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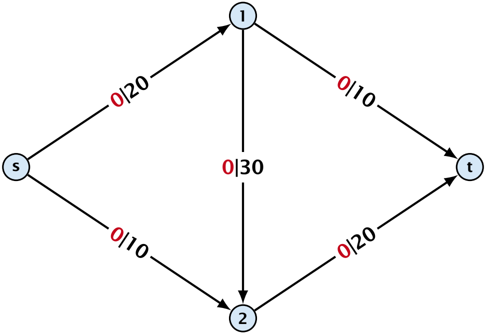

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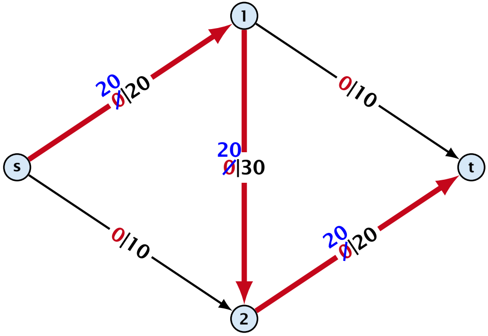

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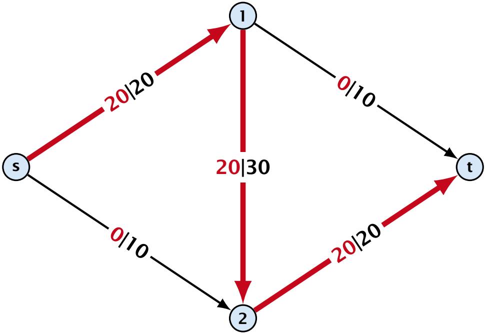

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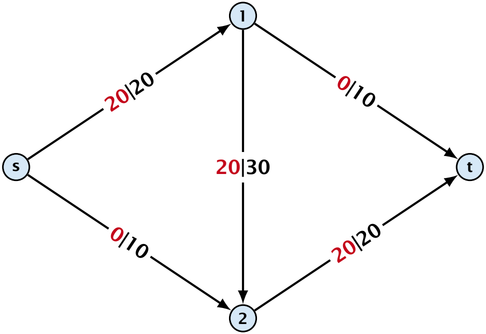

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- G_f has edge e_1' with capacity $\max\{0, c(e_1) - f(e_1) + f(e_2)\}$ and e_2' with with capacity $\max\{0, c(e_2) - f(e_2) + f(e_1)\}$.

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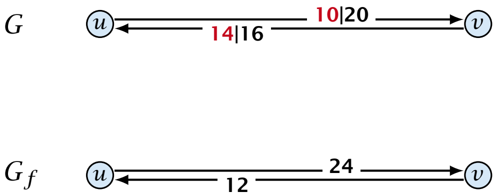

Augmenting Path Algorithm

Definition 1
An augmenting path with respect to flow f, is a path from s to t in the auxiliary graph G_f that contains only edges with non-zero capacity.

Algorithm 46 FordFulkerson($G = (V, E, c)$)

1: Initialize $f(e) \leftarrow 0$ for all edges.
2: **while** $\exists$ augmenting path p in G_f **do**
3: augment as much flow along p as possible.

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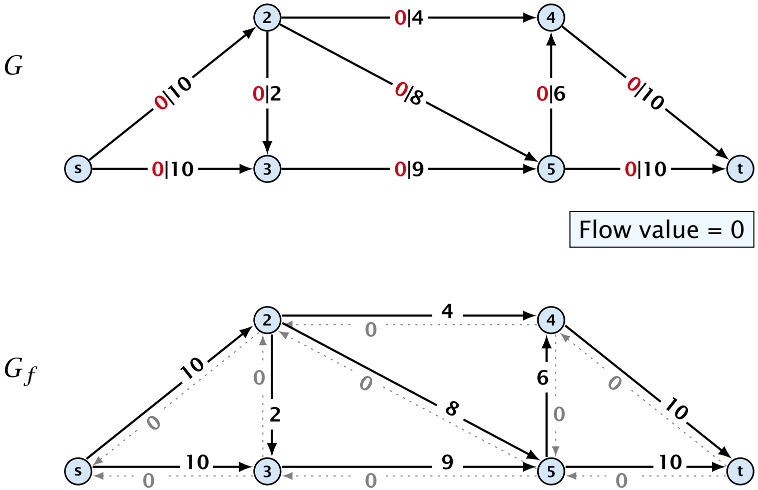

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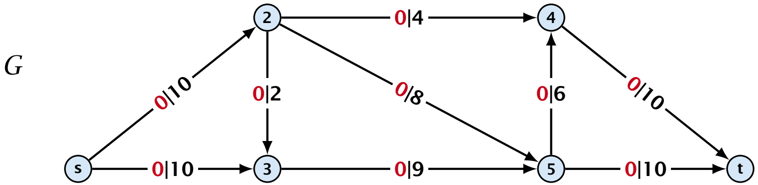
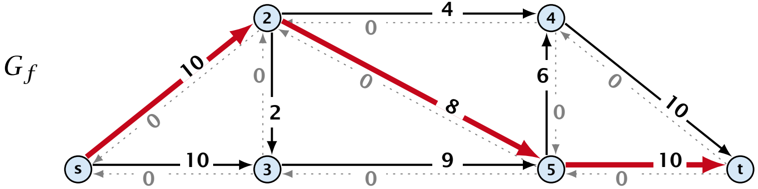


Flow value = 0



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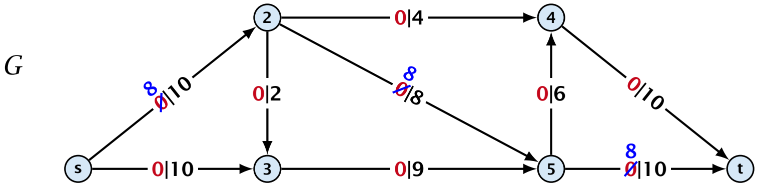
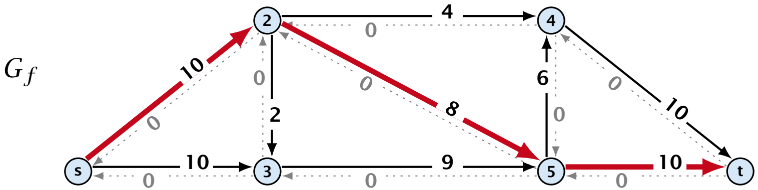
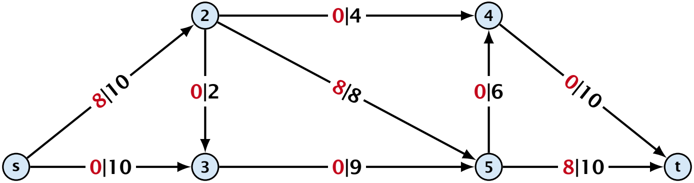


Flow value = 8



Augmenting Path Algorithm

G



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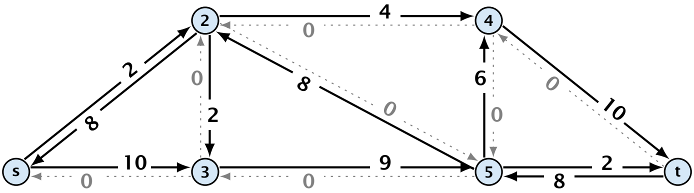

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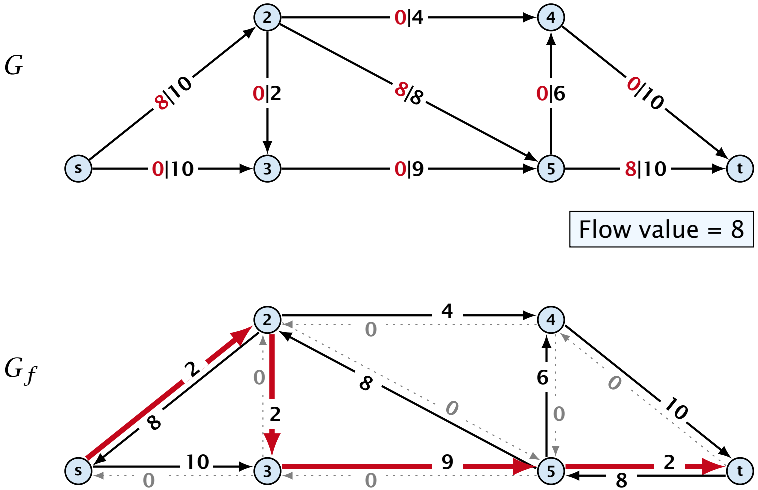

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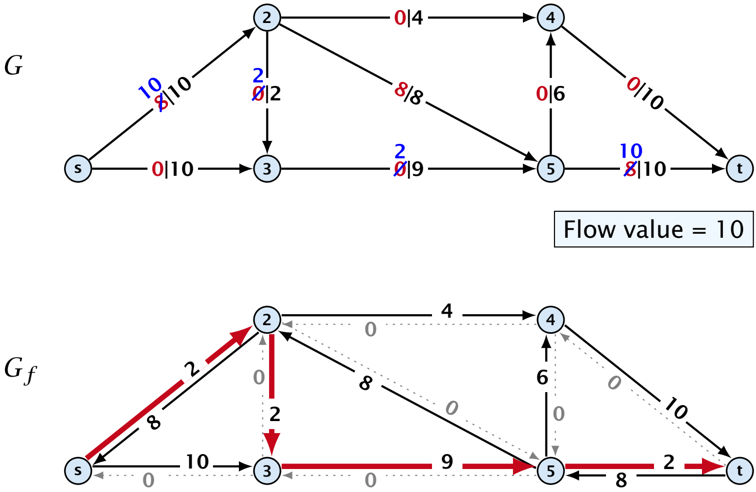

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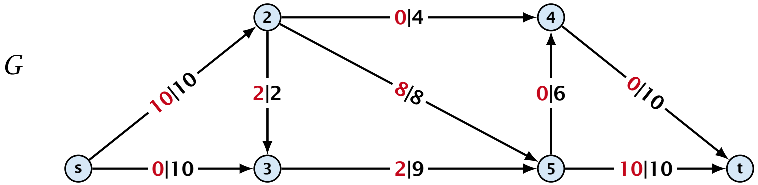
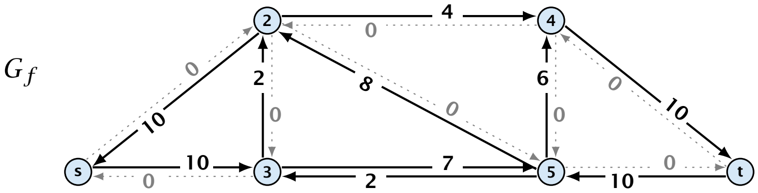


Flow value = 10



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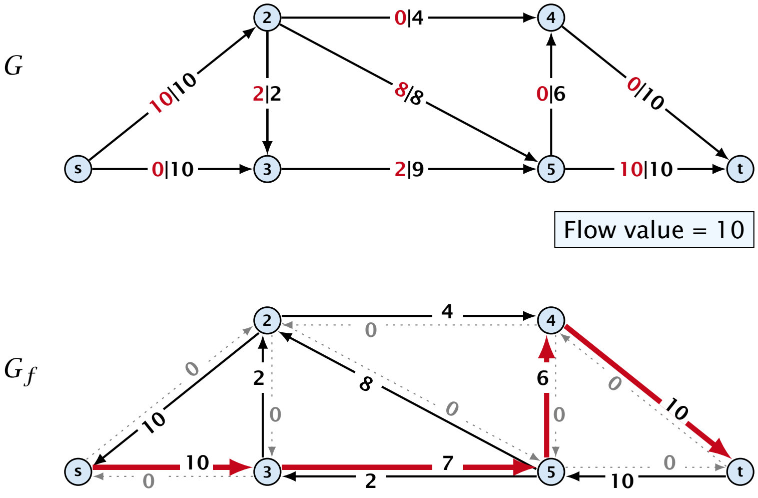

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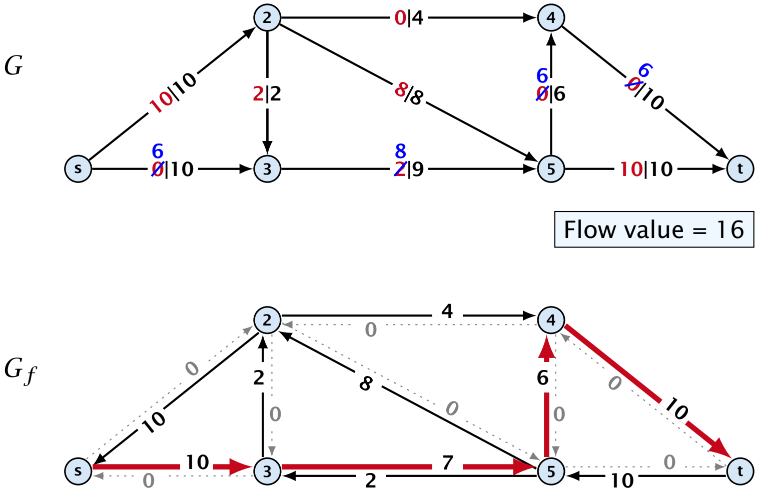

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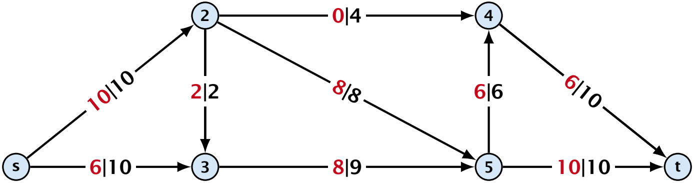
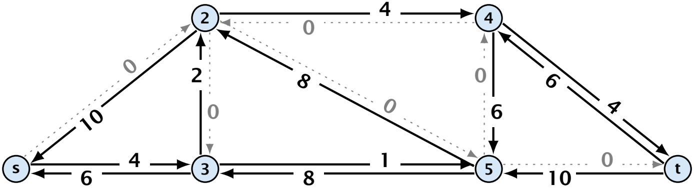


Flow value = 16

G_f



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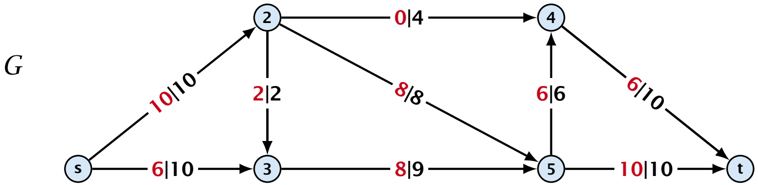

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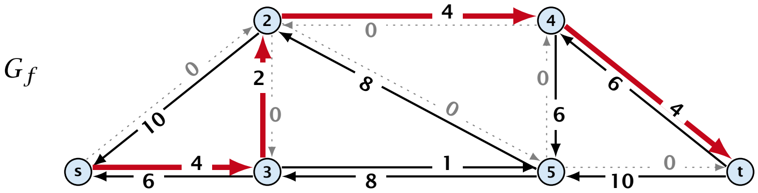

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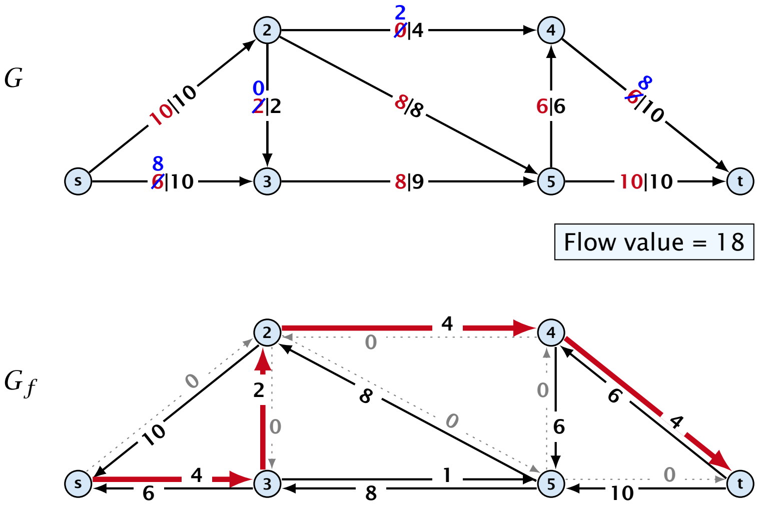

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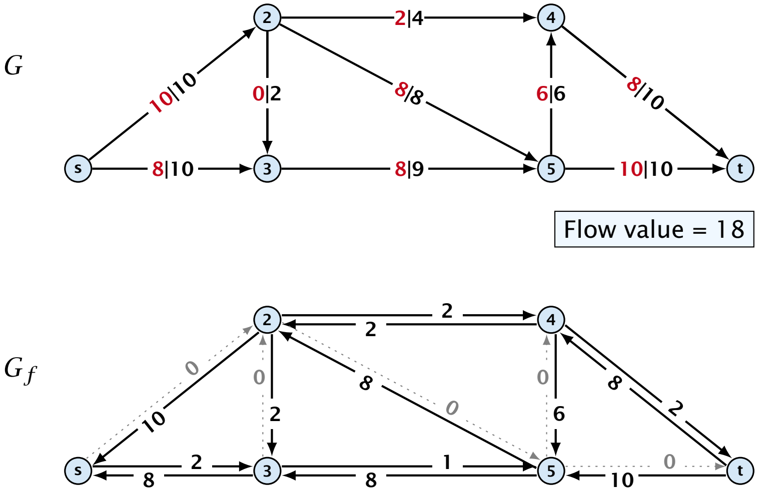

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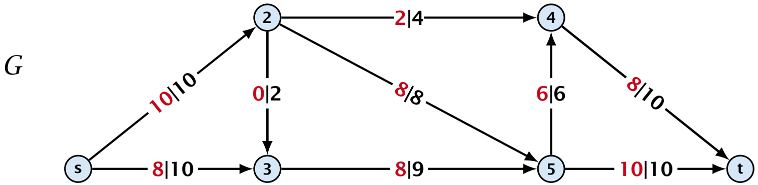
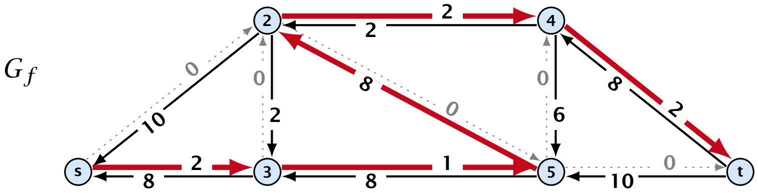


Flow value = 18



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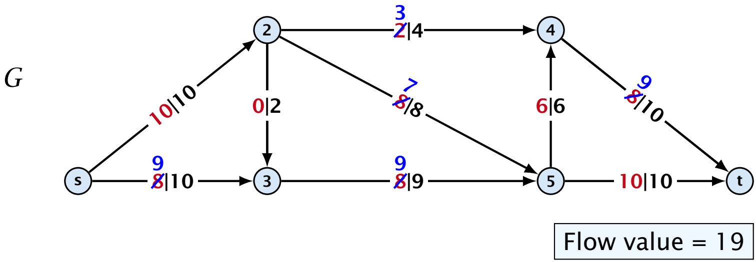

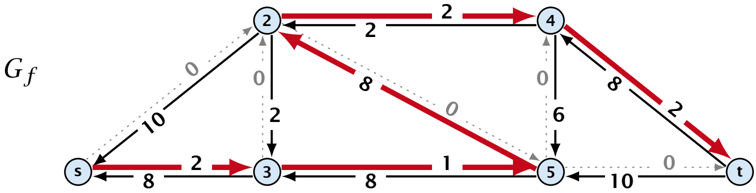

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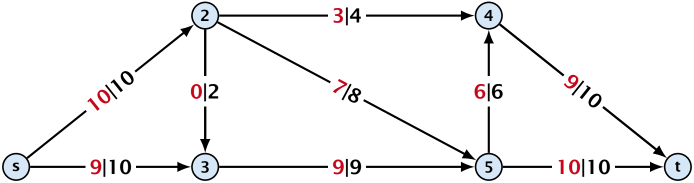

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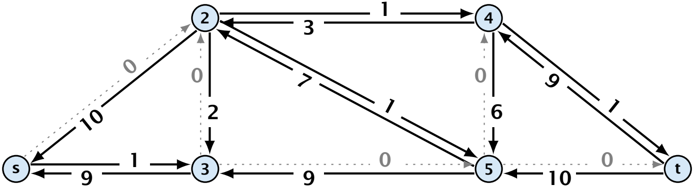

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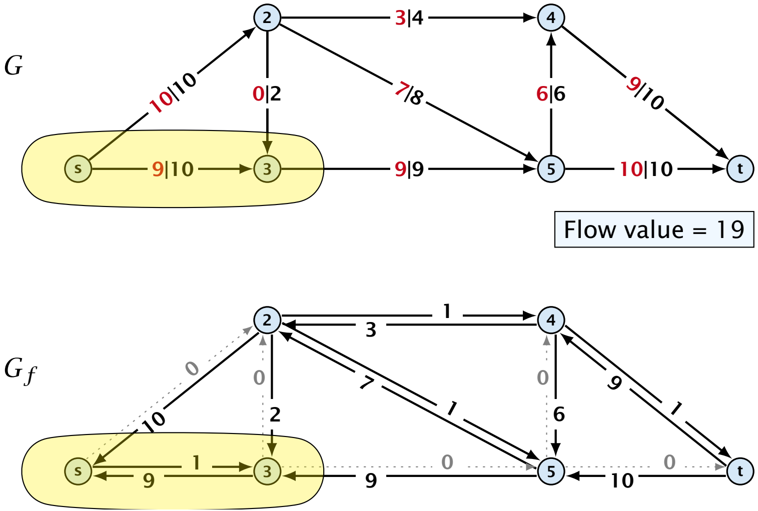

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Theorem 2

A flow f is a maximum flow ***iff*** *there are no augmenting paths.*

Theorem 3

The value of a maximum flow is equal to the value of a minimum cut.

Proof.

Let f be a flow. The following are equivalent:

1. There exists a cut A, B such that $\text{val}(f) = \text{cap}(A, B)$.
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2. $\Longrightarrow$ 3.
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Augmenting Path Algorithm

$$\mathrm{val}(f)$$

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This finishes the proof.

Here the first equality uses the flow value lemma, and the second exploits the fact that the flow along incoming edges must be 0 as the residual graph does not have edges leaving A.

Analysis

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All capacities are integers between 1 and C.

Invariant:
Every flow value $f(e)$ and every residual capacity $c_f(e)$ remains integral troughout the algorithm.

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The algorithm terminates in at most $\mathrm{val}(f^*) \le nC$ *iterations, where* f^* *denotes the maximum flow. Each iteration can be implemented in time* $\mathcal{O}(m)$. *This gives a total running time of* $\mathcal{O}(nmC)$.

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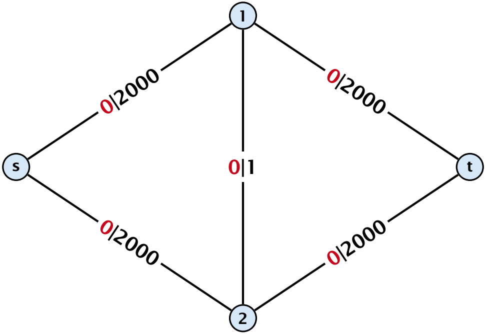
A Bad Input

Problem: The running time may not be polynomial.



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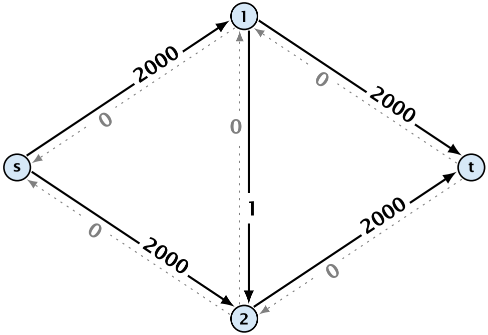

Question:
Can we tweak the algorithm so that the running time is polynomial in the input length?

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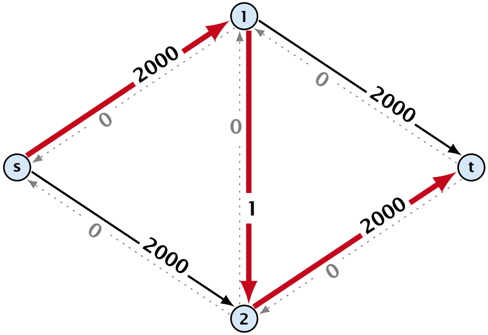

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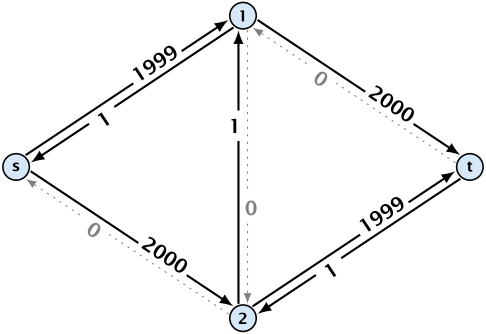

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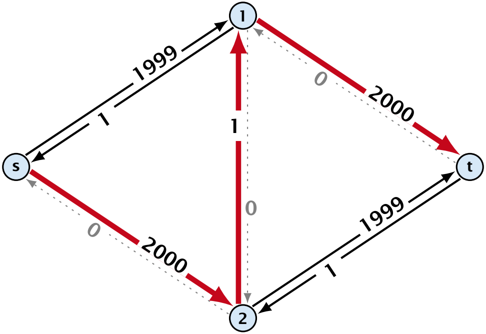

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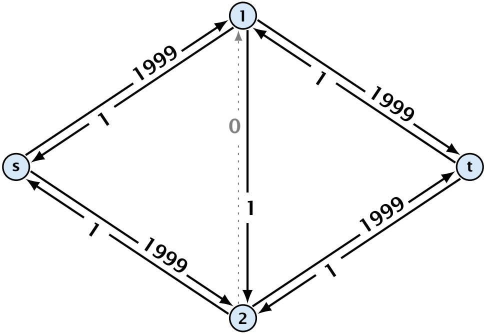

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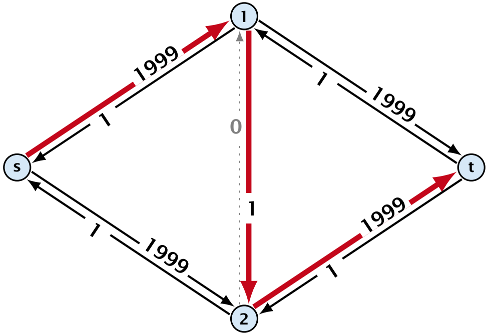

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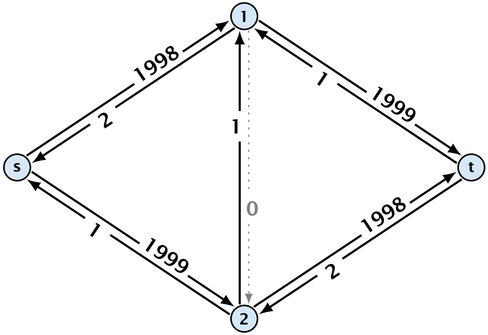

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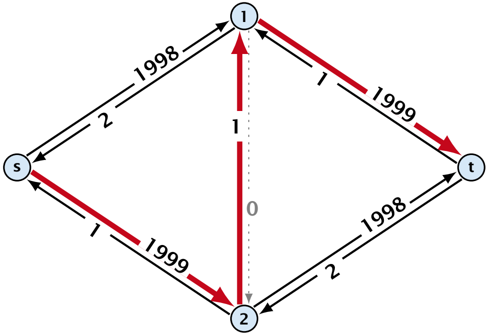

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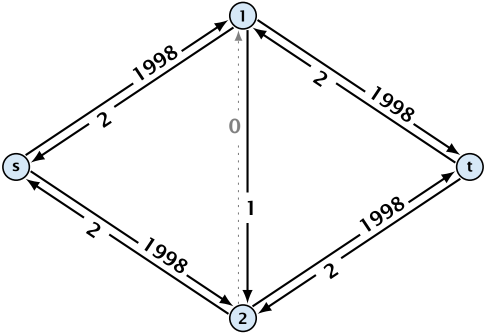

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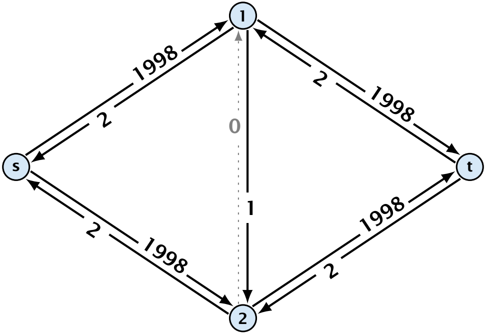

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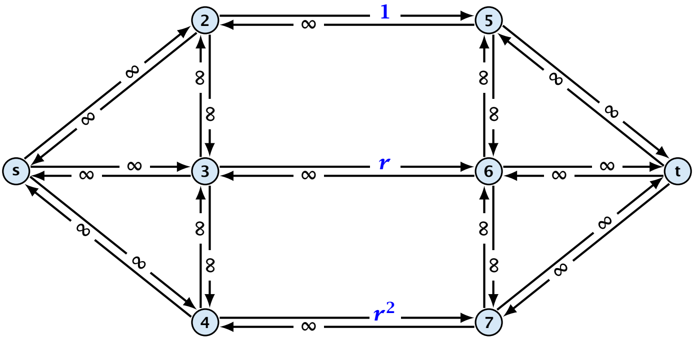
A Pathological Input

Let $r = \frac{1}{2}(\sqrt{5} - 1)$. Then $r^{n+2} = r^n - r^{n+1}$.



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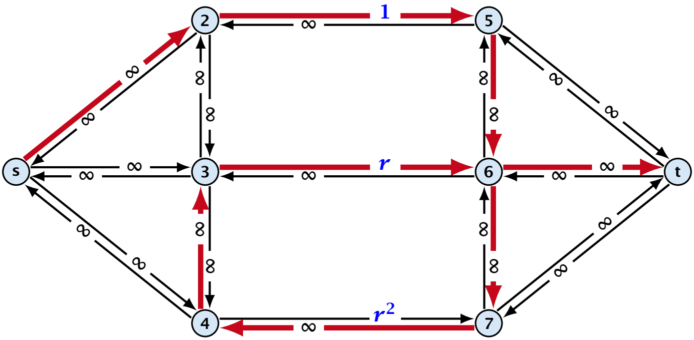

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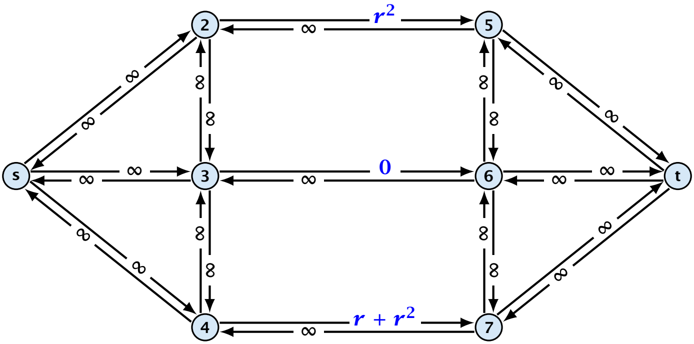

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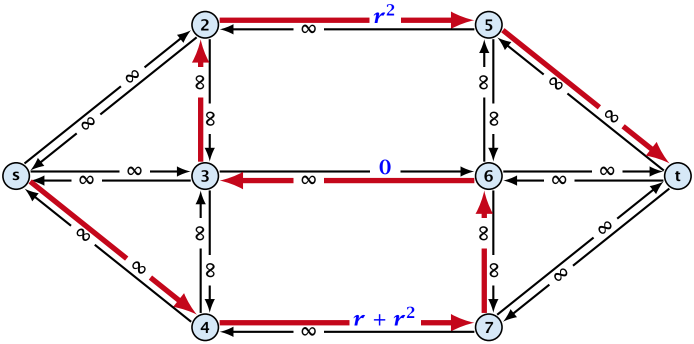

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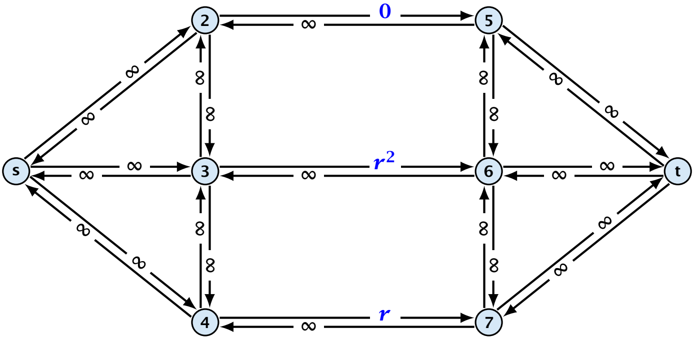

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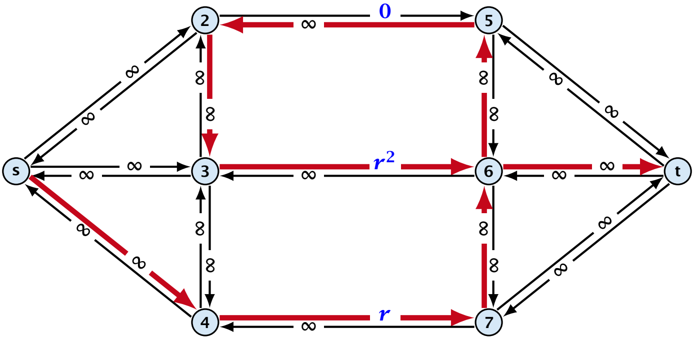

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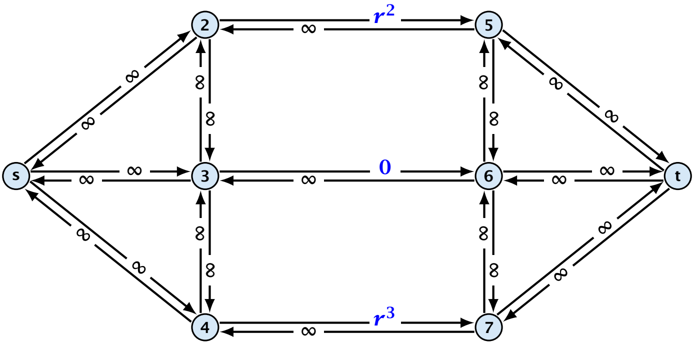

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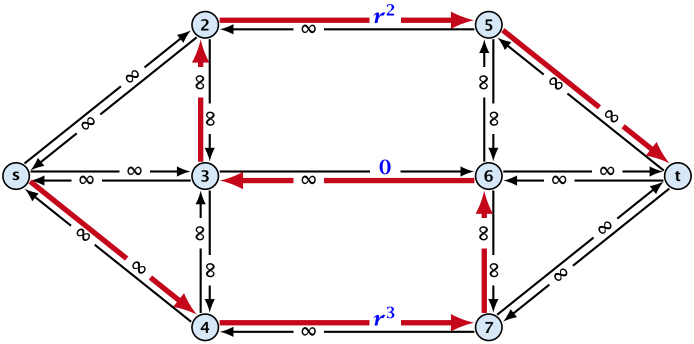

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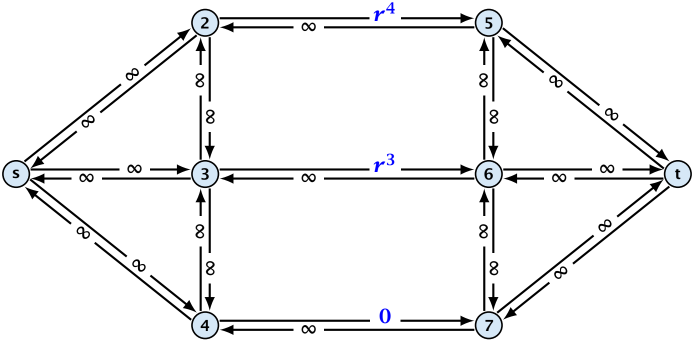

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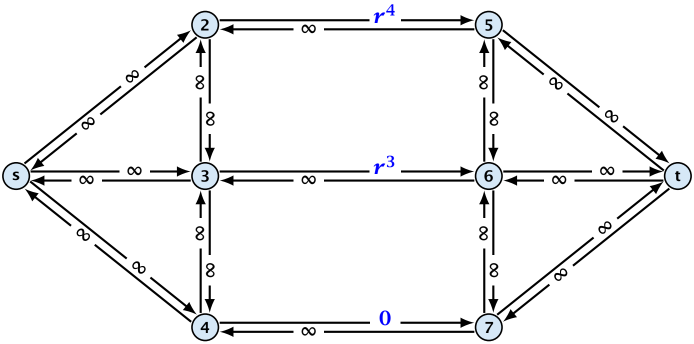

Running time may be infinite!!!

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- We need to find paths efficiently.
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Several possibilities:

- Choose path with maximum bottleneck capacity.
- Choose path with sufficiently large bottleneck capacity.
- Choose the shortest augmenting path.