Example: $a_n = 3a_{n-1} + n, a_0 = 1$

5. Write $f(z)$ as a formal power series:

This leads to the following conditions:

$$A + B + C = 1$$
$$2A + 4B + 3C = 1$$
$$A + 3B = 1$$

which gives

$$A = \frac{7}{4} \quad B = -\frac{1}{4} \quad C = -\frac{1}{2}$$

Example: $a_n = 3a_{n-1} + n, a_0 = 1$

5. Write $f(z)$ as a formal power series:

$$\begin{aligned} A(z) &= \frac{7}{4} \cdot \frac{1}{1-3z} - \frac{1}{4} \cdot \frac{1}{1-z} - \frac{1}{2} \cdot \frac{1}{(1-z)^2} \\ &= \frac{7}{4} \cdot \sum_{n\ge 0} 3^n z^n - \frac{1}{4} \cdot \sum_{n\ge 0} z^n - \frac{1}{2} \cdot \sum_{n\ge 0} (n+1) z^n \\ &= \sum_{n\ge 0} \left(\frac{7}{4} \cdot 3^n - \frac{1}{4} - \frac{1}{2}(n+1)\right) z^n \\ &= \sum_{n\ge 0} \left(\frac{7}{4} \cdot 3^n - \frac{1}{2} n - \frac{3}{4}\right) z^n \end{aligned}$$

6. This means $a_n = \frac{7}{4}3^n - \frac{1}{2}n - \frac{3}{4}$.

6.5 Transformation of the Recurrence

Example 6

$$f_0 = 1$$
$$f_1 = 2$$
$$f_n = f_{n-1} \cdot f_{n-2} \text{ for } n \ge 2 .$$

Define

$$g_n := \log f_n .$$

Then

$$g_n = g_{n-1} + g_{n-2} \text{ for } n \ge 2$$
$$g_1 = \log 2 = 1 (\text{for } \log = \log_2),\; g_0 = 0$$
$$g_n = F_n \;(n\text{-th Fibonacci number})$$
$$f_n = 2^{F_n}$$

6.5 Transformation of the Recurrence

Example 7

$$f_1 = 1$$
$$f_n = 3f_{\frac{n}{2}} + n;\text{ for } n = 2^k,\, k \ge 1\;;$$

Define

$$g_k := f_{2^k} .$$

Then:

$$g_0 = 1$$
$$g_k = 3g_{k-1} + 2^k,\; k \ge 1$$

6 Recurrences

We get

$$\begin{aligned} g_k &= 3\left[g_{k-1}\right] + 2^k \\ &= 3\left[3g_{k-2} + 2^{k-1}\right] + 2^k \\ &= 3^2\left[g_{k-2}\right] + 32^{k-1} + 2^k \\ &= 3^2\left[3g_{k-3} + 2^{k-2}\right] + 32^{k-1} + 2^k \\ &= 3^3 g_{k-3} + 3^2 2^{k-2} + 32^{k-1} + 2^k \\ &= 2^k \cdot \sum_{i=0}^{k} \left(\frac{3}{2}\right)^i \\ &= 2^k \cdot \frac{(\frac{3}{2})^{k+1} - 1}{1/2} = 3^{k+1} - 2^{k+1} \end{aligned}$$

6 Recurrences

Let $n = 2^k$:

$$\begin{aligned} g_k &= 3^{k+1} - 2^{k+1}, \text{ hence} \\ f_n &= 3 \cdot 3^k - 2 \cdot 2^k \\ &= 3(2^{\log 3})^k - 2 \cdot 2^k \\ &= 3(2^k)^{\log 3} - 2 \cdot 2^k \\ &= 3n^{\log 3} - 2n \ . \end{aligned}$$

6 Recurrences

Bibliography

[MS08] Kurt Mehlhorn, Peter Sanders:
Algorithms and Data Structures — The Basic Toolbox,
Springer, 2008

[CLRS90] Thomas H. Cormen, Charles E. Leiserson, Ron L. Rivest, Clifford Stein:
Introduction to algorithms (3rd ed.),
MIT Press and McGraw-Hill, 2009

[Liu85] Chung Laung Liu:
Elements of Discrete Mathematics
McGraw-Hill, 1985

The Karatsuba method can be found in [MS08] Chapter 1. Chapter 4.3 of [CLRS90] covers the "Substitution method" which roughly corresponds to "Guessing+induction". Chapters 4.4, 4.5, 4.6 of this book cover the master theorem. Methods using the characteristic polynomial and generating functions can be found in [Liu85] Chapter 10.