

Part II

Foundations

Vocabularies

$a \cdot b$ “ a times b ”

“ a multiplied by b ”

“ a into b ”

$\frac{a}{b}$ “ a divided by b ”

“ a by b ”

“ a over b ”

(a : numerator (Zähler), b : denominator (Nenner))

a^b “ a raised to the b -th power”

“ a to the b -th”

“ a raised to the power of b ”

“ a to the power of b ”

“ a raised to b ”

“ a to the b ”

“ a raised by the exponent of b ”

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$\binom{n}{k}$ “ n choose k ”

x_i “ x subscript i ”

“ x sub i ”

“ x i ”

$\log_b a$ “log to the base b of a ”

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$$f : X \rightarrow Y, x \mapsto x^2$$

f is a function that maps from **domain** (Definitionsbereich) X to **codomain** (Zielmenge) Y . The set $\{y \in Y \mid \exists x \in X : f(x) = y\}$ is the **image** or the **range** of the function (Bildbereich/Wertebereich).

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3 Goals

- ▶ Gain knowledge about efficient algorithms for important problems, i.e., learn how to solve certain types of problems efficiently.
- ▶ Learn how to analyze and judge the efficiency of algorithms.
- ▶ Learn how to design efficient algorithms.

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4 Modelling Issues

1 What do you measure?

- ▶ **Memory requirement**
- ▶ Running time
- ▶ Number of comparisons
- ▶ Number of multiplications
- ▶ Number of hard-disc accesses
- ▶ Program size
- ▶ Power consumption
- ▶ ...

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4 Modelling Issues

How do you measure?

- ▶ Implementing and testing on representative inputs
 - ▶ How do you choose your inputs?
 - ▶ May be very time-consuming.
 - ▶ Very reliable results if done correctly.
 - ▶ Results only hold for a specific machine and for a specific set of inputs.

- ▶ Theoretical analysis in a specific **model of computation**.

Ques: How many comparisons does this algorithm always require?

Ans: $\frac{n(n-1)}{2}$

Typically focuses on the

Can this lower bound also be comparison-based sorting algorithm needs at least $\frac{n \log n}{2}$ comparisons in the worst case.

Worst case

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Ques: How many comparisons does this algorithm always take?

Typical focus on the

Can this lower bound be any comparison-based sorting algorithm needs at least $\log_2(n!)$ comparisons in the worst case.

What's more?

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Ques: How many steps does this algorithm always take?

Typical answer: $O(n^2)$

Can this lower bound be very computer- and/or programming language-specific?
Algorithm needs at least n comparisons in the worst case.

Why not?

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Quick question: How many steps does the following algorithm always take?

Print out the number 1000.

Can this lower bound be achieved by a computer-based sorting algorithm? What is the best asymptotic complexity for this problem?

1000 steps

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 - ▶ Gives **asymptotic bounds** like “this algorithm always runs in time $\mathcal{O}(n^2)$ ”.
 - ▶ Typically focuses on the **worst case**.
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Input length

The theoretical bounds are usually given by a function $f: \mathbb{N} \rightarrow \mathbb{N}$ that maps the **input length** to the running time (or storage space, comparisons, multiplications, program size etc.).

The **input length** may e.g. be

the size of the input (number of bits)

the number of arguments

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Example 1

Suppose n numbers from the interval $\{1, \dots, N\}$ have to be sorted. In this case we usually say that the input length is n instead of e.g. $n \log N$, which would be the number of bits required to encode the input.

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How to measure performance

1. Calculate running time and storage space etc. on a simplified, idealized model of computation, e.g. Random Access Machine (RAM), Turing Machine (TM), . . .
2. Calculate number of certain basic operations: comparisons, multiplications, harddisc accesses, . . .

Version 2. is often easier, but focusing on one type of operation makes it more difficult to obtain meaningful results.

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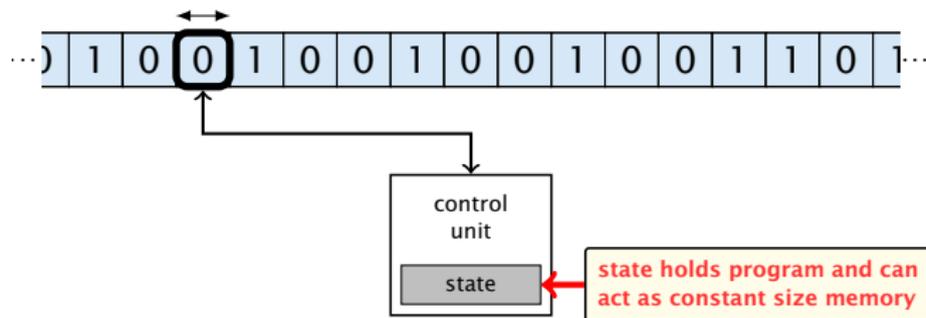
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Turing Machine

- ▶ Very simple model of computation.
- ▶ Only the “current” memory location can be altered.
- ▶ Very good model for discussing computability, or polynomial vs. exponential time.
- ▶ Some simple problems like recognizing whether input is of the form x^x , where x is a string, have quadratic lower bound.

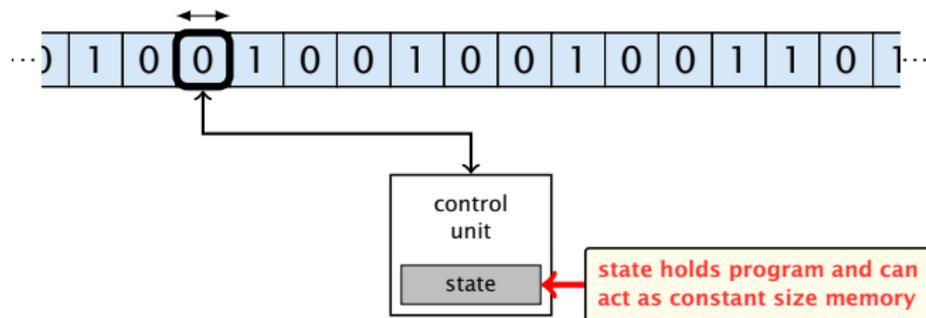
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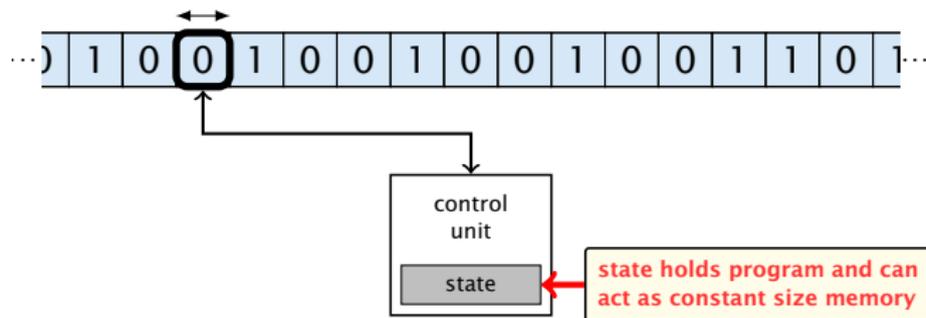
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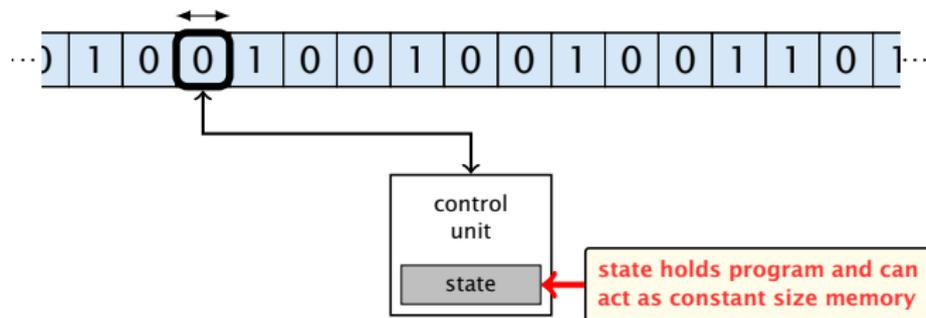
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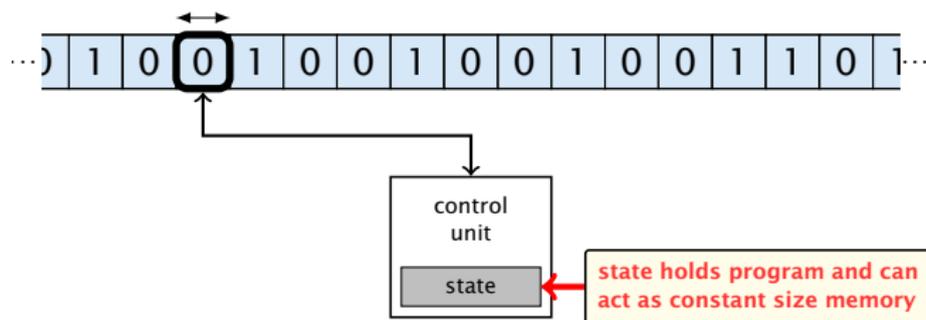
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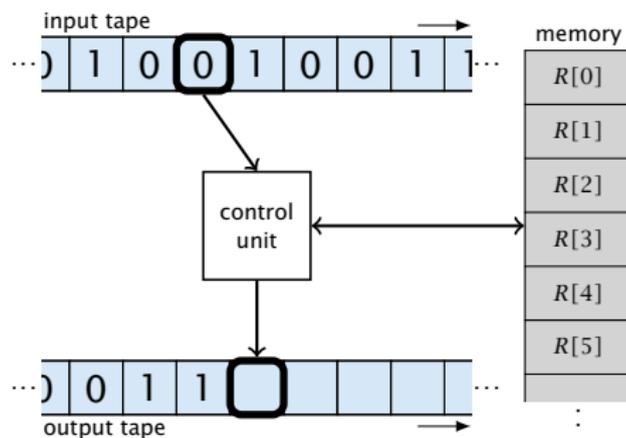
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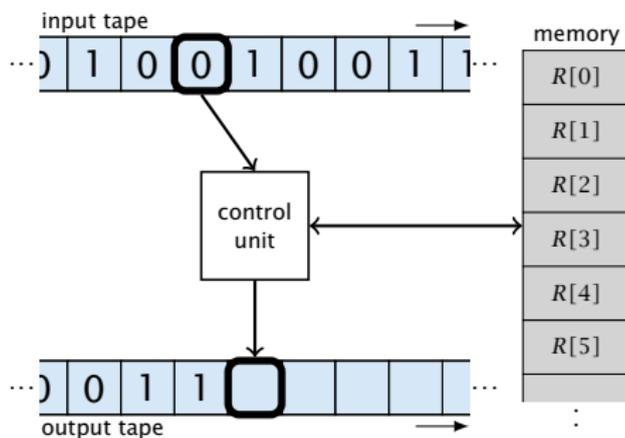
Random Access Machine (RAM)

- ▶ Input tape and output tape (sequences of zeros and ones; unbounded length).
- ▶ Memory unit: infinite but countable number of registers $R[0], R[1], R[2], \dots$
- ▶ Registers hold integers.
- ▶ Indirect addressing.



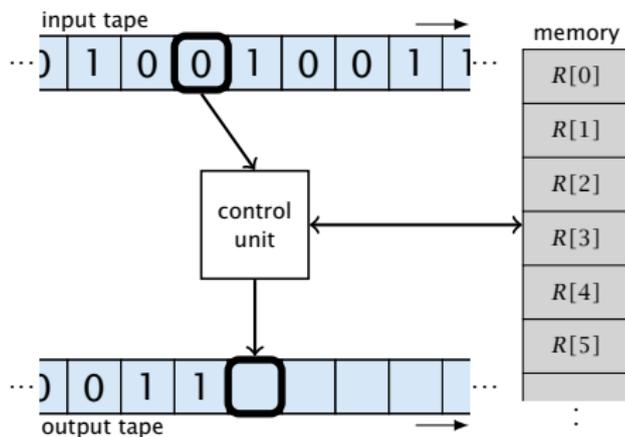
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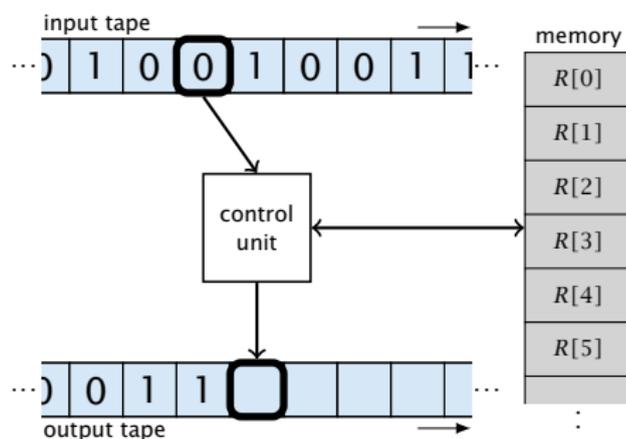
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- ▶ branching (including loops) based on comparisons
 - ▶ `jump x`
jumps to position x in the program;
sets instruction counter to x ;
reads the next operation to perform from register $R[x]$
 - ▶ `jumpz x $R[i]$`
jump to x if $R[i] = 0$
if not the instruction counter is increased by 1;
 - ▶ `jumpi i`
jump to $R[i]$ (indirect jump);
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 - ▶ $R[i] := R[j] + R[k];$
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jump to x if $R[i] = 0$
if not the instruction counter is increased by 1;
 - ▶ jumpi i
jump to $R[i]$ (indirect jump);
- ▶ arithmetic instructions: $+$, $-$, \times , $/$
 - ▶ $R[i] := R[j] + R[k]$;
 - ▶ $R[i] := -R[k]$;

Model of Computation

- ▶ **uniform** cost model

Every operation takes time 1.

- ▶ **logarithmic** cost model

The cost depends on the content of memory cells:

▶ The time for a step is equal to the largest operand involved.

▶ The word size of a register is equal to the length (in bits) of the largest value ever stored in it.

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There are **different types of complexity bounds**:

- ▶ **best-case** complexity:

$$C_{bc}(n) := \min\{C(x) \mid |x| = n\}$$

Usually easy to analyze, but not very meaningful.

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Formal Definition

Let f denote functions from \mathbb{N} to \mathbb{R}^+ .

- ▶ $\mathcal{O}(f) = \{g \mid \exists c > 0 \exists n_0 \in \mathbb{N}_0 \forall n \geq n_0 : [g(n) \leq c \cdot f(n)]\}$
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There is an equivalent definition using limes notation (**assuming that the respective limes exists**). f and g are functions from \mathbb{N}_0 to \mathbb{R}_0^+ .

$$\blacktriangleright g \in \mathcal{O}(f): 0 \leq \lim_{n \rightarrow \infty} \frac{g(n)}{f(n)} < \infty$$

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1. People write $f = \mathcal{O}(g)$, when they mean $f \in \mathcal{O}(g)$. This is **not** an equality (how could a function be equal to a set of functions).
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Asymptotic Notation in Equations

How do we interpret an expression like:

$$2n^2 + 3n + 1 = 2n^2 + \Theta(n)$$

Here, $\Theta(n)$ stands for an **anonymous function** in the set $\Theta(n)$ that makes the expression true.

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How do we interpret an expression like:

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Careful!

“It is understood” that every occurrence of an Θ -symbol (or $\Theta, \Omega, o, \omega$) on the left represents *one anonymous function*.

Hence, the left side is **not** equal to

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We can view an expression containing asymptotic notation as generating a set:

$$n^2 \cdot \mathcal{O}(n) + \mathcal{O}(\log n)$$

represents

$$\left\{ f : \mathbb{N} \rightarrow \mathbb{R}^+ \mid f(n) = n^2 \cdot g(n) + h(n) \right. \\ \left. \text{with } g(n) \in \mathcal{O}(n) \text{ and } h(n) \in \mathcal{O}(\log n) \right\}$$

Asymptotic Notation in Equations

Then an asymptotic equation can be interpreted as containment btw. two sets:

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represents

$$n^2 \cdot \mathcal{O}(n) + \mathcal{O}(\log n) \subseteq \Theta(n^2)$$

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Lemma 3

Let f, g be functions with the property

$\exists n_0 > 0 \forall n \geq n_0 : f(n) > 0$ (the same for g). Then

- ▶ $c \cdot f(n) \in \Theta(f(n))$ for any constant c
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The expressions also hold for Ω . Note that this means that $f(n) + g(n) \in \Theta(\max\{f(n), g(n)\})$.

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The expressions also hold for Ω . Note that this means that $f(n) + g(n) \in \Theta(\max\{f(n), g(n)\})$.

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- ▶ If the running time analysis is tight and actually occurs in practise (i.e., the asymptotic bound is not a purely theoretical worst-case bound), then the algorithm that has better asymptotic running time will always outperform a weaker algorithm for large enough values of n .
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6 Recurrences

Algorithm 2 mergesort(list L)

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1:  $n \leftarrow \text{size}(L)$ 
2: if  $n \leq 1$  return  $L$ 
3:  $L_1 \leftarrow L[1 \cdots \lfloor \frac{n}{2} \rfloor]$ 
4:  $L_2 \leftarrow L[\lfloor \frac{n}{2} \rfloor + 1 \cdots n]$ 
5: mergesort( $L_1$ )
6: mergesort( $L_2$ )
7:  $L \leftarrow \text{merge}(L_1, L_2)$ 
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This algorithm requires

$$T(n) = T\left(\left\lceil \frac{n}{2} \right\rceil\right) + T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + \mathcal{O}(n) \leq 2T\left(\left\lceil \frac{n}{2} \right\rceil\right) + \mathcal{O}(n)$$

comparisons when $n > 1$ and 0 comparisons when $n \leq 1$.

Recurrences

How do we bring the expression for the number of comparisons (\approx running time) into a **closed form**?

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Methods for Solving Recurrences

1. Guessing+Induction

Guess the right solution and prove that it is correct via induction. It needs experience to make the right guess.

2. Master Theorem

For a lot of recurrences that appear in the analysis of algorithms this theorem can be used to obtain tight asymptotic bounds. It does not provide exact solutions.

3. Characteristic Polynomial

Linear homogenous recurrences can be solved via this method.

Methods for Solving Recurrences

4. Generating Functions

A more general technique that allows to solve certain types of linear inhomogenous relations and also sometimes non-linear recurrence relations.

5. Transformation of the Recurrence

Sometimes one can transform the given recurrence relations so that it e.g. becomes linear and can therefore be solved with one of the other techniques.

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First we need to get rid of the \mathcal{O} -notation in our recurrence:

$$T(n) \leq \begin{cases} 2T(\lceil \frac{n}{2} \rceil) + cn & n \geq 2 \\ 0 & \text{otherwise} \end{cases}$$

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One way of solving such a recurrence is to **guess** a solution, and check that it is correct by plugging it in.

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Formally one would make an induction proof, where the above is the induction step. The base case is usually trivial.

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Hence, statement is **true** if we choose $d \geq c$.

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Note that we can do this as for constant-sized inputs the running time is always some constant (b in the above case).

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$$\left\lceil \frac{n}{2} \right\rceil \leq \frac{n}{2} + 1$$

$$\leq 2\left(d\left(\frac{n}{2} + 1\right) \log\left(\frac{n}{2} + 1\right)\right) + cn$$

$$\frac{n}{2} + 1 \leq \frac{9}{16}n$$

$$\leq dn \log\left(\frac{9}{16}n\right) + 2d \log n + cn$$

$$\log \frac{9}{16}n = \log n + (\log 9 - 4)$$

$$= dn \log n + (\log 9 - 4)dn + 2d \log n + cn$$

$$\log n \leq \frac{n}{4}$$

$$\leq dn \log n + (\log 9 - 3.5)dn + cn$$

6.1 Guessing+Induction

We also make a guess of $T(n) \leq dn \log n$ and get

$$\begin{aligned}T(n) &\leq 2T\left(\left\lceil \frac{n}{2} \right\rceil\right) + cn \\&\leq 2\left(d\left\lceil \frac{n}{2} \right\rceil \log \left\lceil \frac{n}{2} \right\rceil\right) + cn \\&\leq 2\left(d\left(\frac{n}{2} + 1\right) \log\left(\frac{n}{2} + 1\right)\right) + cn \\&\leq dn \log\left(\frac{9}{16}n\right) + 2d \log n + cn \\&= dn \log n + (\log 9 - 4)dn + 2d \log n + cn \\&\leq dn \log n + (\log 9 - 3.5)dn + cn \\&\leq dn \log n - 0.33dn + cn\end{aligned}$$

$$\left\lceil \frac{n}{2} \right\rceil \leq \frac{n}{2} + 1$$

$$\frac{n}{2} + 1 \leq \frac{9}{16}n$$

$$\log \frac{9}{16}n = \log n + (\log 9 - 4)$$

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$$\leq 2\left(d\left\lceil \frac{n}{2} \right\rceil \log \left\lceil \frac{n}{2} \right\rceil\right) + cn$$

$$\left\lceil \frac{n}{2} \right\rceil \leq \frac{n}{2} + 1$$

$$\leq 2\left(d\left(\frac{n}{2} + 1\right) \log\left(\frac{n}{2} + 1\right)\right) + cn$$

$$\frac{n}{2} + 1 \leq \frac{9}{16}n$$

$$\leq dn \log\left(\frac{9}{16}n\right) + 2d \log n + cn$$

$$\log \frac{9}{16}n = \log n + (\log 9 - 4)$$

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$$\log n \leq \frac{n}{4}$$

$$\leq dn \log n + (\log 9 - 3.5)dn + cn$$

$$\leq dn \log n - 0.33dn + cn$$

$$\leq dn \log n$$

for a suitable choice of d .

6.2 Master Theorem

Lemma 4

Let $a \geq 1$, $b \geq 1$ and $\epsilon > 0$ denote constants. Consider the recurrence

$$T(n) = aT\left(\frac{n}{b}\right) + f(n) .$$

Case 1.

If $f(n) = \mathcal{O}(n^{\log_b(a)-\epsilon})$ then $T(n) = \Theta(n^{\log_b a})$.

Case 2.

If $f(n) = \Theta(n^{\log_b(a)} \log^k n)$ then $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$,
 $k \geq 0$.

Case 3.

If $f(n) = \Omega(n^{\log_b(a)+\epsilon})$ and for sufficiently large n
 $af\left(\frac{n}{b}\right) \leq cf(n)$ for some constant $c < 1$ then $T(n) = \Theta(f(n))$.

6.2 Master Theorem

We prove the Master Theorem for the case that n is of the form b^{ℓ} , and we assume that the non-recursive case occurs for problem size 1 and incurs cost 1.

The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:

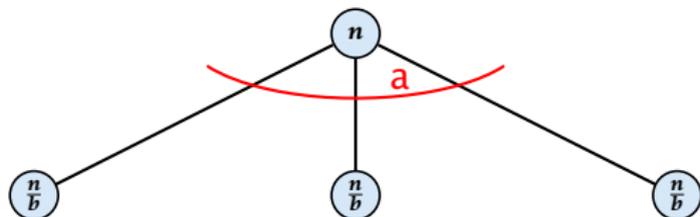
The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:



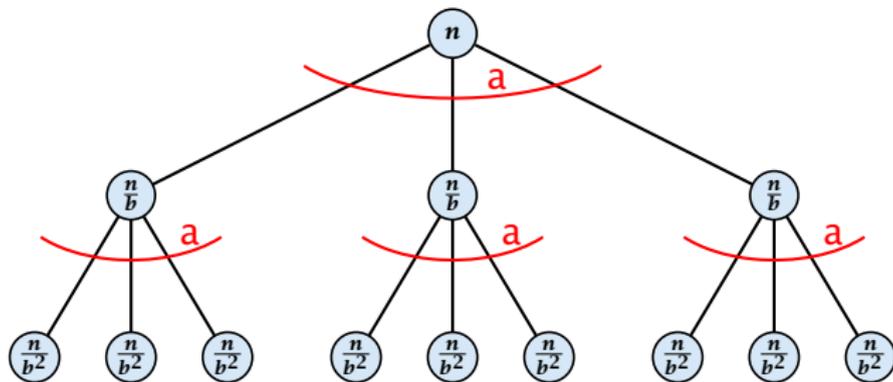
The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:



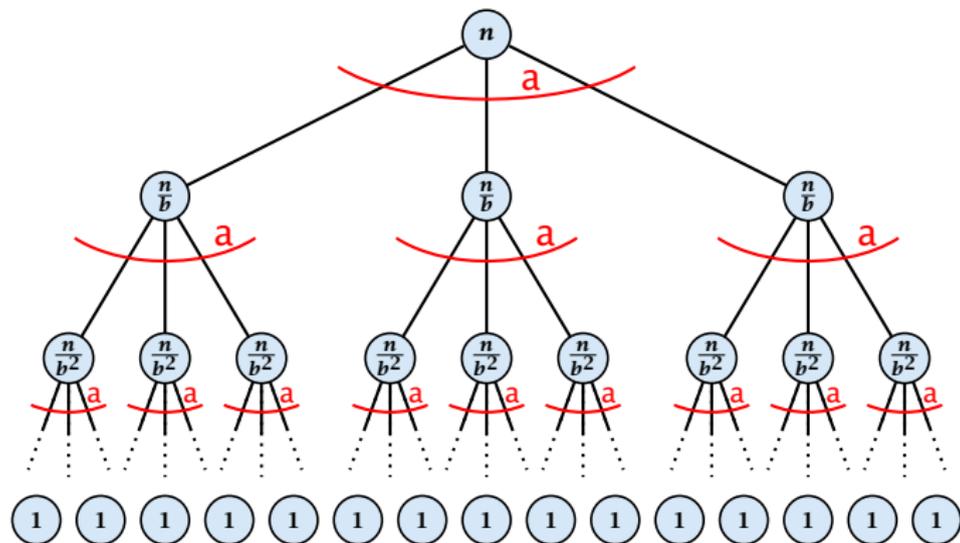
The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:



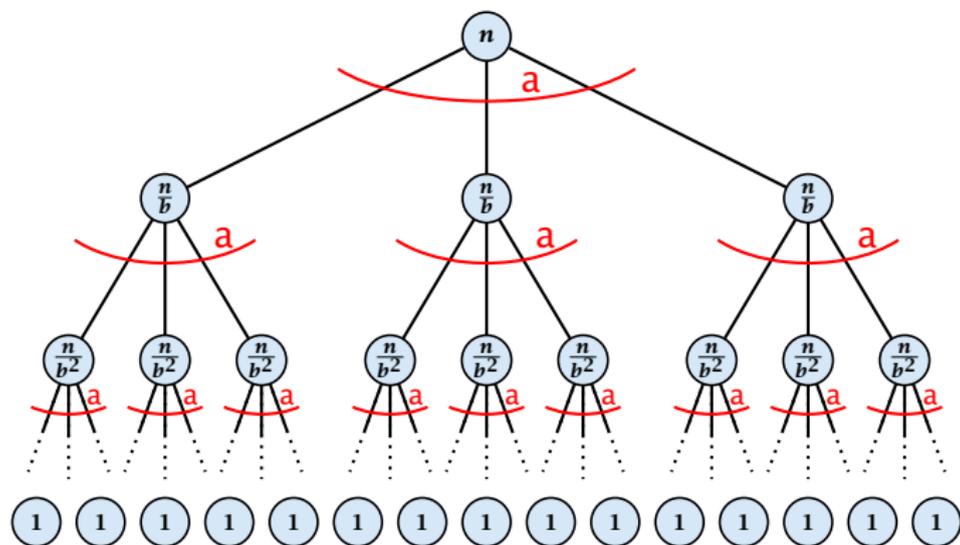
The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:



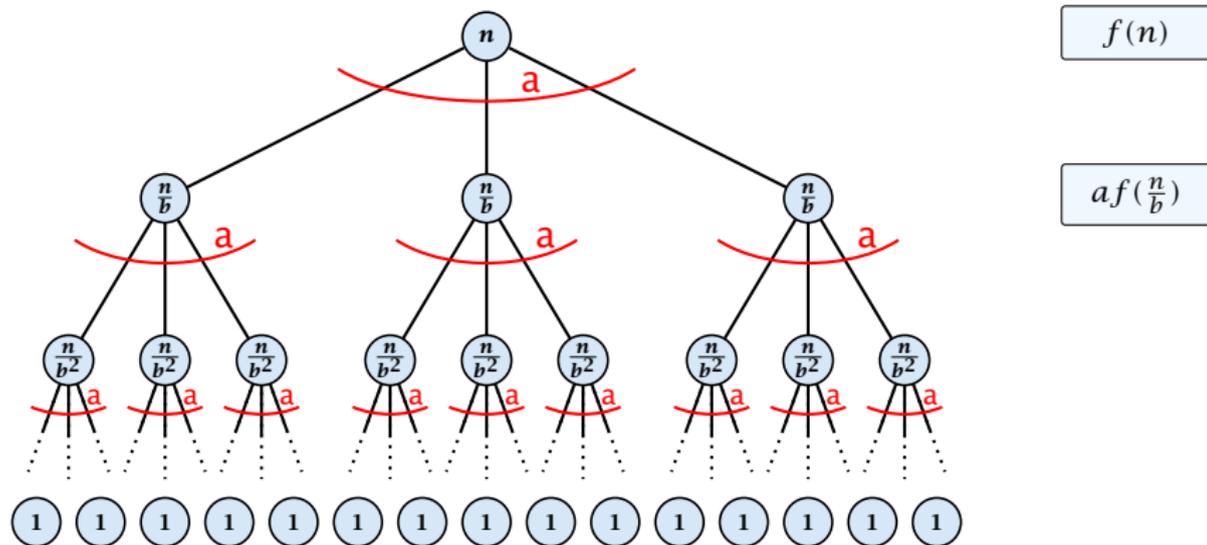
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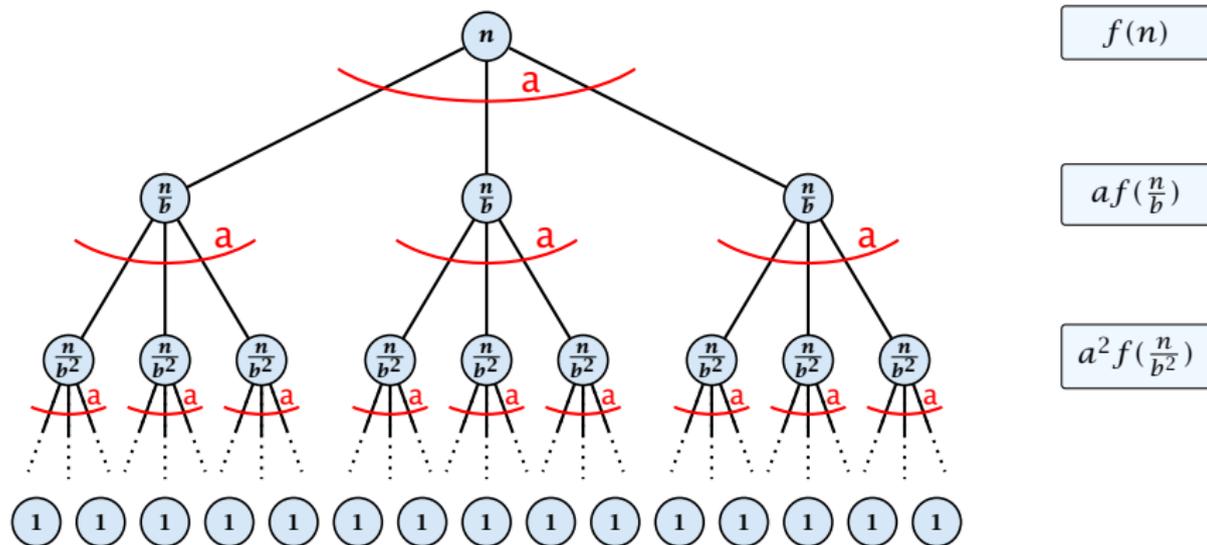
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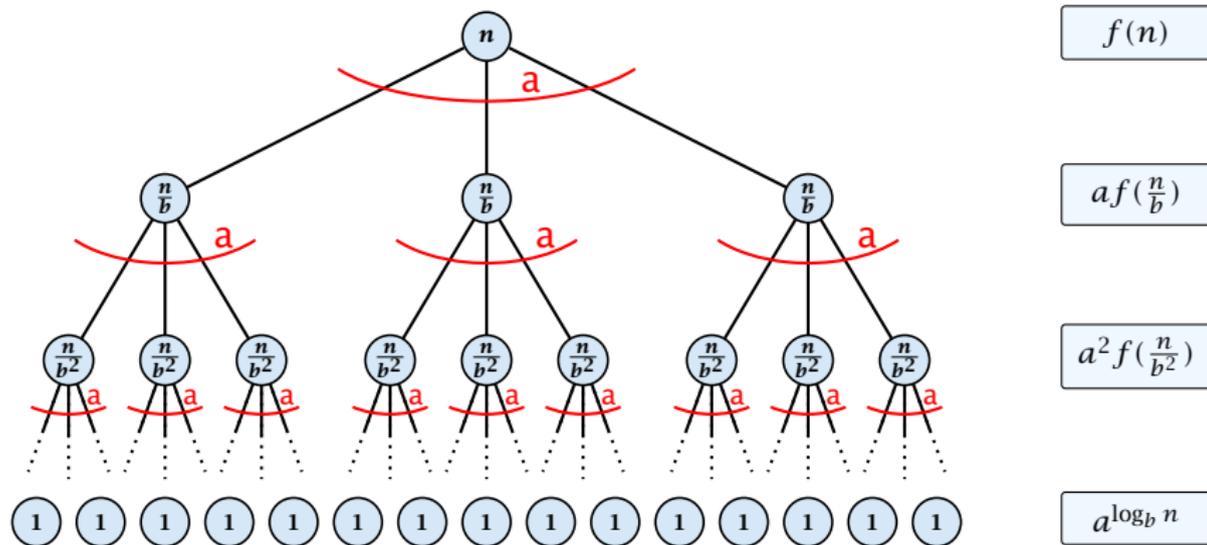
The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:



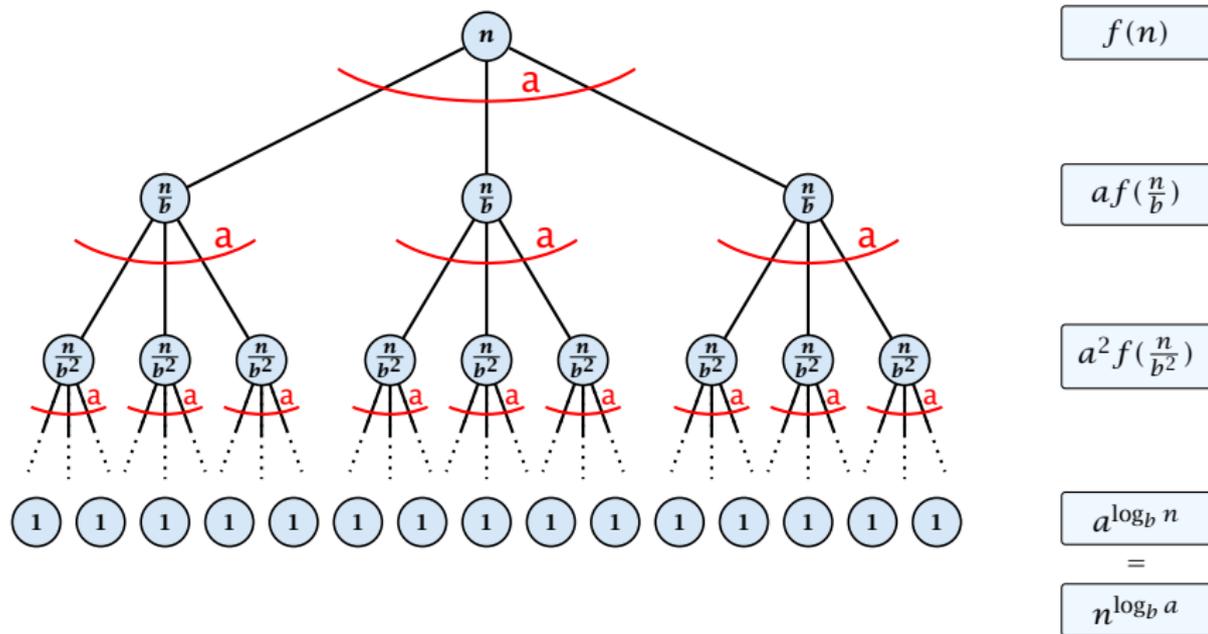
The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:



The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:



6.2 Master Theorem

This gives

$$T(n) = n^{\log_b a} + \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right).$$

Case 1. Now suppose that $f(n) \leq cn^{\log_b a - \epsilon}$.

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$$T(n) = n^{\log_b a}$$

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$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

Case 1. Now suppose that $f(n) \leq cn^{\log_b a - \epsilon}$.

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon} \end{aligned}$$

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$$b^{-i(\log_b a - \epsilon)} = b^{\epsilon i} (b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}$$

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$$\boxed{\sum_{i=0}^k q^i = \frac{q^{k+1} - 1}{q - 1}}$$

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$$\boxed{\sum_{i=0}^k q^i = \frac{q^{k+1} - 1}{q - 1}} = cn^{\log_b a - \epsilon} (b^{\epsilon \log_b n} - 1) / (b^{\epsilon} - 1)$$

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$$\begin{aligned} \boxed{\sum_{i=0}^k q^i = \frac{q^{k+1} - 1}{q - 1}} &= cn^{\log_b a - \epsilon} (b^{\epsilon \log_b n} - 1) / (b^{\epsilon} - 1) \\ &= cn^{\log_b a - \epsilon} (n^{\epsilon} - 1) / (b^{\epsilon} - 1) \end{aligned}$$

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Case 1. Now suppose that $f(n) \leq cn^{\log_b a - \epsilon}$.

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon} \end{aligned}$$

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Hence,

$$T(n) \leq \left(\frac{c}{b^{\epsilon} - 1} + 1 \right) n^{\log_b(a)}$$

Case 1. Now suppose that $f(n) \leq cn^{\log_b a - \epsilon}$.

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Hence,

$$T(n) \leq \left(\frac{c}{b^\epsilon - 1} + 1 \right) n^{\log_b a} \quad \Rightarrow T(n) = \mathcal{O}(n^{\log_b a}).$$

Case 2. Now suppose that $f(n) \leq cn^{\log_b a}$.

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Hence,

$$T(n) = \mathcal{O}(n^{\log_b a} \log_b n)$$

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Hence,

$$T(n) = \mathcal{O}(n^{\log_b a} \log_b n)$$

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Hence,

$$T(n) = \Omega(n^{\log_b a} \log_b n)$$

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Hence,

$$T(n) = \Omega(n^{\log_b a} \log_b n)$$

$$\Rightarrow T(n) = \Omega(n^{\log_b a} \log n).$$

Case 2. Now suppose that $f(n) \leq cn^{\log_b a} (\log_b(n))^k$.

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$$n = b^\ell \Rightarrow \ell = \log_b n$$

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$n = b^\ell \Rightarrow \ell = \log_b n$
--

$$= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k$$

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$$n = b^\ell \Rightarrow \ell = \log_b n$$

$$\begin{aligned} &= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k \\ &= cn^{\log_b a} \sum_{i=0}^{\ell-1} (\ell - i)^k \end{aligned}$$

Case 2. Now suppose that $f(n) \leq cn^{\log_b a} (\log_b(n))^k$.

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Case 2. Now suppose that $f(n) \leq cn^{\log_b a} (\log_b(n))^k$.

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \cdot \left(\log_b\left(\frac{n}{b^i}\right)\right)^k \end{aligned}$$

$$n = b^\ell \Rightarrow \ell = \log_b n$$

$$= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k$$

$$= cn^{\log_b a} \sum_{i=0}^{\ell-1} (\ell - i)^k$$

$$= cn^{\log_b a} \sum_{i=1}^{\ell} i^k \approx \frac{1}{k} \ell^{k+1}$$

Case 2. Now suppose that $f(n) \leq cn^{\log_b a} (\log_b(n))^k$.

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \cdot \left(\log_b\left(\frac{n}{b^i}\right)\right)^k \end{aligned}$$

$$n = b^\ell \Rightarrow \ell = \log_b n$$

$$\begin{aligned} &= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k \\ &= cn^{\log_b a} \sum_{i=0}^{\ell-1} (\ell - i)^k \\ &= cn^{\log_b a} \sum_{i=1}^{\ell} i^k \\ &\approx \frac{c}{k} n^{\log_b a} \ell^{k+1} \end{aligned}$$

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$$\Rightarrow T(n) = \mathcal{O}(n^{\log_b a} \log^{k+1} n).$$

Case 3. Now suppose that $f(n) \geq dn^{\log_b a + \epsilon}$, and that for sufficiently large n : $af(n/b) \leq cf(n)$, for $c < 1$.

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$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

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$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq \sum_{i=0}^{\log_b n - 1} c^i f(n) + \mathcal{O}(n^{\log_b a}) \end{aligned}$$

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$$q < 1 : \sum_{i=0}^n q^i = \frac{1 - q^{n+1}}{1 - q} \leq \frac{1}{1 - q}$$

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$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq \sum_{i=0}^{\log_b n - 1} c^i f(n) + \mathcal{O}(n^{\log_b a}) \\ &\leq \frac{1}{1-c} f(n) + \mathcal{O}(n^{\log_b a}) \end{aligned}$$

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$$q < 1 : \sum_{i=0}^n q^i = \frac{1-q^{n+1}}{1-q} \leq \frac{1}{1-q}$$

Hence,

$$T(n) \leq \mathcal{O}(f(n))$$

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$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq \sum_{i=0}^{\log_b n - 1} c^i f(n) + \mathcal{O}(n^{\log_b a}) \\ &\leq \frac{1}{1-c} f(n) + \mathcal{O}(n^{\log_b a}) \end{aligned}$$

$$q < 1 : \sum_{i=0}^n q^i = \frac{1-q^{n+1}}{1-q} \leq \frac{1}{1-q}$$

Hence,

$$T(n) \leq \mathcal{O}(f(n))$$

$$\Rightarrow T(n) = \Theta(f(n)).$$

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

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For this we first need to be able to add two integers A and B :

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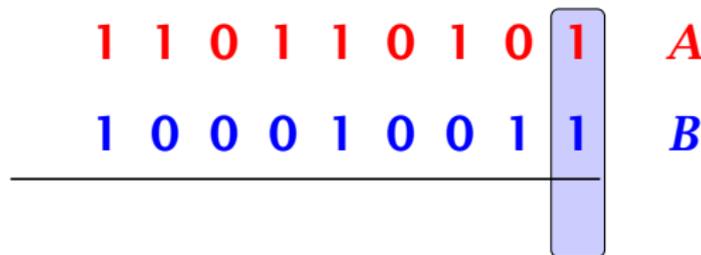
For this we first need to be able to add two integers A and B :

$$\begin{array}{r} 1\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ A \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ B \\ \hline \end{array}$$

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1	1	0	1	1	0	1	0	1	A
1	0	0	0	1	0	0	1	1	B
<hr/>								0	

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For this we first need to be able to add two integers A and B :

1	1	0	1	1	0	1	0	1	1	A
1	0	0	0	1	0	0	1	1		B
<hr/>										
							1			
									0	

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1	1	0	1	1	0	1	0	1	<i>A</i>
1	0	0	0	1	0	0	1	1	<i>B</i>
							1	1	
							0	0	

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1	1	0	1	1	0	1	0	1	A
1	0	0	0	1	0	0	1	1	B
							1		
								1	
									0
									0
									0

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						1	1	1	
						0	0	0	

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1	1	0	1	1	0	1	0	1	A
1	0	0	0	1	0	0	1	1	B
<hr/>									
					1	1	1		
						0	0	0	

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Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

$$\begin{array}{rcccccccc} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & A \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & B \\ \hline & & & & 0 & 1 & 1 & 1 & & \\ & & & & & 1 & 0 & 0 & 0 & \end{array}$$

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1	0	0	0	1	0	0	1	1	B
<hr/>									
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					1	0	0	0	

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1	0	0	0	1	0	0	1	1	B
				1	0	1	1	1	
-----				0	1	0	0	0	

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1	1	0	1	1	0	1	0	1	A
1	0	0	0	1	0	0	1	1	B
		1	1	0	1	1	1		
<hr/>									
			0	0	1	0	0	0	

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<hr/>									
			1	1	0	1	1	1	
			0	0	1	0	0	0	

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<hr/>									
		1	0	0	1	0	0	0	

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1	1	0	1	1	0	1	0	1	A
1	0	0	0	1	0	0	1	1	B
<hr/>									
		0	1	1	0	1	1	1	
		1	0	0	1	0	0	0	

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For this we first need to be able to add two integers A and B :

	1	1	0	1	1	0	1	0	1	A
	1	0	0	0	1	0	0	1	1	B
	<small>0</small>	<small>0</small>	<small>1</small>	<small>1</small>	<small>0</small>	<small>1</small>	<small>1</small>	<small>1</small>		
<hr/>										
	1	1	0	0	1	0	0	0		

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1	1	0	1	1	0	1	0	1	A
1	0	0	0	1	0	0	1	1	B
<hr/>									
1	1	0	0	1	0	0	0	0	

The diagram illustrates the addition of two 8-bit integers, A and B. The bits of A are 1, 1, 0, 1, 1, 0, 1, 0, 1. The bits of B are 1, 0, 0, 0, 1, 0, 0, 1, 1. The sum is 1, 1, 0, 0, 1, 0, 0, 0, 0. A carry of 1 is shown in a blue box on the left.

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For this we first need to be able to add two integers A and B :

	1	1	0	1	1	0	1	0	1	<i>A</i>
	1	0	0	0	1	0	0	1	1	<i>B</i>
	0	1	1	0	0	1	0	0	0	

Note: In the original image, a vertical blue box highlights the first column of bits (1, 1, 0) and the carry bit 1 to its left. Green numbers 1, 0, 0, 1, 1, 0, 1, 1, 1 are placed below the second row of bits.

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For this we first need to be able to add two integers A and B :

	1	1	0	1	1	0	1	0	1	A
	1	0	0	0	1	0	0	1	1	B
	<hr/>									
	0	1	1	0	0	1	0	0	0	

Note: In the original image, a vertical blue bar on the left side of the addition shows carry propagation. The carry starts at 1 in the first column, then becomes 0, then 1, then 0, then 1, then 0, then 1, then 1, then 1, and finally 0 in the final column.

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	1	1	0	1	1	0	1	0	1	A
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	<hr/>									
1	0	1	1	0	0	1	0	0	0	

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Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

$$\begin{array}{r} 1\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ A \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ B \\ \hline 1\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ 0 \end{array}$$

This gives that two n -bit integers can be added in time $\mathcal{O}(n)$.

Example: Multiplying Two Integers

Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

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$$\begin{array}{r} 10001 \times 1011 \\ \hline \end{array}$$

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Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \\ \times 1\ 0\ 1\ 1 \\ \hline \end{array}$$

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Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \end{array}$$

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$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0 \end{array}$$

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$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0 \\ 0\ 0 \end{array}$$

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$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \end{array}$$

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$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \end{array}$$

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$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0 \end{array}$$

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$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0 \\ \hline 1\ 0\ 1\ 1\ 1\ 0\ 1\ 1 \end{array}$$

Time requirement:

Example: Multiplying Two Integers

Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0 \\ \hline 1\ 0\ 1\ 1\ 1\ 0\ 1\ 1 \end{array}$$

Time requirement:

- ▶ Computing intermediate results: $\mathcal{O}(nm)$.

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Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0 \\ \hline 1\ 0\ 1\ 1\ 1\ 0\ 1\ 1 \end{array}$$

Time requirement:

- ▶ Computing intermediate results: $\mathcal{O}(nm)$.
- ▶ Adding m numbers of length $\leq 2n$:
 $\mathcal{O}((m+n)m) = \mathcal{O}(nm)$.

Example: Multiplying Two Integers

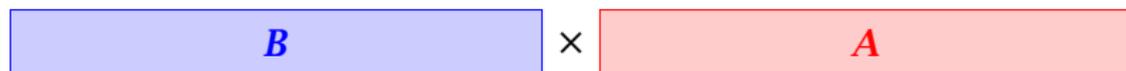
A recursive approach:

Suppose that integers A and B are of length $n = 2^k$, for some k .

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$$\boxed{b_n \quad \dots \quad b_0} \times \boxed{a_n \quad \dots \quad a_0}$$

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A recursive approach:

Suppose that integers A and B are of length $n = 2^k$, for some k .

$$\boxed{b_n \quad \cdots \quad b_{\frac{n}{2}} \quad b_{\frac{n}{2}-1} \quad \cdots \quad b_0} \times \boxed{a_n \quad \cdots \quad a_{\frac{n}{2}} \quad a_{\frac{n}{2}-1} \quad \cdots \quad a_0}$$

Example: Multiplying Two Integers

A recursive approach:

Suppose that integers A and B are of length $n = 2^k$, for some k .

$$\begin{array}{|c|c|} \hline B_1 & B_0 \\ \hline \end{array} \times \begin{array}{|c|c|} \hline A_1 & A_0 \\ \hline \end{array}$$

Example: Multiplying Two Integers

A recursive approach:

Suppose that integers A and B are of length $n = 2^k$, for some k .



Then it holds that

$$A = A_1 \cdot 2^{\frac{n}{2}} + A_0 \text{ and } B = B_1 \cdot 2^{\frac{n}{2}} + B_0$$

Example: Multiplying Two Integers

A recursive approach:

Suppose that integers A and B are of length $n = 2^k$, for some k .



Then it holds that

$$A = A_1 \cdot 2^{\frac{n}{2}} + A_0 \text{ and } B = B_1 \cdot 2^{\frac{n}{2}} + B_0$$

Hence,

$$A \cdot B = A_1 B_1 \cdot 2^n + (A_1 B_0 + A_0 B_1) \cdot 2^{\frac{n}{2}} + A_0 \cdot B_0$$

Example: Multiplying Two Integers

Algorithm 3 $\text{mult}(A, B)$

```
1: if  $|A| = |B| = 1$  then  
2:   return  $a_0 \cdot b_0$   
3: split  $A$  into  $A_0$  and  $A_1$   
4: split  $B$  into  $B_0$  and  $B_1$   
5:  $Z_2 \leftarrow \text{mult}(A_1, B_1)$   
6:  $Z_1 \leftarrow \text{mult}(A_1, B_0) + \text{mult}(A_0, B_1)$   
7:  $Z_0 \leftarrow \text{mult}(A_0, B_0)$   
8: return  $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$ 
```

Example: Multiplying Two Integers

Algorithm 3 $\text{mult}(A, B)$

```
1: if  $|A| = |B| = 1$  then  
2:     return  $a_0 \cdot b_0$   
3: split  $A$  into  $A_0$  and  $A_1$   
4: split  $B$  into  $B_0$  and  $B_1$   
5:  $Z_2 \leftarrow \text{mult}(A_1, B_1)$   
6:  $Z_1 \leftarrow \text{mult}(A_1, B_0) + \text{mult}(A_0, B_1)$   
7:  $Z_0 \leftarrow \text{mult}(A_0, B_0)$   
8: return  $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$ 
```

$\mathcal{O}(1)$

Example: Multiplying Two Integers

Algorithm 3 $\text{mult}(A, B)$

1: **if** $|A| = |B| = 1$ **then**

$\mathcal{O}(1)$

2: **return** $a_0 \cdot b_0$

$\mathcal{O}(1)$

3: split A into A_0 and A_1

4: split B into B_0 and B_1

5: $Z_2 \leftarrow \text{mult}(A_1, B_1)$

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$\mathcal{O}(n)$

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Example: Multiplying Two Integers

Algorithm 3 $\text{mult}(A, B)$

- 1: **if** $|A| = |B| = 1$ **then** $\mathcal{O}(1)$
- 2: **return** $a_0 \cdot b_0$ $\mathcal{O}(1)$
- 3: split A into A_0 and A_1 $\mathcal{O}(n)$
- 4: split B into B_0 and B_1 $\mathcal{O}(n)$
- 5: $Z_2 \leftarrow \text{mult}(A_1, B_1)$
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$\mathcal{O}(1)$

$\mathcal{O}(1)$

$\mathcal{O}(n)$

$\mathcal{O}(n)$

$T\left(\frac{n}{2}\right)$

Example: Multiplying Two Integers

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$\mathcal{O}(1)$

$\mathcal{O}(1)$

$\mathcal{O}(n)$

$\mathcal{O}(n)$

$T(\frac{n}{2})$

$2T(\frac{n}{2}) + \mathcal{O}(n)$

Example: Multiplying Two Integers

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$\mathcal{O}(n)$

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$T(\frac{n}{2})$

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Example: Multiplying Two Integers

Algorithm 3 $\text{mult}(A, B)$

1: if $ A = B = 1$ then	$\mathcal{O}(1)$
2: return $a_0 \cdot b_0$	$\mathcal{O}(1)$
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8: return $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$	$\mathcal{O}(n)$

We get the following recurrence:

$$T(n) = 4T\left(\frac{n}{2}\right) + \mathcal{O}(n) .$$

Example: Multiplying Two Integers

Master Theorem: Recurrence: $T[n] = aT(\frac{n}{b}) + f(n)$.

- ▶ Case 1: $f(n) = \mathcal{O}(n^{\log_b a - \epsilon})$ $T(n) = \Theta(n^{\log_b a})$
- ▶ Case 2: $f(n) = \Theta(n^{\log_b a} \log^k n)$ $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$
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In our case $a = 4$, $b = 2$, and $f(n) = \Theta(n)$. Hence, we are in Case 1, since $n = \mathcal{O}(n^{2-\epsilon}) = \mathcal{O}(n^{\log_b a - \epsilon})$.

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We get a running time of $\mathcal{O}(n^2)$ for our algorithm.

Example: Multiplying Two Integers

Master Theorem: Recurrence: $T[n] = aT(\frac{n}{b}) + f(n)$.

- ▶ Case 1: $f(n) = \mathcal{O}(n^{\log_b a - \epsilon})$ $T(n) = \Theta(n^{\log_b a})$
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⇒ Not better than the “school method”.

Example: Multiplying Two Integers

We can use the following identity to compute Z_1 :

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$$Z_1 = A_1B_0 + A_0B_1$$

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We can use the following identity to compute Z_1 :

$$\begin{aligned}Z_1 &= A_1B_0 + A_0B_1 \\ &= (A_0 + A_1) \cdot (B_0 + B_1) - A_1B_1 - A_0B_0\end{aligned}$$

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We can use the following identity to compute Z_1 :

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Hence,

Algorithm 4 mult(A, B)

```
1: if  $|A| = |B| = 1$  then
2:   return  $a_0 \cdot b_0$ 
3: split  $A$  into  $A_0$  and  $A_1$ 
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2: **return** $a_0 \cdot b_0$

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$\mathcal{O}(n)$

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$\mathcal{O}(1)$

$\mathcal{O}(n)$

$\mathcal{O}(n)$

$T(\frac{n}{2})$

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$\mathcal{O}(1)$

$\mathcal{O}(1)$

$\mathcal{O}(n)$

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$T(\frac{n}{2})$

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Example: Multiplying Two Integers

We get the following recurrence:

$$T(n) = 3T\left(\frac{n}{2}\right) + \mathcal{O}(n) .$$

Master Theorem: Recurrence: $T[n] = aT\left(\frac{n}{b}\right) + f(n)$.

- ▶ Case 1: $f(n) = \mathcal{O}(n^{\log_b a - \epsilon})$ $T(n) = \Theta(n^{\log_b a})$
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Again we are in Case 1. We get a running time of $\Theta(n^{\log_2 3}) \approx \Theta(n^{1.59})$.

A huge improvement over the "school method".

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6.3 The Characteristic Polynomial

Consider the recurrence relation:

$$c_0T(n) + c_1T(n-1) + c_2T(n-2) + \dots + c_kT(n-k) = f(n)$$

This is the general form of a **linear** recurrence relation of **order k** with constant coefficients ($c_0, c_k \neq 0$).

The recurrence is **linear** as it only depends on the preceding values. That is, the recurrence relation is of the form

The recurrence is **linear** as there are no products of variables. If $f(n) = 0$, then the recurrence relation becomes a **linear homogeneous** recurrence relation of order k .

Note that we ignore **boundary conditions** for the moment.

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Consider the recurrence relation:

$$c_0T(n) + c_1T(n-1) + c_2T(n-2) + \dots + c_kT(n-k) = f(n)$$

This is the general form of a **linear** recurrence relation of **order k** with constant coefficients ($c_0, c_k \neq 0$).

- ▶ $T(n)$ only depends on the k preceding values. This means the recurrence relation is of **order k** .
- ▶ The recurrence is linear as there are no products of $T[n]$'s.
- ▶ If $f(n) = 0$ then the recurrence relation becomes a linear, **homogenous** recurrence relation of order k .

Note that we ignore **boundary conditions** for the moment.

6.3 The Characteristic Polynomial

Observations:

- ▶ The solution $T[1], T[2], T[3], \dots$ is completely determined by a set of **boundary conditions** that specify values for $T[1], \dots, T[k]$.
- ▶ In fact, any k consecutive values completely determine the solution.
- ▶ k non-consecutive values might not be an appropriate set of boundary conditions (depends on the problem).

Approach:

- ▶ First determine all solutions that satisfy recurrence relation.
- ▶ Then pick the right one by analyzing boundary conditions.
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The Homogenous Case

The solution space

$$S = \left\{ \mathcal{T} = T[1], T[2], T[3], \dots \mid \mathcal{T} \text{ fulfills recurrence relation} \right\}$$

is a **vector space**. This means that if $\mathcal{T}_1, \mathcal{T}_2 \in S$, then also $\alpha\mathcal{T}_1 + \beta\mathcal{T}_2 \in S$, for arbitrary constants α, β .

How do we find a non-trivial solution?

We guess that the solution is of the form λ^n , $\lambda \neq 0$, and see what happens. In order for this guess to fulfill the recurrence we need

$$c_0\lambda^n + c_1\lambda^{n-1} + c_2 \cdot \lambda^{n-2} + \dots + c_k \cdot \lambda^{n-k} = 0$$

for all $n \geq k$.

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Dividing by λ^{n-k} gives that all these constraints are identical to

$$c_0\lambda^k + c_1\lambda^{k-1} + c_2 \cdot \lambda^{k-2} + \dots + c_k = 0$$

This means that if λ_i is a root (Nullstelle) of $P[\lambda]$ then $T[n] = \lambda_i^n$ is a solution to the recurrence relation.

Let $\lambda_1, \dots, \lambda_k$ be the k (complex) roots of $P[\lambda]$. Then, because of the vector space property

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Lemma 5

Assume that the characteristic polynomial has k *distinct* roots $\lambda_1, \dots, \lambda_k$. Then *all* solutions to the recurrence relation are of the form

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Proof.

There is one solution for every possible choice of boundary conditions for $T[1], \dots, T[k]$.

We show that the above set of solutions contains one solution for every choice of boundary conditions.

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Proof (cont.).

Suppose I am given boundary conditions $T[i]$ and I want to see whether I can choose the α'_i 's such that these conditions are met:

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We show that the column vectors are linearly independent. Then the above equation has a solution.

Computing the Determinant

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{k-1}^2 & \lambda_k^2 \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^k & \lambda_2^k & \cdots & \lambda_{k-1}^k & \lambda_k^k \end{vmatrix} =$$

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$$\begin{vmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1 & (\lambda_2 - \lambda_1) \cdot 1 & \cdots & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-3} & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & (\lambda_k - \lambda_1) \cdot 1 & \cdots & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-3} & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-2} \end{vmatrix}$$

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$$\prod_{i=2}^k (\lambda_i - \lambda_1) \cdot \begin{vmatrix} 1 & \lambda_2 & \cdots & \lambda_2^{k-3} & \lambda_2^{k-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_k & \cdots & \lambda_k^{k-3} & \lambda_k^{k-2} \end{vmatrix}$$

Computing the Determinant

Repeating the above steps gives:

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{k-1}^2 & \lambda_k^2 \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^k & \lambda_2^k & \cdots & \lambda_{k-1}^k & \lambda_k^k \end{vmatrix} = \prod_{i=1}^k \lambda_i \cdot \prod_{i>\ell} (\lambda_i - \lambda_\ell)$$

Hence, if all λ_i 's are different, then the determinant is non-zero.

The Homogeneous Case

What happens if the roots are not all distinct?

Suppose we have a root λ_i with multiplicity (Vielfachheit) at least 2. Then not only is λ_i^n a solution to the recurrence but also $n\lambda_i^{n-1}$.

To see this consider the polynomial

$$P[\lambda] \cdot \lambda^{n-k} = c_0\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \dots + c_k\lambda^{n-k}$$

Since λ_i is a root we can write this as $Q[\lambda] \cdot (\lambda - \lambda_i)^2$.

Calculating the derivative gives a polynomial that still has root λ_i .

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This means

$$c_0 n \lambda_i^{n-1} + c_1 (n-1) \lambda_i^{n-2} + \dots + c_k (n-k) \lambda_i^{n-k-1} = 0$$

Hence,

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Suppose λ_i has multiplicity j . We know that

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(after taking the derivative; multiplying with λ ; plugging in λ_i)

Doing this again gives

$$c_0 n^2 \lambda_i^n + c_1 (n-1)^2 \lambda_i^{n-1} + \dots + c_k (n-k)^2 \lambda_i^{n-k} = 0$$

We can continue $j-1$ times.

Hence, $n^\ell \lambda_i^n$ is a solution for $\ell \in 0, \dots, j-1$.

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The Homogeneous Case

Lemma 6

Let $P[\lambda]$ denote the characteristic polynomial to the recurrence

$$c_0T[n] + c_1T[n-1] + \dots + c_kT[n-k] = 0$$

Let $\lambda_i, i = 1, \dots, m$ be the (complex) roots of $P[\lambda]$ with multiplicities ℓ_i . Then the general solution to the recurrence is given by

$$T[n] = \sum_{i=1}^m \sum_{j=0}^{\ell_i-1} \alpha_{ij} \cdot (n^j \lambda_i^n) .$$

The full proof is omitted. We have only shown that any choice of α_{ij} 's is a solution to the recurrence.

Example: Fibonacci Sequence

$$T[0] = 0$$

$$T[1] = 1$$

$$T[n] = T[n - 1] + T[n - 2] \text{ for } n \geq 2$$

The characteristic polynomial is

$$\lambda^2 - \lambda - 1$$

Finding the roots, gives

$$\lambda_{1/2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + 1} = \frac{1}{2} (1 \pm \sqrt{5})$$

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$$\alpha \left(\frac{1 + \sqrt{5}}{2} \right) + \beta \left(\frac{1 - \sqrt{5}}{2} \right) = 1 \implies \alpha - \beta = \frac{2}{\sqrt{5}}$$

Example: Fibonacci Sequence

Hence, the solution is

$$\frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^n - \left(\frac{1 - \sqrt{5}}{2} \right)^n \right]$$

The Inhomogeneous Case

Consider the recurrence relation:

$$c_0T(n) + c_1T(n-1) + c_2T(n-2) + \cdots + c_kT(n-k) = f(n)$$

with $f(n) \neq 0$.

While we have a fairly general technique for solving **homogeneous**, linear recurrence relations the inhomogeneous case is different.

The Inhomogeneous Case

The general solution of the recurrence relation is

$$T(n) = T_h(n) + T_p(n) ,$$

where T_h is **any** solution to the homogeneous equation, and T_p is **one** particular solution to the inhomogeneous equation.

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The Inhomogeneous Case

Example:

$$T[n] = T[n - 1] + 1 \quad T[0] = 1$$

Then,

$$T[n - 1] = T[n - 2] + 1 \quad (n \geq 2)$$

Subtracting the first from the second equation gives,

$$T[n] - T[n - 1] = T[n - 1] - T[n - 2] \quad (n \geq 2)$$

or

$$T[n] = 2T[n - 1] - T[n - 2] \quad (n \geq 2)$$

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$T[1] = 2$ gives $1 + \beta = 2 \Rightarrow \beta = 1$.

The Inhomogeneous Case

If $f(n)$ is a polynomial of degree r this method can be applied $r + 1$ times to obtain a homogeneous equation:

$$T[n] = T[n - 1] + n^2$$

Shift:

$$T[n - 1] = T[n - 2] + (n - 1)^2$$

Difference:

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Shift:

$$T[n - 1] = 2T[n - 2] - T[n - 3] + 2(n - 1) - 1$$

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Shift:

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and so on...

6.4 Generating Functions

Definition 7 (Generating Function)

Let $(a_n)_{n \geq 0}$ be a sequence. The corresponding

- ▶ **generating function** (Erzeugendenfunktion) is

$$F(z) := \sum_{n \geq 0} a_n z^n;$$

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There are two different views:

A generating function is a **formal power series** (formale Potenzreihe).

Then the generating function is an **algebraic object**.

Let $f = \sum_{n \geq 0} a_n z^n$ and $g = \sum_{n \geq 0} b_n z^n$.

- ▶ **Equality:** f and g are equal if $a_n = b_n$ for all n .
- ▶ **Addition:** $f + g := \sum_{n \geq 0} (a_n + b_n) z^n$.
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We view a power series as a function $f : \mathbb{C} \rightarrow \mathbb{C}$.

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What does $\sum_{n \geq 0} z^n = \frac{1}{1-z}$ mean in the algebraic view?

It means that the power series $1 - z$ and the power series $\sum_{n \geq 0} z^n$ are invers, i.e.,

$$(1 - z) \cdot \left(\sum_{n \geq 0} z^n \right) = 1 .$$

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It means that the power series $1 - z$ and the power series $\sum_{n \geq 0} z^n$ are inverses, i.e.,

$$(1 - z) \cdot \left(\sum_{n \geq 0} z^n \right) = 1 .$$

This is well-defined.

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Hence, the generating function of the sequence $a_n = n + 1$ is $1/(1-z)^2$.

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Hence, the generating function of the sequence

$$a_n = (n+1)(n+2) \text{ is } \frac{2}{(1-z)^3} .$$

6.4 Generating Functions

Computing the k -th derivative of $\sum z^n$.

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The generating function of the sequence $a_n = \binom{n+k}{k}$ is $\frac{1}{(1-z)^{k+1}}$.

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The generating function of the sequence $a_n = n$ is $\frac{z}{(1-z)^2}$.

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We know

$$\sum_{n \geq 0} y^n = \frac{1}{1-y}$$

Hence,

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Example: $a_n = a_{n-1} + 1$, $a_0 = 1$

Suppose we have the recurrence $a_n = a_{n-1} + 1$ for $n \geq 1$ and $a_0 = 1$.

$A(z)$

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Hence, $a_n = n + 1$.

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Solving Recursions with Generating Functions

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6. The coefficients of the resulting power series are the a_n .

Example: $a_n = 2a_{n-1}$, $a_0 = 1$

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2. Plug in:

$$A(z) = 1 + \sum_{n \geq 1} (2a_{n-1})z^n$$

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$$\begin{aligned}A(z) &= \sum_{n \geq 0} a_n z^n \\&= a_0 + \sum_{n \geq 1} a_n z^n \\&= 1 + \sum_{n \geq 1} (3a_{n-1} + n) z^n \\&= 1 + 3z \sum_{n \geq 1} a_{n-1} z^{n-1} + \sum_{n \geq 1} n z^n \\&= 1 + 3z \sum_{n \geq 0} a_n z^n + \sum_{n \geq 0} n z^n \\&= 1 + 3zA(z) + \frac{z}{(1-z)^2}\end{aligned}$$

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which gives

$$A = \frac{7}{4} \quad B = -\frac{1}{4} \quad C = -\frac{1}{2}$$

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6. This means $a_n = \frac{7}{4}3^n - \frac{1}{2}n - \frac{3}{4}$.

6.5 Transformation of the Recurrence

Example 9

$$f_0 = 1$$

$$f_1 = 2$$

$$f_n = f_{n-1} \cdot f_{n-2} \text{ for } n \geq 2 .$$

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Let $n = 2^k$:

$$g_k = 3^{k+1} - 2^{k+1}, \text{ hence}$$

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