infty} s^3 \left(\frac{1}{1+\epsilon}\right)^s \leq \mathcal{O}\left(\frac{1}{m^2}\right). \end{aligned}$$

Here we used the fact that  $(1 + \epsilon)m \leq n$ .

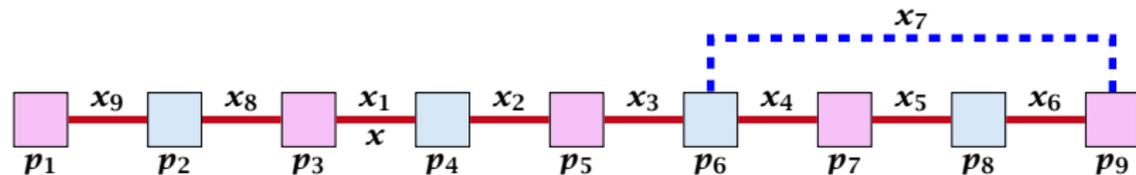
Hence,

$$\Pr[\text{cycle}] = \mathcal{O}\left(\frac{1}{m^2}\right).$$

# Cuckoo Hashing

Now, we analyze the probability that a phase is not successful without running into a closed cycle.

# Cuckoo Hashing



Sequence of visited keys:

$x = x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_3, x_2, x_1 = x, x_8, x_9, \dots$

# Cuckoo Hashing

Consider the sequence of not necessarily distinct keys starting with  $x$  in the order that they are visited during the phase.

## Lemma 22

*If the sequence is of length  $p$  then there exists a sub-sequence of at least  $\frac{p+2}{3}$  keys starting with  $x$  of *distinct* keys.*

# Cuckoo Hashing

Taking  $x_1 \rightarrow \dots \rightarrow x_i$  twice, and  $x_1 \rightarrow x_{i+1} \rightarrow \dots \rightarrow x_j$  once gives  $2i + (j - i + 1) = i + j + 1 \geq p + 2$  keys. Hence, one of the sequences contains at least  $(p + 2)/3$  keys.

## Proof.

Let  $i$  be the number of keys (including  $x$ ) that we see before the first repeated key. Let  $j$  denote the total number of distinct keys.

The sequence is of the form:

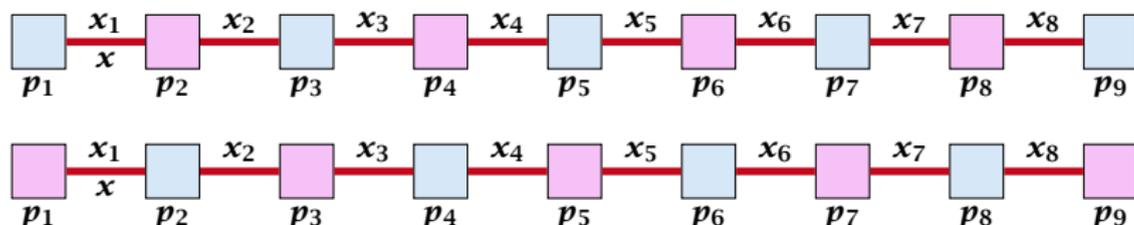
$$x = x_1 \rightarrow x_2 \rightarrow \dots \rightarrow x_i \rightarrow x_r \rightarrow x_{r-1} \rightarrow \dots \rightarrow x_1 \rightarrow x_{i+1} \rightarrow \dots \rightarrow x_j$$

As  $r \leq i - 1$  the length  $p$  of the sequence is

$$p = i + r + (j - i) \leq i + j - 1 .$$

Either sub-sequence  $x_1 \rightarrow x_2 \rightarrow \dots \rightarrow x_i$  or sub-sequence  $x_1 \rightarrow x_{i+1} \rightarrow \dots \rightarrow x_j$  has at least  $\frac{p+2}{3}$  elements. □

# Cuckoo Hashing



A path-structure of size  $s$  is defined by

- ▶  $s + 1$  different cells (alternating btw. cells from  $T_1$  and  $T_2$ ).
- ▶  $s$  distinct keys  $x = x_1, x_2, \dots, x_s$ , linking the cells.
- ▶ The leftmost cell is either from  $T_1$  or  $T_2$ .

# Cuckoo Hashing

A path-structure is **active** if for every key  $x_\ell$  (linking a cell  $p_i$  from  $T_1$  and a cell  $p_j$  from  $T_2$ ) we have

$$h_1(x_\ell) = p_i \quad \text{and} \quad h_2(x_\ell) = p_j$$

## Observation:

If a phase takes at least  $t$  steps without running into a cycle there must exist an active path-structure of size  $(2t + 2)/3$ .

Note that we count **complete** steps. A search that touches  $2t$  or  $2t + 1$  keys takes  $t$  steps.

# Cuckoo Hashing

The probability that a given path-structure of size  $s$  is active is at most  $\frac{\mu^2}{n^{2s}}$ .

The probability that there exists an active path-structure of size  $s$  is at most

$$\begin{aligned} & 2 \cdot n^{s+1} \cdot m^{s-1} \cdot \frac{\mu^2}{n^{2s}} \\ & \leq 2\mu^2 \left(\frac{m}{n}\right)^{s-1} \leq 2\mu^2 \left(\frac{1}{1+\epsilon}\right)^{s-1} \\ & \leq 2\mu^2 \left(\frac{1}{1+\epsilon}\right)^{(2t+2)/3-1} = 2\mu^2 \left(\frac{1}{1+\epsilon}\right)^{(2t-1)/3}. \end{aligned}$$

# Cuckoo Hashing

We choose  $\text{maxsteps} \geq 3\ell/2 + 2$ . Then the probability that a phase terminates unsuccessfully without running into a cycle is at most

$$\begin{aligned} & \Pr[\text{unsuccessful} \mid \text{no cycle}] \\ & \leq \Pr[\exists \text{ active path-structure of size at least } \frac{2\text{maxsteps}-1}{3}] \\ & \leq \Pr[\exists \text{ active path-structure of size at least } \ell + 1] \\ & \leq \Pr[\exists \text{ active path-structure of size exactly } \ell + 1] \\ & \leq 2\mu^2 \left(\frac{1}{1+\epsilon}\right)^\ell \leq \frac{1}{m^2} \end{aligned}$$

by choosing  $\ell \geq \log\left(\frac{1}{2\mu^2 m^2}\right) / \log\left(\frac{1}{1+\epsilon}\right) = \log(2\mu^2 m^2) / \log(1+\epsilon)$

This gives  $\text{maxsteps} = \Theta(\log m)$ .

Note that the existence of a path structure of size larger than  $s$  implies the existence of a path structure of size exactly  $s$ .

# Cuckoo Hashing

So far we estimated

$$\Pr[\text{cycle}] \leq \mathcal{O}\left(\frac{1}{m^2}\right)$$

and

$$\Pr[\text{unsuccessful} \mid \text{no cycle}] \leq \mathcal{O}\left(\frac{1}{m^2}\right)$$

Observe that

$$\begin{aligned}\Pr[\text{successful}] &= \Pr[\text{no cycle}] - \Pr[\text{unsuccessful} \mid \text{no cycle}] \\ &\geq c \cdot \Pr[\text{no cycle}]\end{aligned}$$

for a suitable constant  $c > 0$ .

This is a very weak (and trivial) statement but still sufficient for our asymptotic analysis.

# Cuckoo Hashing

The expected number of complete steps in the **successful phase** of an insert operation is:

$$\begin{aligned} & E[\text{number of steps} \mid \text{phase successful}] \\ &= \sum_{t \geq 1} \Pr[\text{search takes at least } t \text{ steps} \mid \text{phase successful}] \end{aligned}$$

We have

$$\begin{aligned} & \Pr[\text{search at least } t \text{ steps} \mid \text{successful}] \\ &= \Pr[\text{search at least } t \text{ steps} \wedge \text{successful}] / \Pr[\text{successful}] \\ &\leq \frac{1}{c} \Pr[\text{search at least } t \text{ steps} \wedge \text{successful}] / \Pr[\text{no cycle}] \\ &\leq \frac{1}{c} \Pr[\text{search at least } t \text{ steps} \wedge \text{no cycle}] / \Pr[\text{no cycle}] \\ &= \frac{1}{c} \Pr[\text{search at least } t \text{ steps} \mid \text{no cycle}] . \end{aligned}$$

$$\Pr[A \mid B] = \frac{\Pr[A \wedge B]}{\Pr[B]}$$

# Cuckoo Hashing

Hence,

$E[\text{number of steps} \mid \text{phase successful}]$

$$\begin{aligned} &\leq \frac{1}{c} \sum_{t \geq 1} \Pr[\text{search at least } t \text{ steps} \mid \text{no cycle}] \\ &\leq \frac{1}{c} \sum_{t \geq 1} 2\mu^2 \left(\frac{1}{1+\epsilon}\right)^{(2t-1)/3} = \frac{1}{c} \sum_{t \geq 0} 2\mu^2 \left(\frac{1}{1+\epsilon}\right)^{(2(t+1)-1)/3} \\ &= \frac{2\mu^2}{c(1+\epsilon)^{1/3}} \sum_{t \geq 0} \left(\frac{1}{(1+\epsilon)^{2/3}}\right)^t = \mathcal{O}(1) . \end{aligned}$$

This means the expected cost for a successful phase is constant (even after accounting for the cost of the incomplete step that finishes the phase).

# Cuckoo Hashing

A phase that is not successful induces cost for doing a complete rehash (this dominates the cost for the steps in the phase).

The probability that a phase is not successful is  $p = \mathcal{O}(1/m^2)$  (probability  $\mathcal{O}(1/m^2)$  of running into a cycle and probability  $\mathcal{O}(1/m^2)$  of reaching **maxsteps** without running into a cycle).

A rehash try requires  $m$  insertions and takes expected constant time per insertion. It fails with probability  $p := \mathcal{O}(1/m)$ .

The expected number of unsuccessful rehashes is

$$\sum_{i \geq 1} p^i = \frac{1}{1-p} - 1 = \frac{p}{1-p} = \mathcal{O}(p).$$

Therefore the expected cost for re-hashes is

$$\mathcal{O}(m) \cdot \mathcal{O}(p) = \mathcal{O}(1).$$

## Formal Proof

Let  $Y_i$  denote the event that the  $i$ -th rehash does not lead to a valid configuration (assuming  $i$ -th rehash occurs) (i.e., one of the  $m + 1$  insertions fails):

$$\Pr[Y_i] \leq (m + 1) \cdot \mathcal{O}(1/m^2) \leq \mathcal{O}(1/m) =: p .$$

Let  $Z_i$  denote the event that the  $i$ -th rehash occurs:

$$\Pr[Z_i] \leq \Pr[\wedge_{j=1}^{i-1} Y_j] \leq p^{i-1}$$

Let  $X_i^s$ ,  $s \in \{1, \dots, m + 1\}$  denote the cost for inserting the  $s$ -th element during the  $i$ -th rehash (assuming  $i$ -th rehash occurs):

$$\begin{aligned} \mathbb{E}[X_i^s] &= \mathbb{E}[\text{steps} \mid \text{phase successful}] \cdot \Pr[\text{phase successful}] \\ &\quad + \text{maxsteps} \cdot \Pr[\text{not successful}] = \mathcal{O}(1) . \end{aligned}$$

The expected cost for all rehashes is

$$\mathbb{E} \left[ \sum_i \sum_s Z_i X_i^s \right]$$

Note that  $Z_i$  is independent of  $X_j^s$ ,  $j \geq i$  (however, it is not independent of  $X_j^s$ ,  $j < i$ ). Hence,

$$\begin{aligned} \mathbb{E} \left[ \sum_i \sum_s Z_i X_i^s \right] &= \sum_i \sum_s \mathbb{E}[Z_i] \cdot \mathbb{E}[X_i^s] \\ &\leq \mathcal{O}(1) \cdot \sum_i p^{i-1} \\ &\leq \mathcal{O}(1) \cdot \frac{1}{1-p} \\ &= \mathcal{O}(1) . \end{aligned}$$

# Cuckoo Hashing

## What kind of hash-functions do we need?

Since  $\text{maxsteps}$  is  $\Theta(\log m)$  the largest size of a path-structure or cycle-structure contains just  $\Theta(\log m)$  different keys.

Therefore, it is sufficient to have  $(\mu, \Theta(\log m))$ -independent hash-functions.

# Cuckoo Hashing

How do we make sure that  $n \geq (1 + \epsilon)m$ ?

- ▶ Let  $\alpha := 1/(1 + \epsilon)$ .
- ▶ Keep track of the number of elements in the table. When  $m \geq \alpha n$  we double  $n$  and do a complete re-hash (**table-expand**).
- ▶ Whenever  $m$  drops below  $\alpha n/4$  we divide  $n$  by 2 and do a rehash (**table-shrink**).
- ▶ Note that right after a change in table-size we have  $m = \alpha n/2$ . In order for a table-expand to occur at least  $\alpha n/2$  insertions are required. Similar, for a table-shrink at least  $\alpha n/4$  deletions must occur.
- ▶ Therefore we can amortize the rehash cost after a change in table-size against the cost for insertions and deletions.

# Cuckoo Hashing

## Lemma 23

*Cuckoo Hashing has an expected constant insert-time and a worst-case constant search-time.*

Note that the above lemma only holds if the fill-factor (number of keys/total number of hash-table slots) is at most  $\frac{1}{2(1+\epsilon)}$ .

The  $1/(2(1+\epsilon))$  fill-factor comes from the fact that the total hash-table is of size  $2n$  (because we have two tables of size  $n$ ); moreover  $m \leq (1+\epsilon)n$ .

## Bibliography

- [MS08] Kurt Mehlhorn, Peter Sanders:  
*Algorithms and Data Structures — The Basic Toolbox*,  
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- [CLRS90] Thomas H. Cormen, Charles E. Leiserson, Ron L. Rivest, Clifford Stein:  
*Introduction to algorithms (3rd ed.)*,  
MIT Press and McGraw-Hill, 2009

Chapter 4 of [MS08] contains a detailed description about Hashing with Linear Probing and Hashing with Chaining. Also the Perfect Hashing scheme can be found there.

The analysis of Hashing with Chaining under the assumption of uniform hashing can be found in Chapter 11.2 of [CLRS90]. Chapter 11.3.3 describes Universal Hashing. Collision resolution with Open Addressing is described in Chapter 11.4. Chapter 11.5 describes the Perfect Hashing scheme.

Reference for Cuckoo Hashing???