Amortized Analysis

Definition 1
A data structure with operations $\mathrm{op}_1(), \ldots, \mathrm{op}_k()$ has amortized running times $t_1, \ldots, t_k$ for these operations if the following holds.

Suppose you are given a sequence of operations (starting with an empty data-structure) that operate on at most n elements, and let k_i denote the number of occurences of $\mathrm{op}_i()$ within this sequence. Then the actual running time must be at most $\sum_i k_i \cdot t_i(n)$.

Potential Method

Introduce a potential for the data structure.

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- Amortized cost of the i-th operation is

$$\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1}) \ .$$

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- Show that $\Phi(D_i) \geq \Phi(D_0)$.

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Then

$$\sum_{i=1}^{k} c_i$$

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- Show that $\Phi(D_i) \geq \Phi(D_0)$.

Then

$$\sum_{i=1}^{k} c_i \leq \sum_{i=1}^{k} c_i + \Phi(D_k) - \Phi(D_0) = \sum_{i=1}^{k} \hat{c}_i$$

This means the amortized costs can be used to derive a bound on the total cost.

Example: Stack

Stack

- $S.\mathbf{push}()$
- $S.\mathbf{pop}()$
- $S.\mathbf{multipop}(k)$: removes k items from the stack. If the stack currently contains less than k items it empties the stack.
- The user has to ensure that pop and multipop do not generate an underflow.

Actual cost:

- $S.\mathbf{push}()$: cost 1.
- $S.\mathbf{pop}()$: cost 1.
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Amortized cost:

- $S.\mathbf{push}()$: cost

$$\hat{C}_{\text{push}} = C_{\text{push}} + \Delta\Phi = 1 + 1 \le 2 \enspace .$$

- $S.\mathbf{pop}()$: cost

$$\hat{C}_{\text{pop}} = C_{\text{pop}} + \Delta\Phi = 1 - 1 \le 0 \enspace .$$

- $S.\mathbf{multipop}(k)$: cost

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Example: Binary Counter

Incrementing a binary counter:
Consider a computational model where each bit-operation costs one time-unit.

Incrementing an n-bit binary counter may require to examine n-bits, and maybe change them.

Actual cost:

- Changing bit from 0 to 1: cost 1.
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- Increment: cost is $k + 1$, where k is the number of consecutive ones in the least significant bit-positions (e.g, 001101 has $k = 1$).

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Example: Binary Counter

Choose potential function $\Phi(x) = k$, where k denotes the number of ones in the binary representation of x.

Amortized cost:

$$\hat{C}_{0\to1} = C_{0\to1} + \Delta\Phi = 1 + 1 \le 2$$

$$\hat{C}_{1\to0} = C_{1\to0} + \Delta\Phi = 1 - 1 \le 0$$

Let k denotes the number of consecutive ones in the least significant bit-positions. An increment involves k -operations, and one -operation.

Hence, the amortized cost is $k\hat{C}_{1\to0} + \hat{C}_{0\to1} \le 2$.

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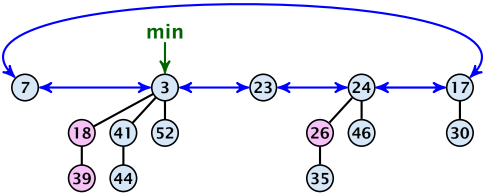
8.3 Fibonacci Heaps

Collection of trees that fulfill the heap property.

Structure is much more relaxed than binomial heaps.



8.3 Fibonacci Heaps

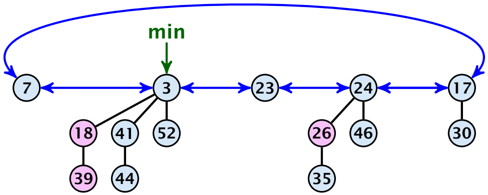
Additional implementation details:

- Every node x stores its degree in a field $x.\mathrm{degree}$. Note that this can be updated in constant time when adding a child to x.
- Every node stores a boolean value $x.\mathrm{marked}$ that specifies whether x is marked or not.

8.3 Fibonacci Heaps

The potential function:

- $t(S)$ denotes the number of trees in the heap.
- $m(S)$ denotes the number of marked nodes.
- We use the potential function $\Phi(S) = t(S) + 2m(S)$.



The potential is $\Phi(S) = 5 + 2 \cdot 3 = 11$.

8.3 Fibonacci Heaps

We assume that one unit of potential can pay for a constant amount of work, where the constant is chosen “big enough” (to take care of the constants that occur).

To make this more explicit we use c to denote the amount of work that a unit of potential can pay for.

8.3 Fibonacci Heaps

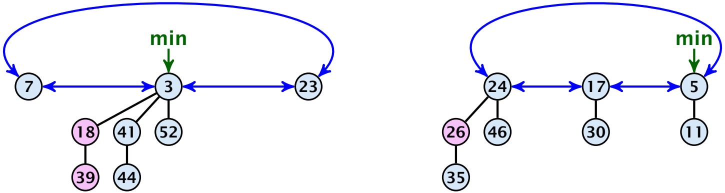
S. minimum()

- Access through the min-pointer.
- Actual cost $\mathcal{O}(1)$.
- No change in potential.
- Amortized cost $\mathcal{O}(1)$.

8.3 Fibonacci Heaps

$S.\mathbf{merge}(S')$

- Merge the root lists.
- Adjust the min-pointer



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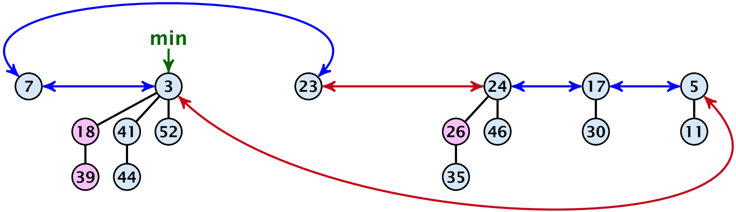

Running time:

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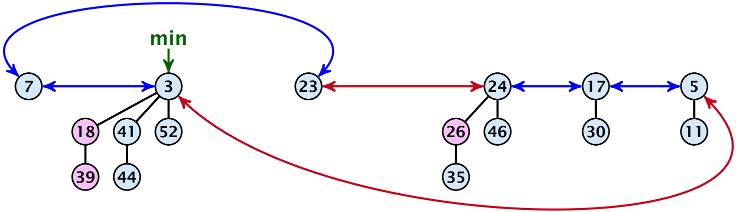

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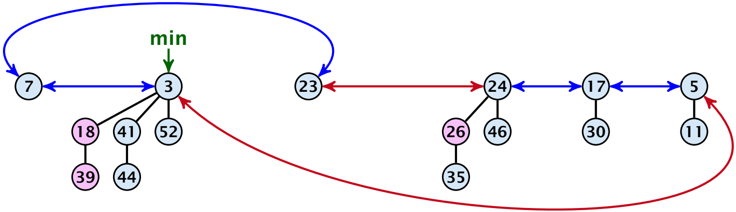


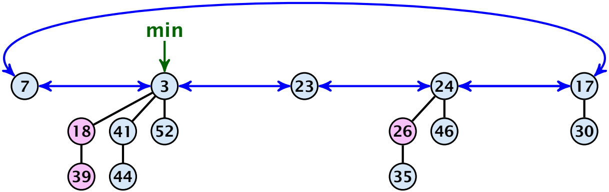
Running time:

- Actual cost $\mathcal{O}(1)$.
- No change in potential.
- Hence, amortized cost is $\mathcal{O}(1)$.

8.3 Fibonacci Heaps

$S.\text{insert}(x)$

- Create a new tree containing x.
- Insert x into the root-list.
- Update min-pointer, if necessary.



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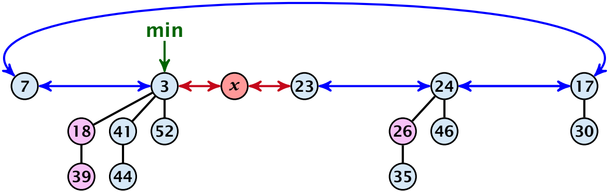

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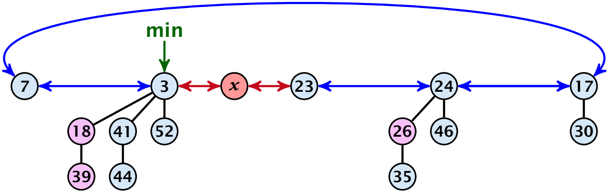
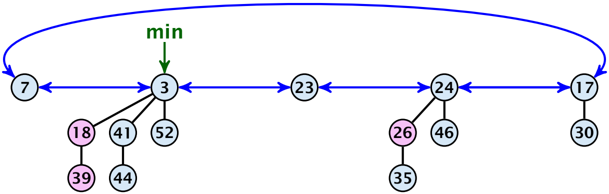


Running time:

- Actual cost $\mathcal{O}(1)$.
- Change in potential is $+1$.
- Amortized cost is $c + \mathcal{O}(1) = \mathcal{O}(1)$.

8.3 Fibonacci Heaps

S. delete-min(x)



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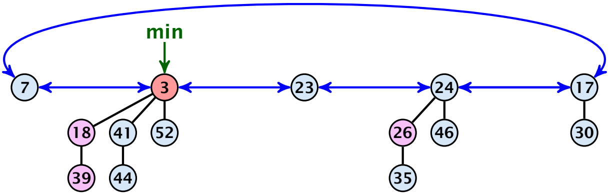
S. delete-min(x)

- Delete minimum; add child-trees to heap; time: $D(\min) \cdot \mathcal{O}(1)$.



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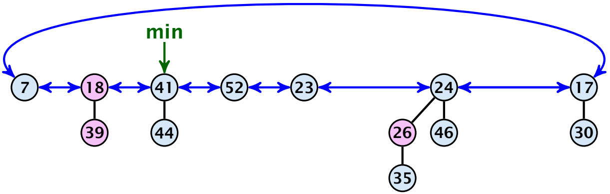
***S*. delete-min(x)**

- Delete minimum; add child-trees to heap; time: $D(\min) \cdot \mathcal{O}(1)$.
- Update min-pointer; time: $(t + D(\min)) \cdot \mathcal{O}(1)$.



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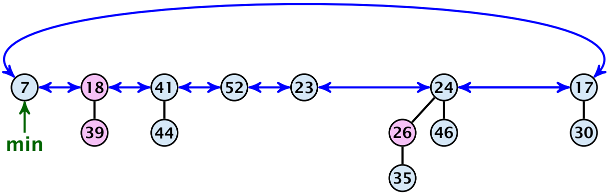

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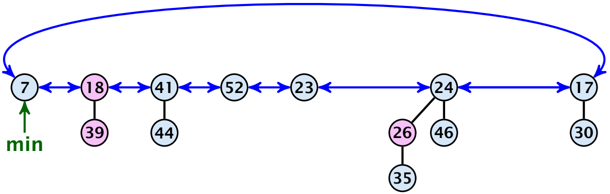
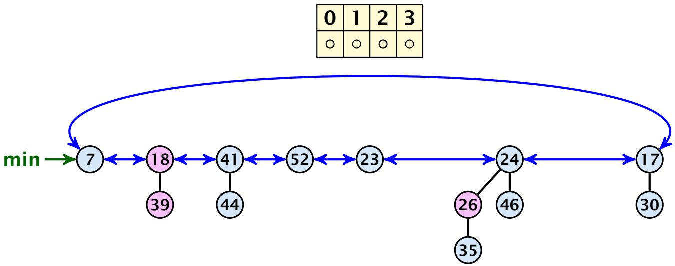


- Consolidate root-list so that no roots have the same degree.
  Time $t \cdot \mathcal{O}(1)$ (see next slide).

8.3 Fibonacci Heaps

Consolidate:



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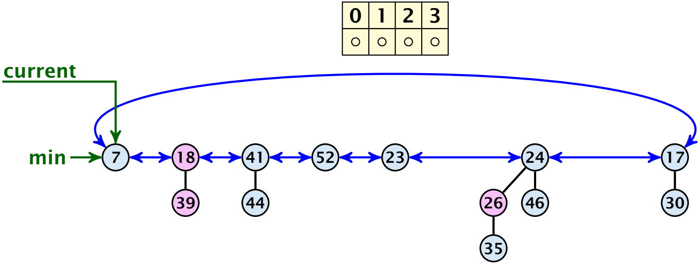

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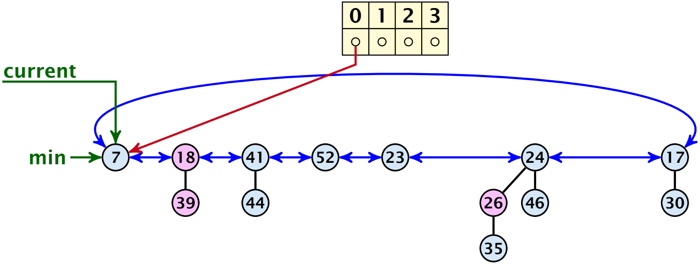

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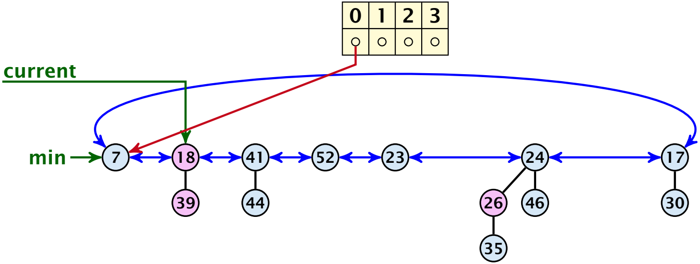

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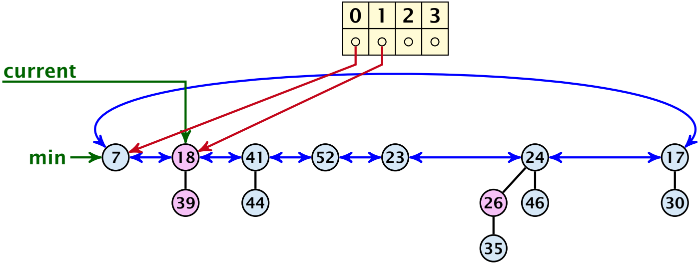

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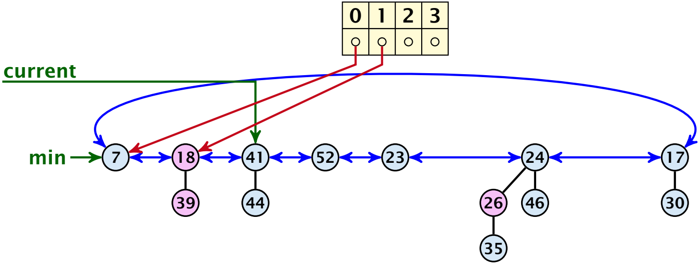

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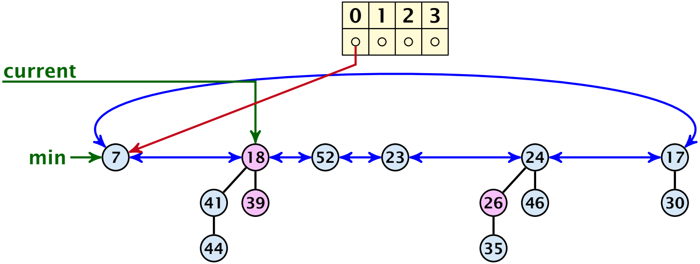

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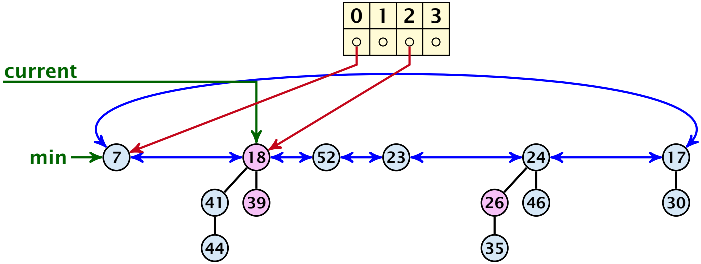

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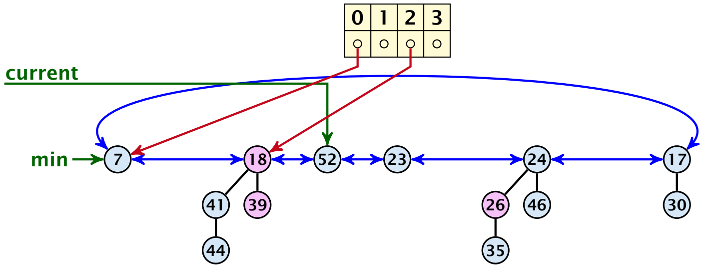

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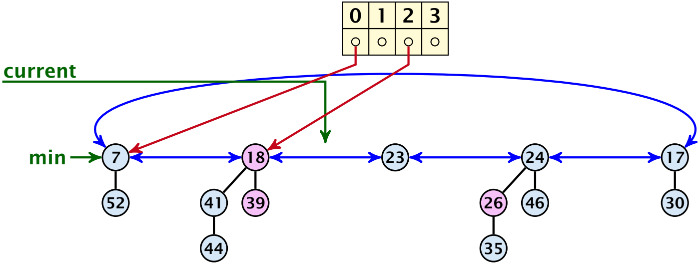

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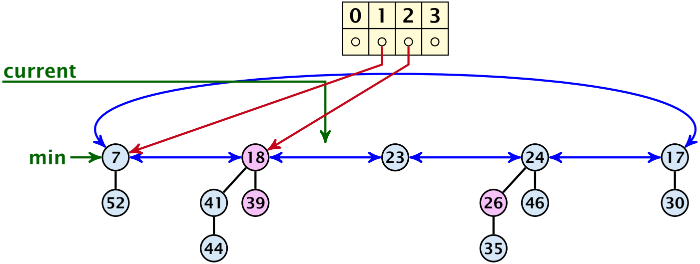

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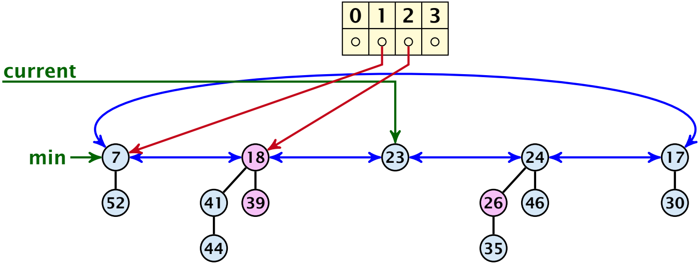

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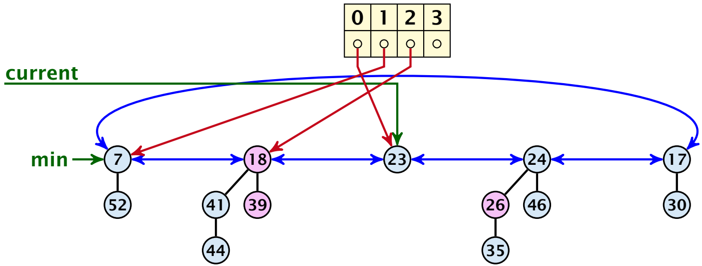

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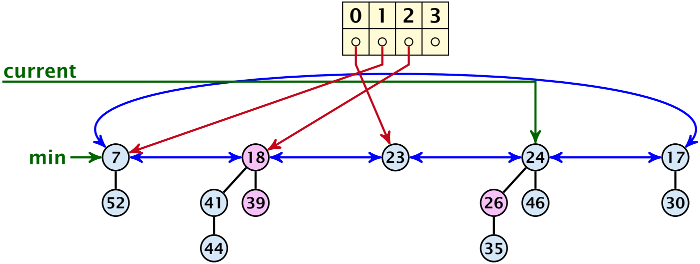

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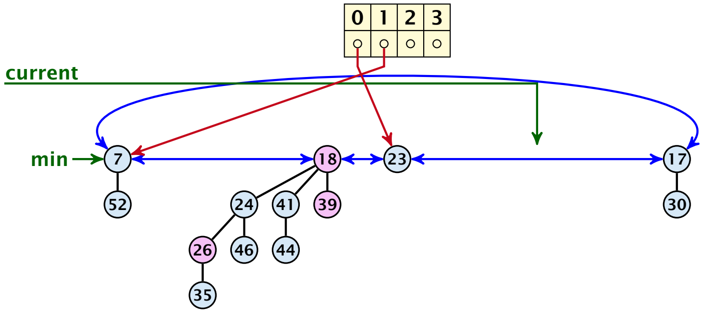

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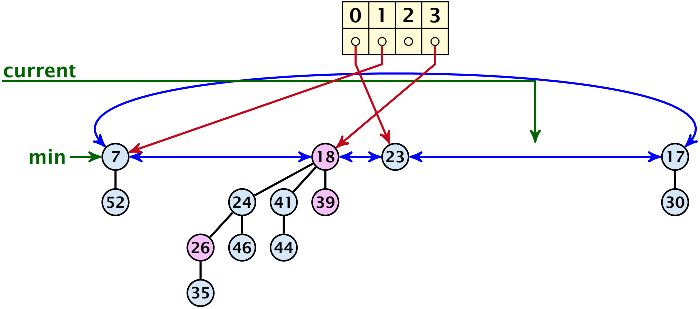

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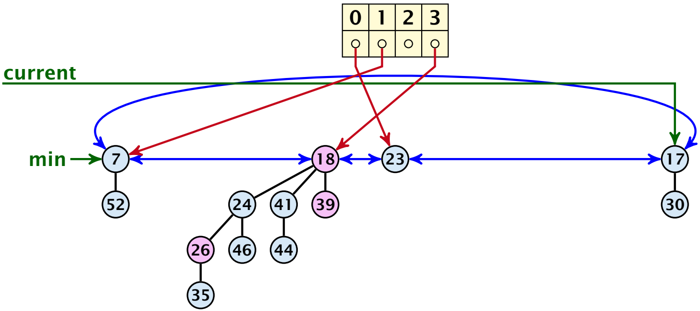

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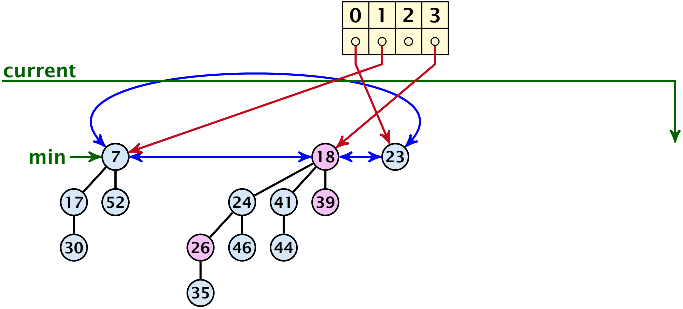

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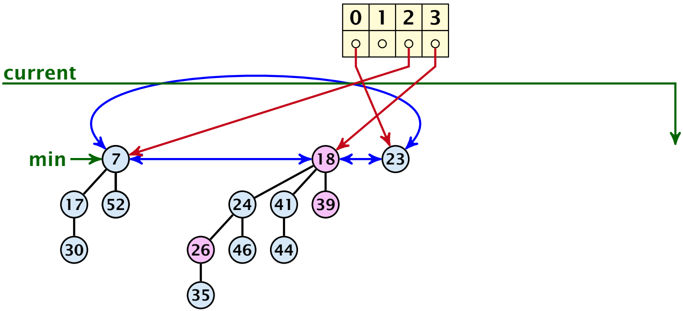

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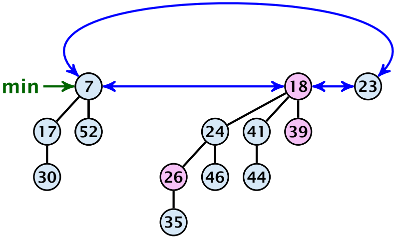

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8.3 Fibonacci Heaps

If the input trees of the consolidation procedure are binomial trees (for example only singleton vertices) then the output will be a set of distinct binomial trees, and, hence, the Fibonacci heap will be (more or less) a Binomial heap right after the consolidation.

If we do not have delete or decrease-key operations then $D_n \leq \log n$.

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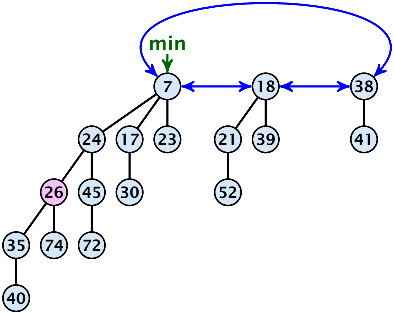
Fibonacci Heaps: decrease-key(handle h, v)



Case 1: decrease-key does not violate heap-property

- Just decrease the key-value of element referenced by h. Nothing else to do.

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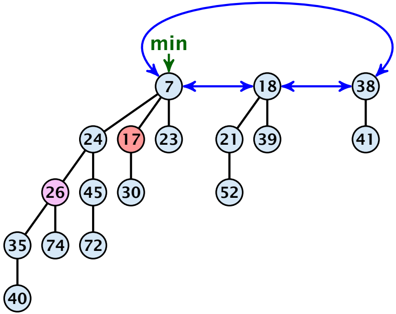

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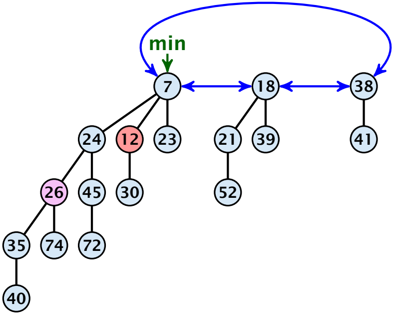

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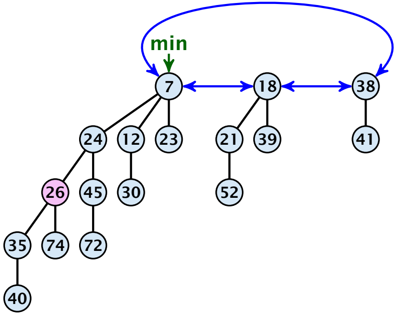

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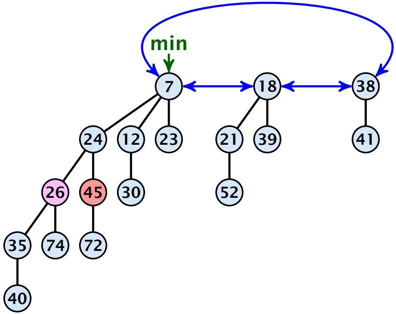

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- Mark the (previous) parent of x (unless it's a root).

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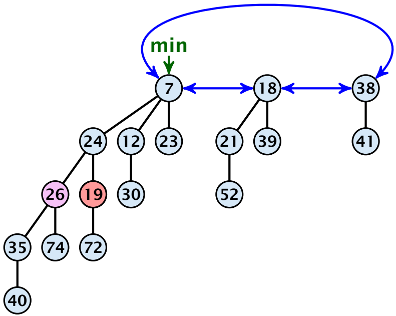

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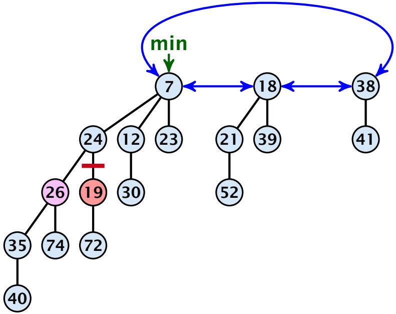

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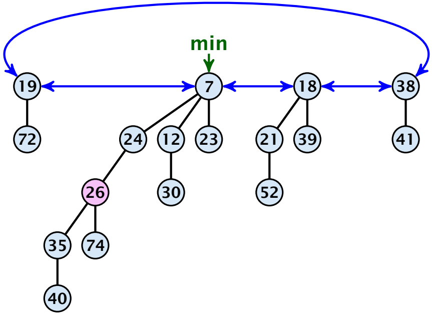

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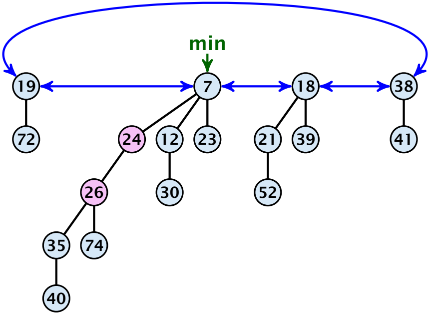

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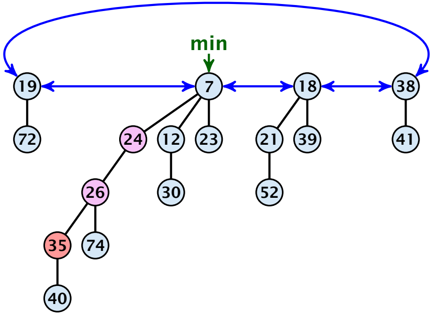

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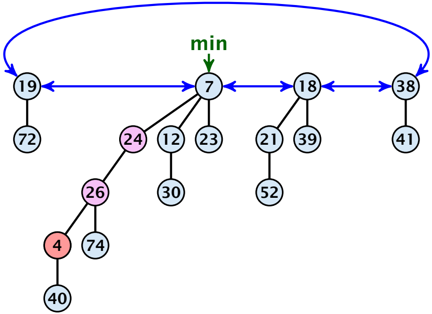

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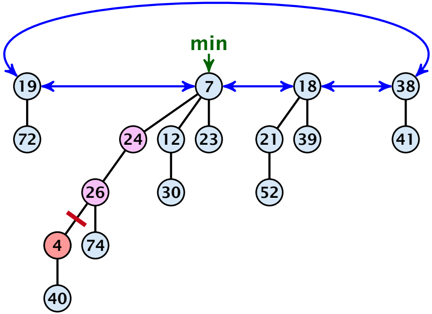

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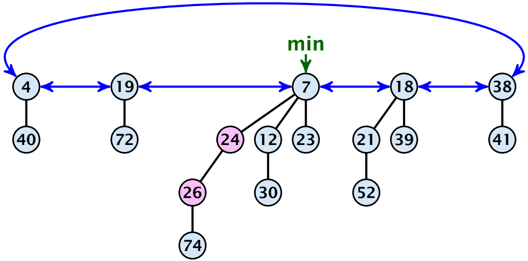

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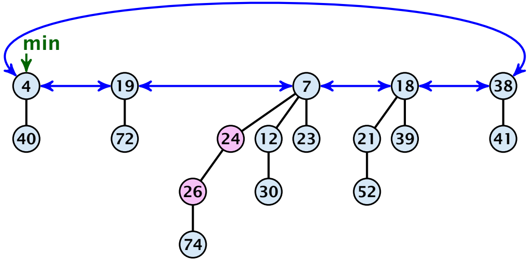

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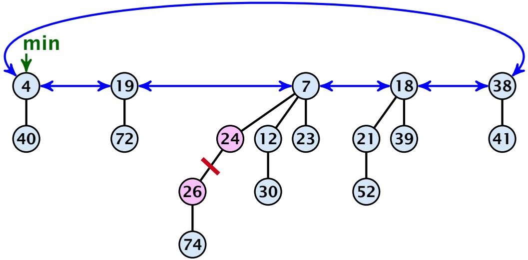

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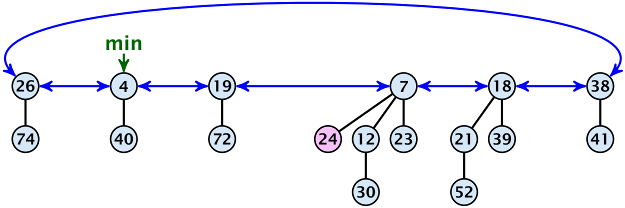

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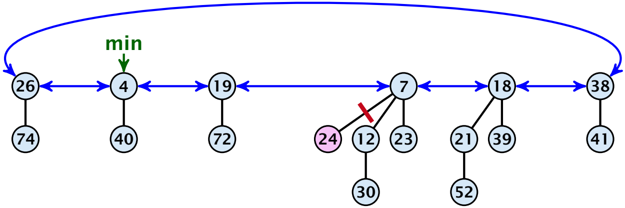

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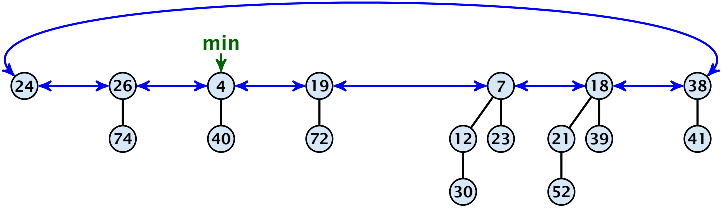

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- Adjust min-pointers, if necessary.
- Execute the following:

```
p ← parent[x];
while (p is marked)
    pp ← parent[p];
    cut of p; make it into a root; unmark it;
    p ← pp;
if p is unmarked and not a root mark it;
```

Fibonacci Heaps: decrease-key(handle h, v)

Actual cost:

- Constant cost for decreasing the value.
- Constant cost for each of ℓ cuts.
- Hence, cost is at most $c_2 \cdot (\ell + 1)$, for some constant c_2.

Amortized cost:

- $t' = t + \ell$, as every cut creates one new root.
- $m' \leq m - (\ell - 1) + 1 = m - \ell + 2$, since all but the first cut unmarks a node; the last cut may mark a node.
- $\Delta\Phi \leq \ell + 2(-\ell + 2) = 4 - \ell$
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  $$c_2(\ell+1)+c(4-\ell) \leq (c_2-c)\ell+4c+c_2 = \mathcal{O}(1)\,,$$

  if $c \geq c_2$.

Delete node

$H.\mathbf{delete}(x)$:

- decrease value of x to $-\infty$.
- delete-min.

Amortized cost: $\mathcal{O}(D_n)$

- $\mathcal{O}(1)$ for decrease-key.
- $\mathcal{O}(Dn)$ for delete-min.

8.3 Fibonacci Heaps

Lemma 2

Let x be a node with degree k and let $y_1, \ldots, y_k$ denote the children of x in the order that they were linked to x. Then

$$\mathrm{degree}(y_i) \geq \begin{cases} 0 & \text{if } i = 1 \\ i - 2 & \text{if } i > 1 \end{cases}$$

8.3 Fibonacci Heaps

Proof

- When y_i was linked to x, at least $y_1, \ldots, y_{i-1}$ were already linked to x.
- Hence, at this time $\text{degree}(x) \geq i - 1$, and therefore also $\text{degree}(y_i) \geq i - 1$ as the algorithm links nodes of equal degree only.
- Since, then y_i has lost at most one child.
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8.3 Fibonacci Heaps

Definition 3

Consider the following non-standard Fibonacci type sequence:

$$F_k = \begin{cases} 1 & \text{if } k = 0 \\ 2 & \text{if } k = 1 \\ F_{k-1} + F_{k-2} & \text{if } k \geq 2 \end{cases}$$

Facts:

1. $F_k \geq \phi^k$.
2. For $k \geq 2$: $F_k = 2 + \sum_{i=0}^{k-2} F_i$.

The above facts can be easily proved by induction. From this it follows that $s_k \geq F_k \geq \phi^k$, which gives that the maximum degree in a Fibonacci heap is logarithmic.

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