

7 Dictionary

Dictionary:

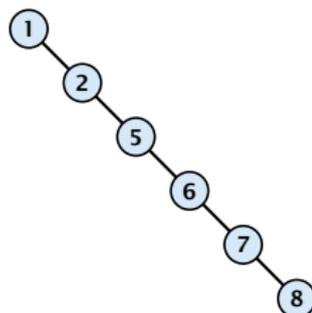
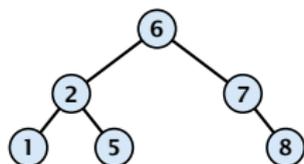
- ▶ **$S.insert(x)$** : Insert an element x .
- ▶ **$S.delete(x)$** : Delete the element pointed to by x .
- ▶ **$S.search(k)$** : Return a pointer to an element e with $key[e] = k$ in S if it exists; otherwise return null.

7.1 Binary Search Trees

An (**internal**) **binary search tree** stores the elements in a binary tree. Each tree-node corresponds to an element. All elements in the left sub-tree of a node v have a smaller key-value than $\text{key}[v]$ and elements in the right sub-tree have a larger-key value. We assume that all key-values are different.

(**External** Search Trees store objects only at leaf-vertices)

Examples:

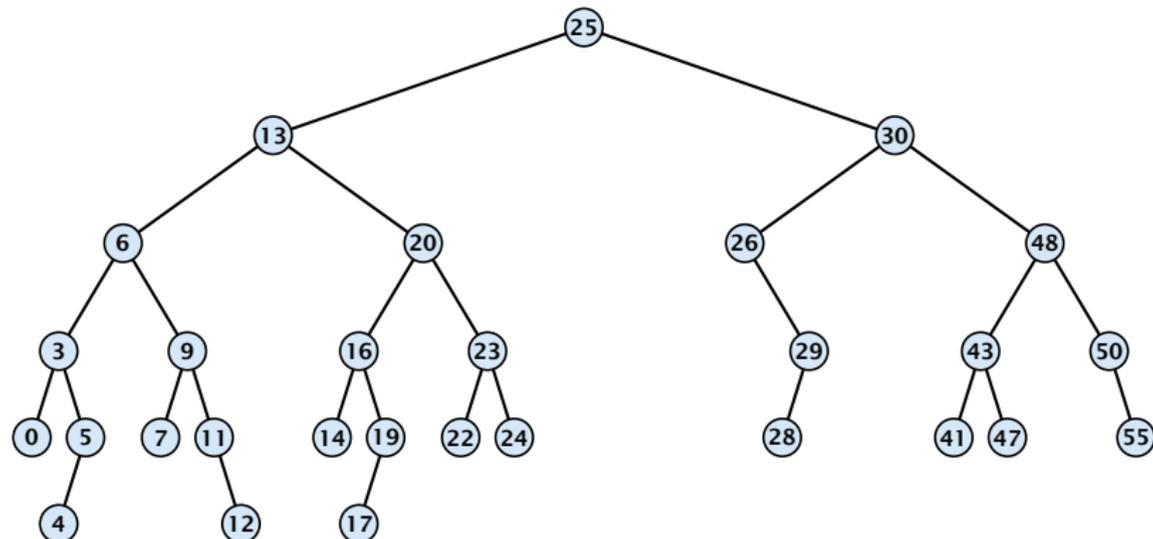


7.1 Binary Search Trees

We consider the following operations on binary search trees. Note that this is a super-set of the dictionary-operations.

- ▶ $T.$ **insert**(x)
- ▶ $T.$ **delete**(x)
- ▶ $T.$ **search**(k)
- ▶ $T.$ **successor**(x)
- ▶ $T.$ **predecessor**(x)
- ▶ $T.$ **minimum**()
- ▶ $T.$ **maximum**()

Binary Search Trees: Searching

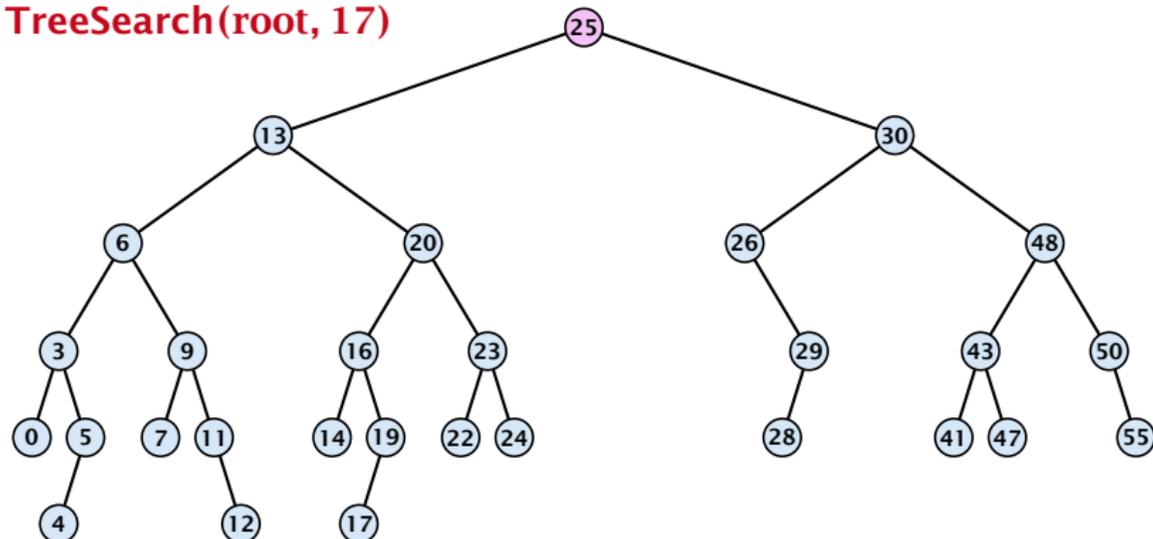


Algorithm 5 TreeSearch(x, k)

- 1: **if** $x = \text{null}$ **or** $k = \text{key}[x]$ **return** x
- 2: **if** $k < \text{key}[x]$ **return** TreeSearch(left[x], k)
- 3: **else return** TreeSearch(right[x], k)

Binary Search Trees: Searching

TreeSearch(root, 17)

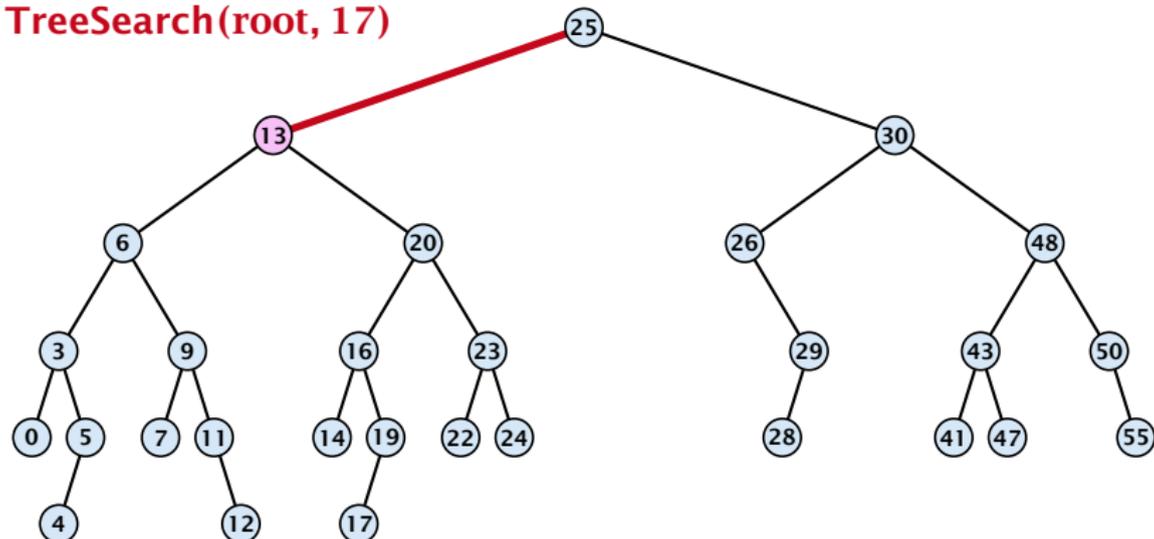


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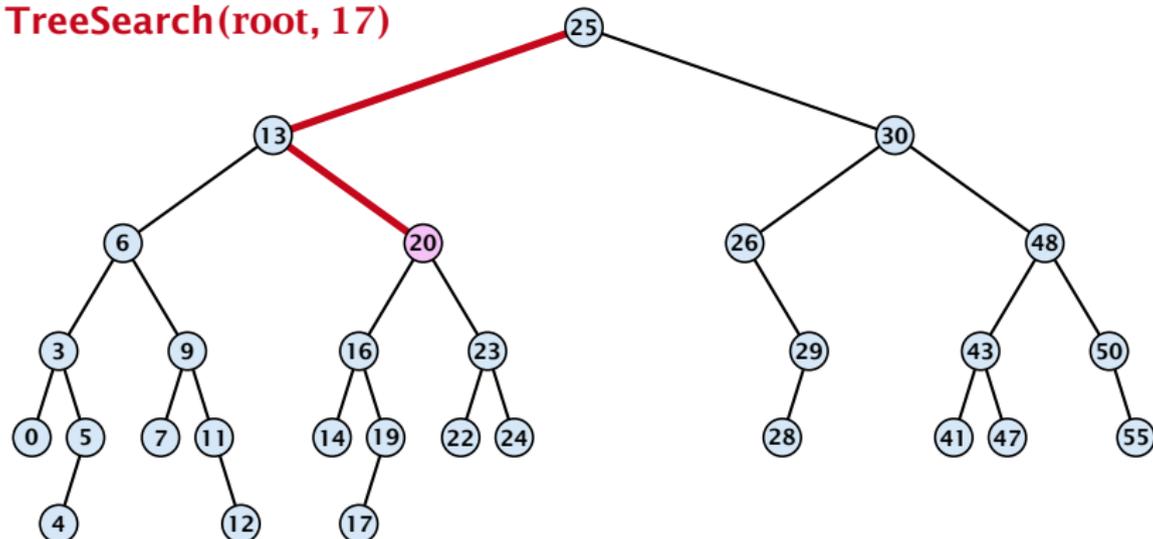


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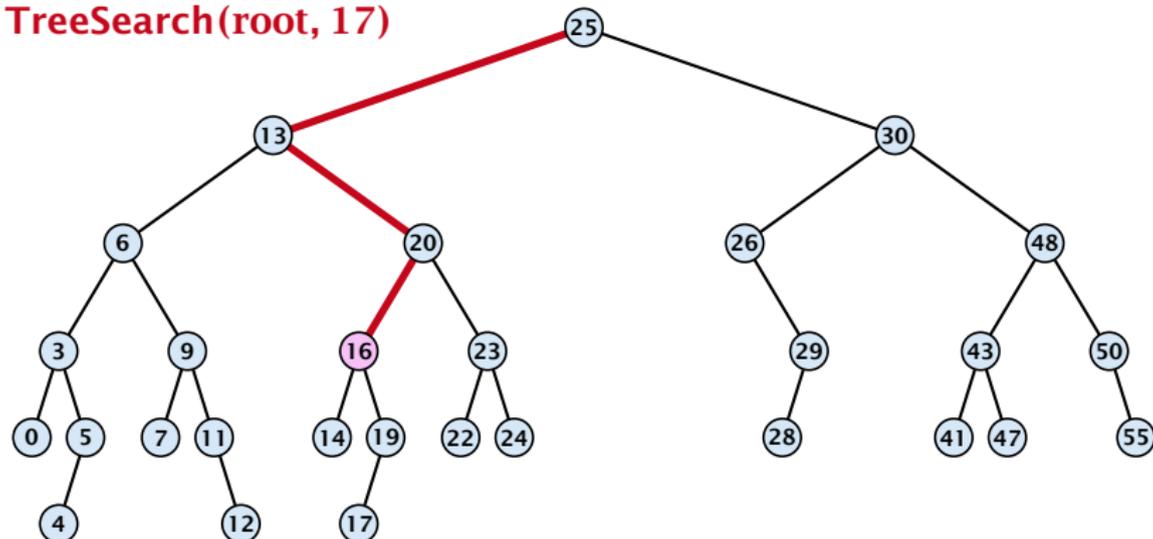


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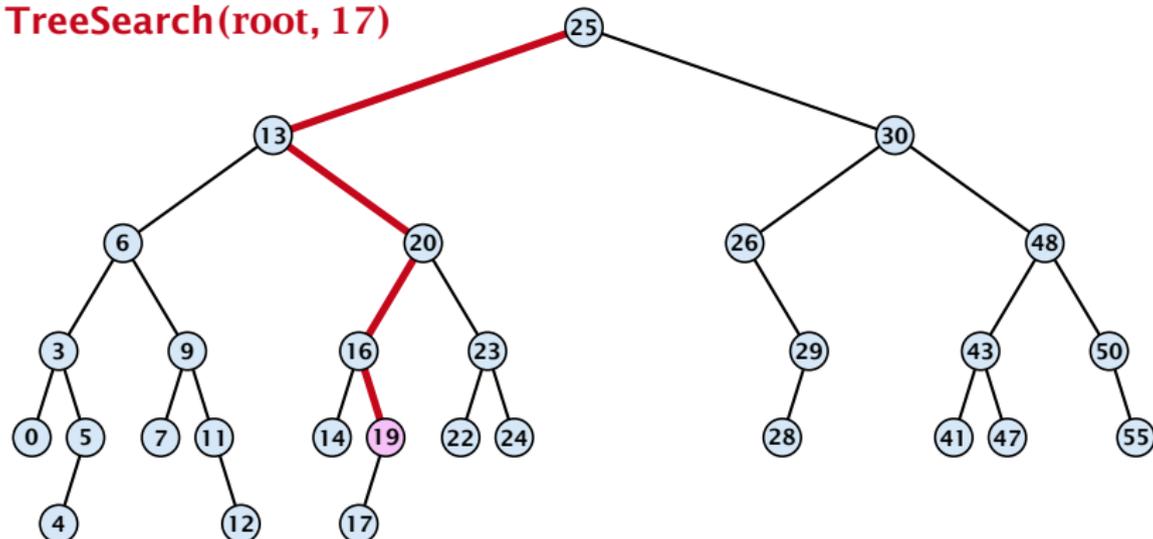


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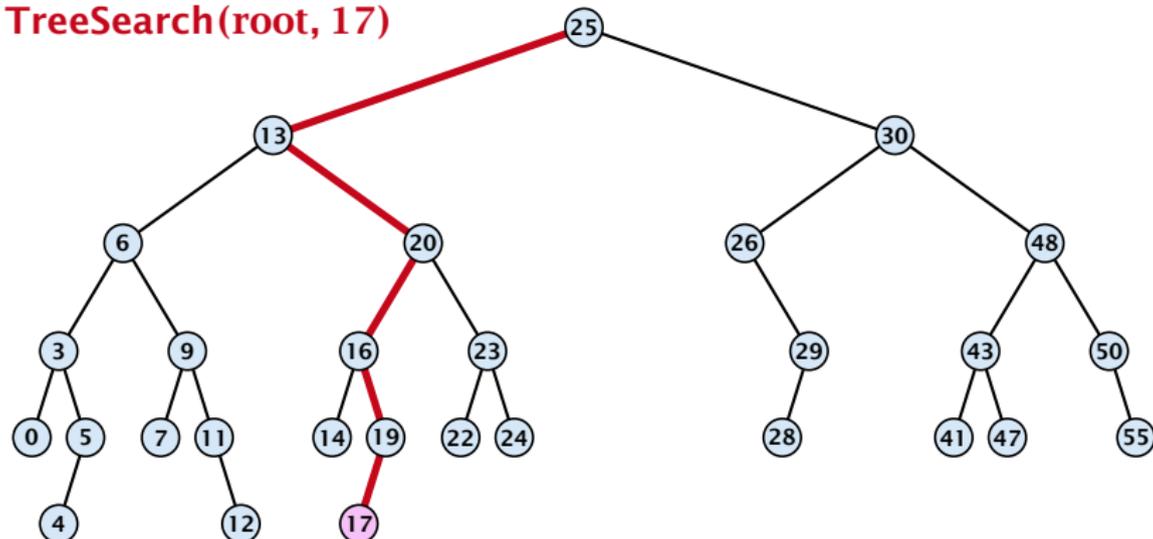


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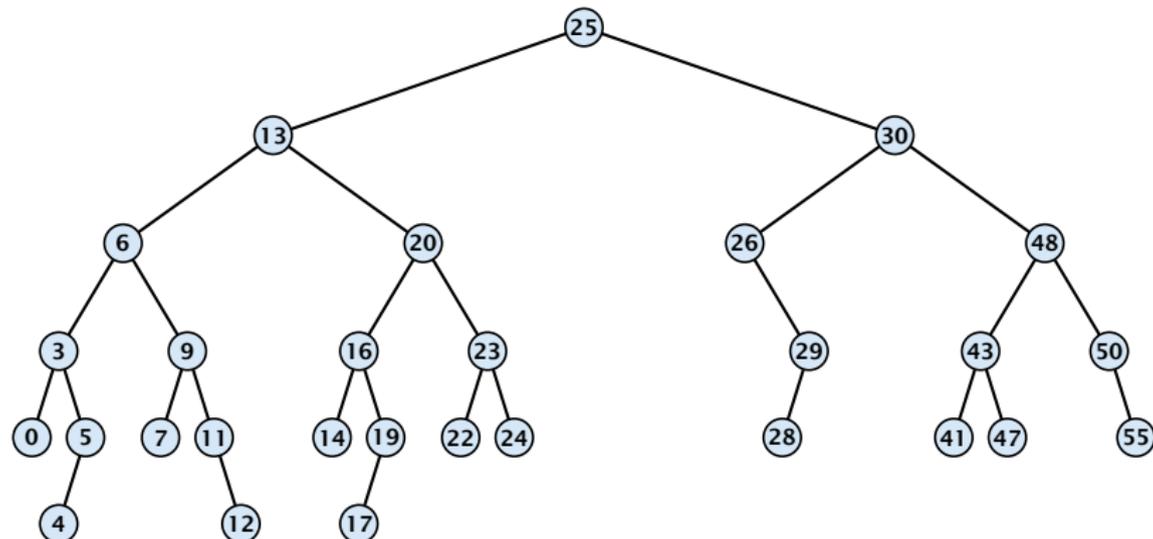
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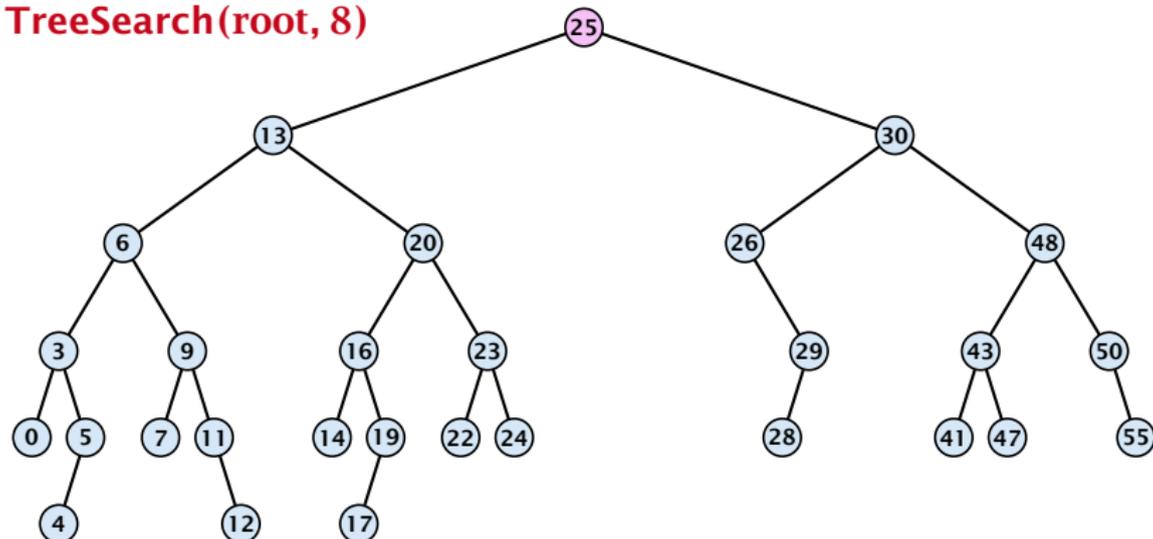


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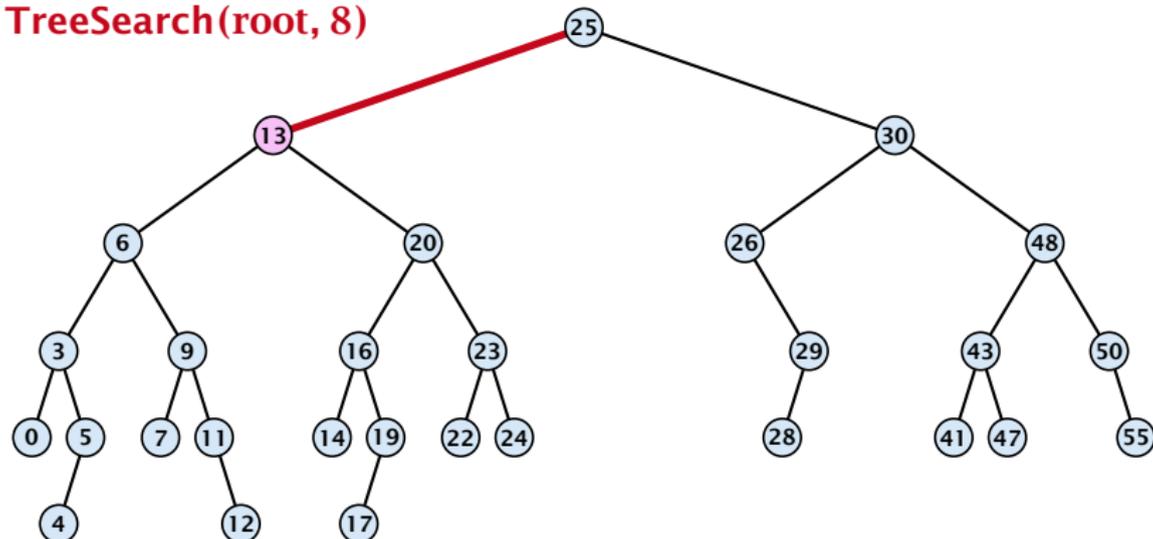


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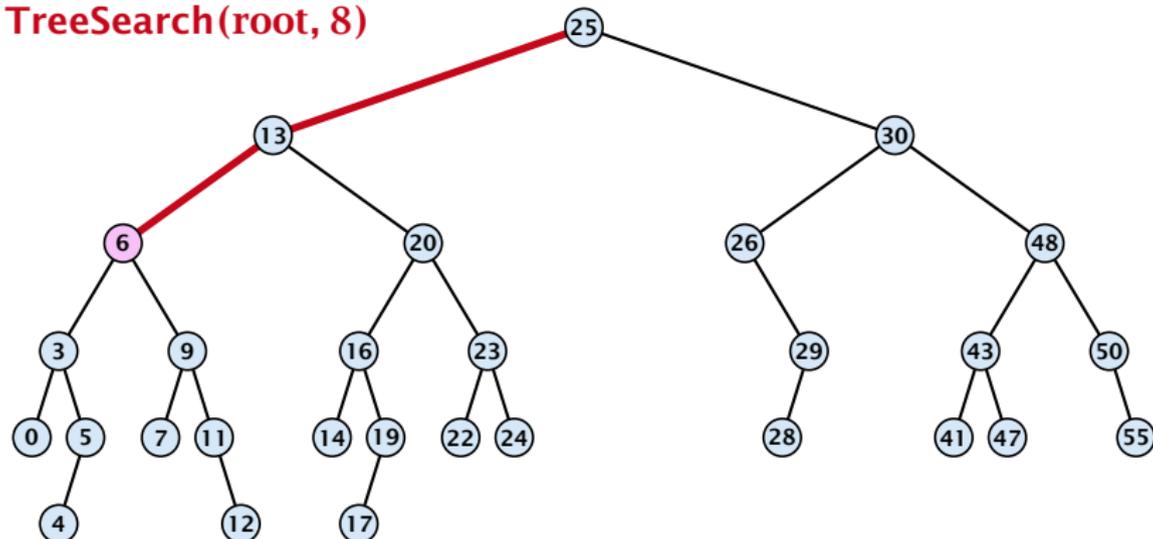


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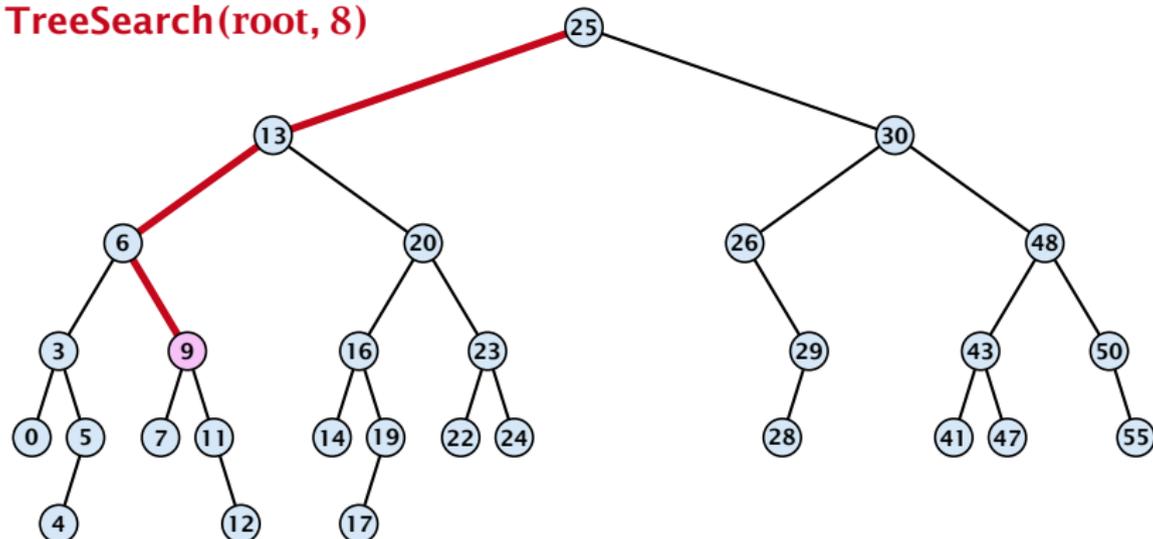


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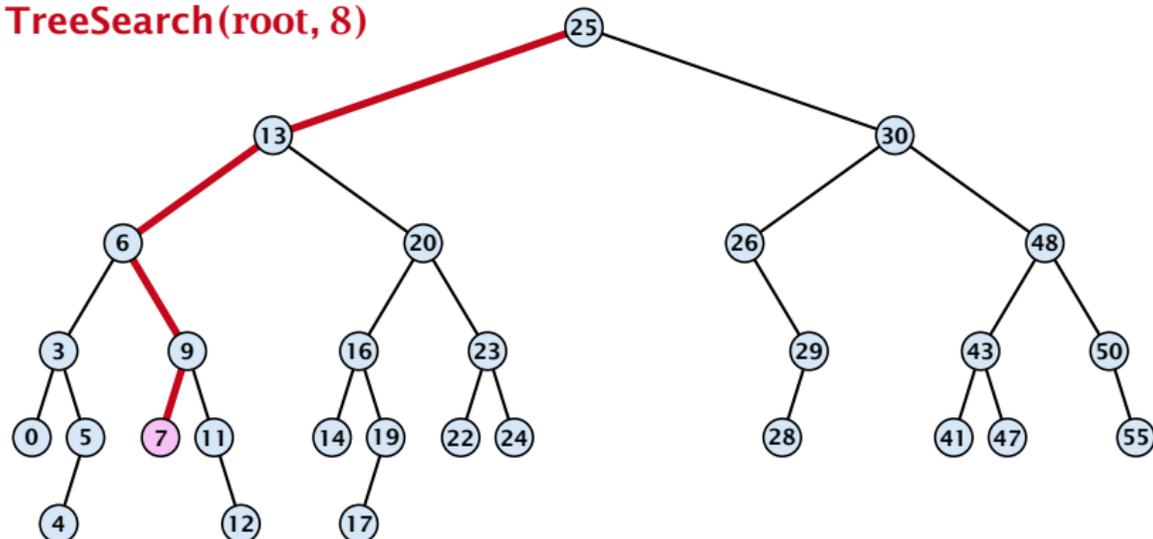


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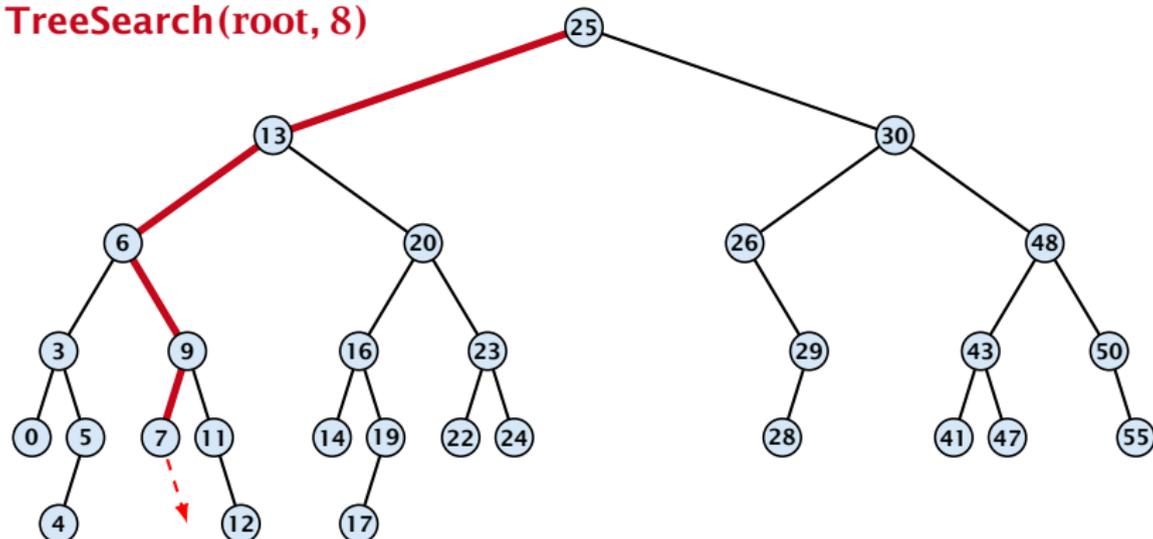


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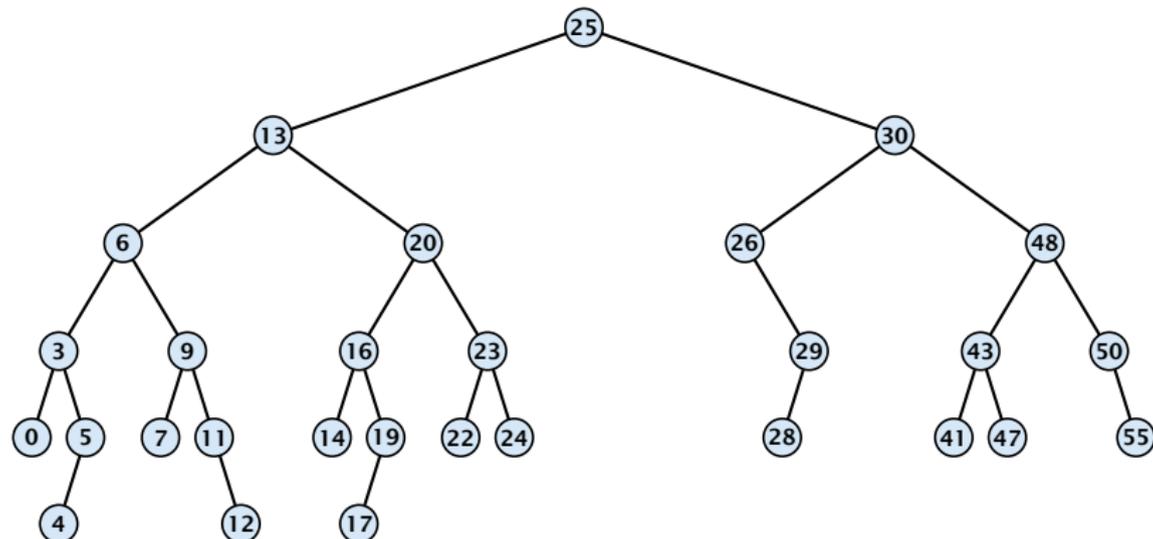
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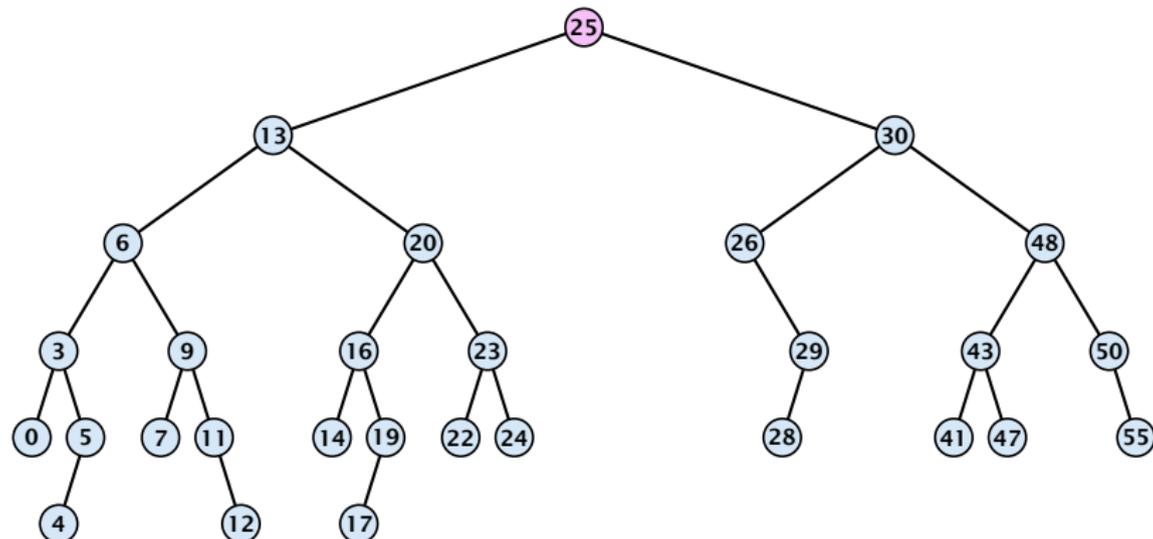
Binary Search Trees: Minimum



Algorithm 6 TreeMin(x)

- 1: **if** $x = \text{null}$ **or** $\text{left}[x] = \text{null}$ **return** x
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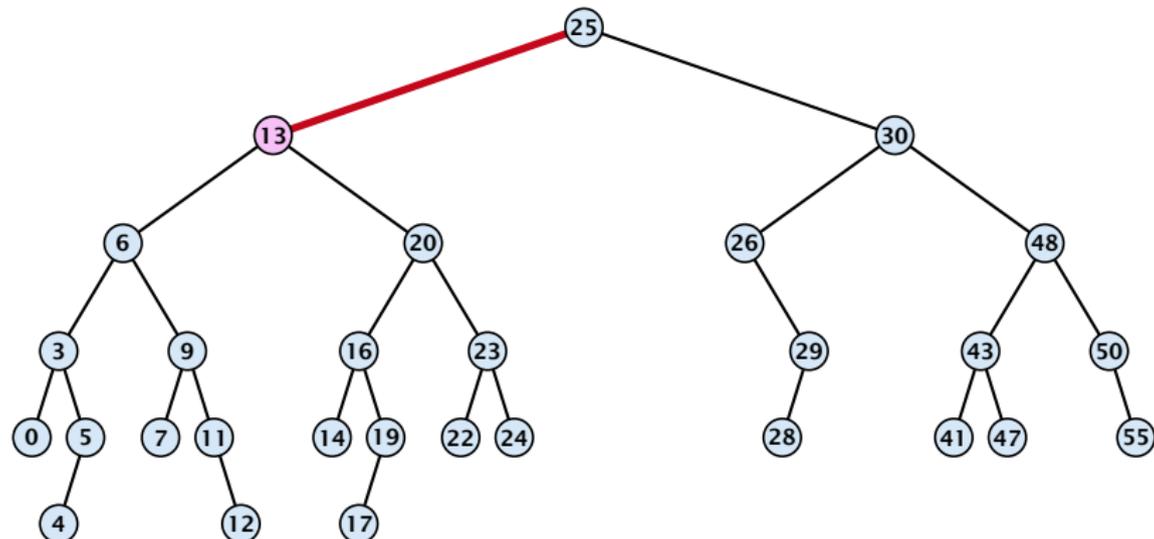
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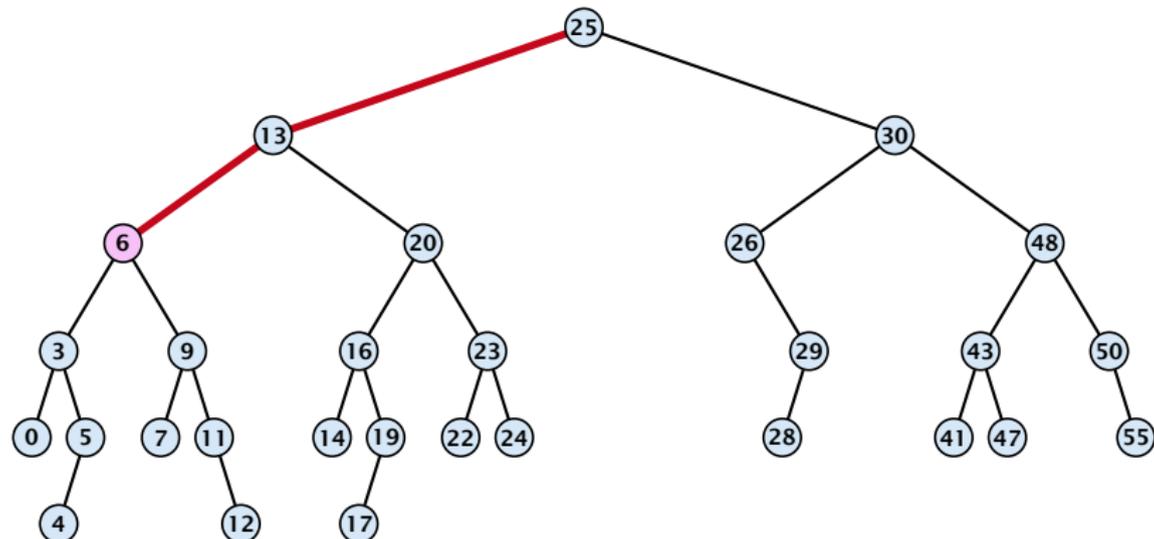
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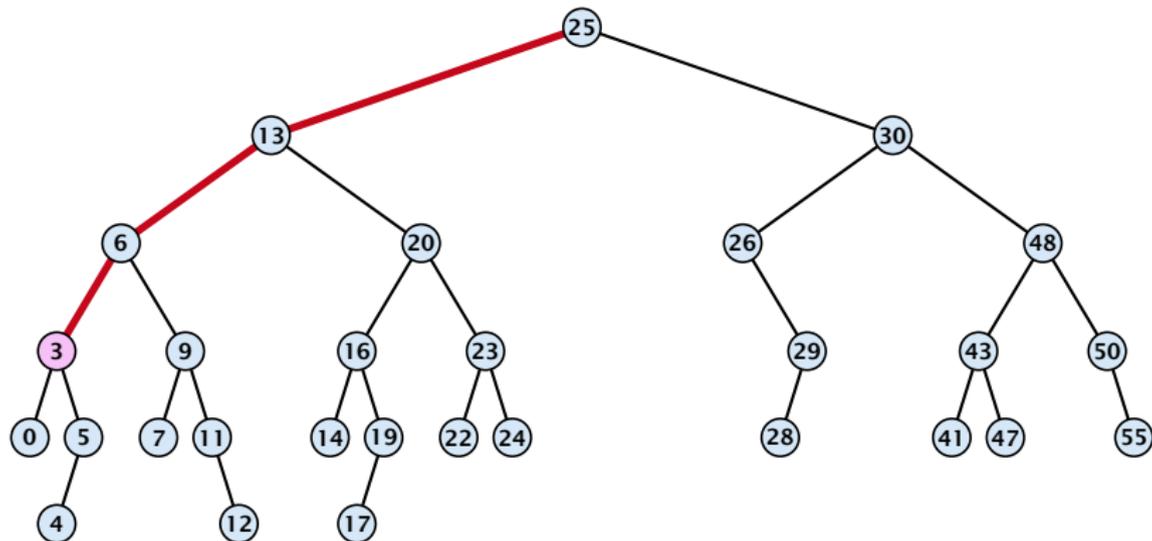
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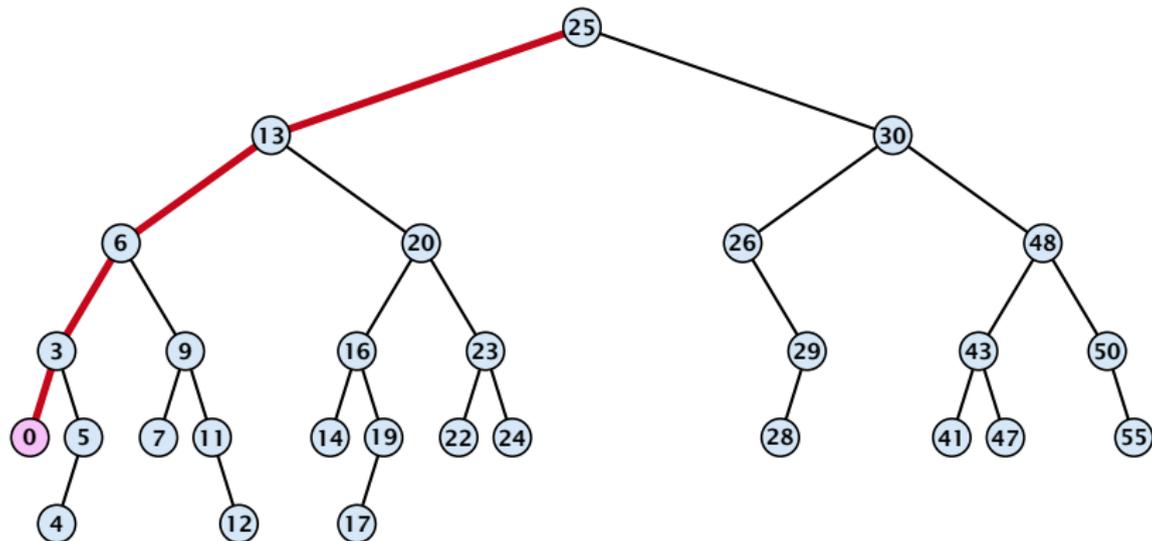
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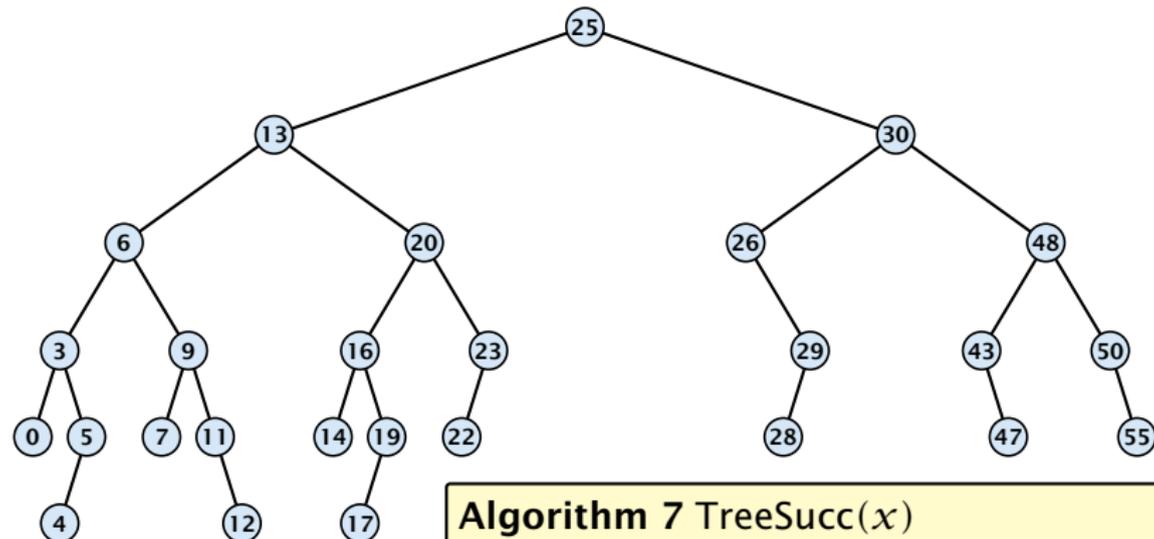
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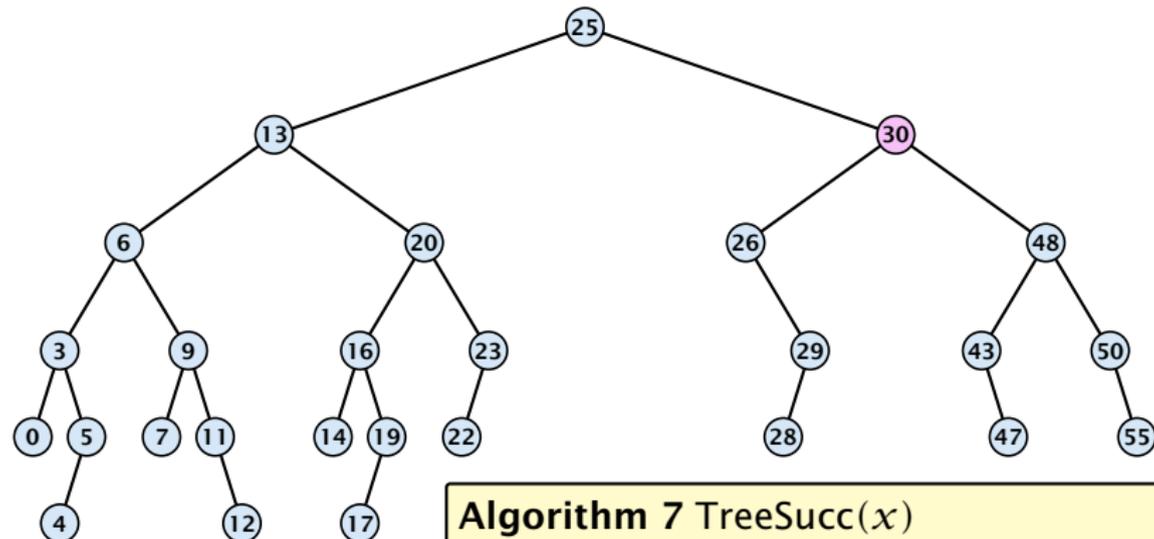
Binary Search Trees: Successor



Algorithm 7 TreeSucc(x)

- 1: **if** right[x] \neq null **return** TreeMin(right[x])
- 2: $y \leftarrow$ parent[x]
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- 4: $x \leftarrow y$; $y \leftarrow$ parent[x]
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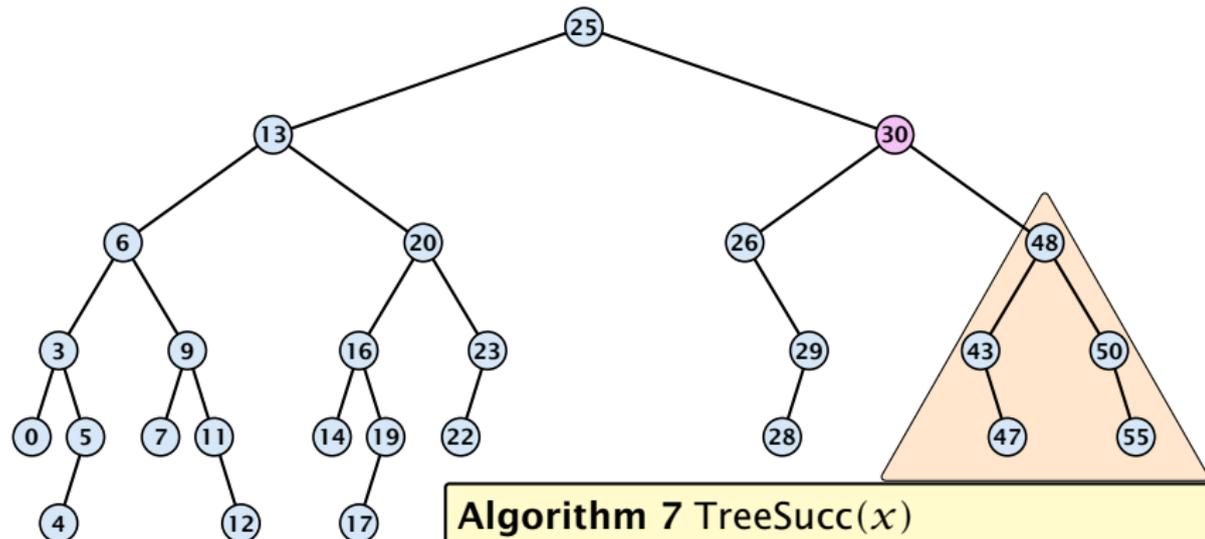
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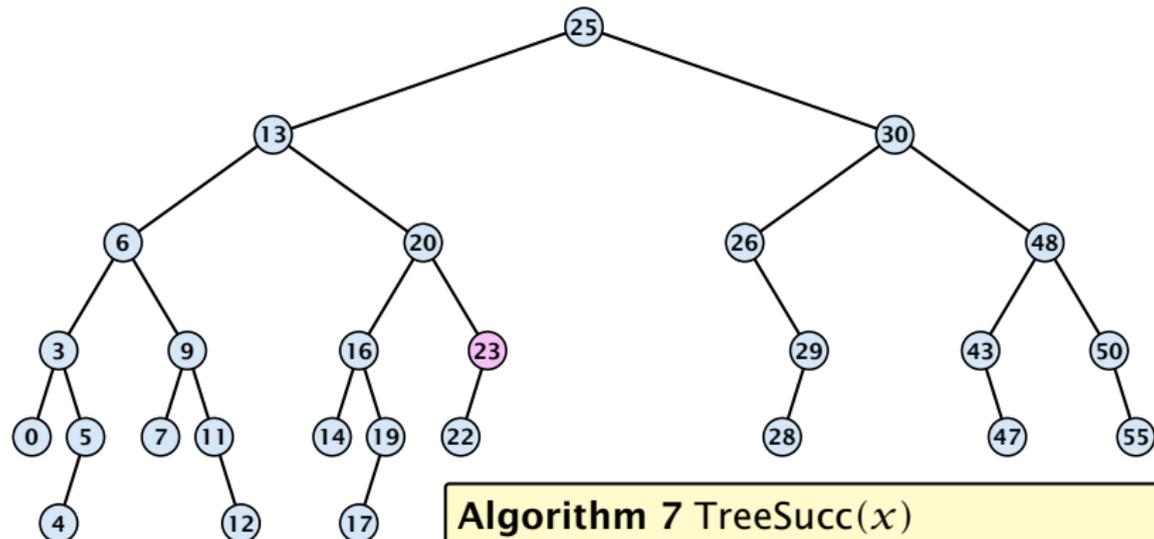
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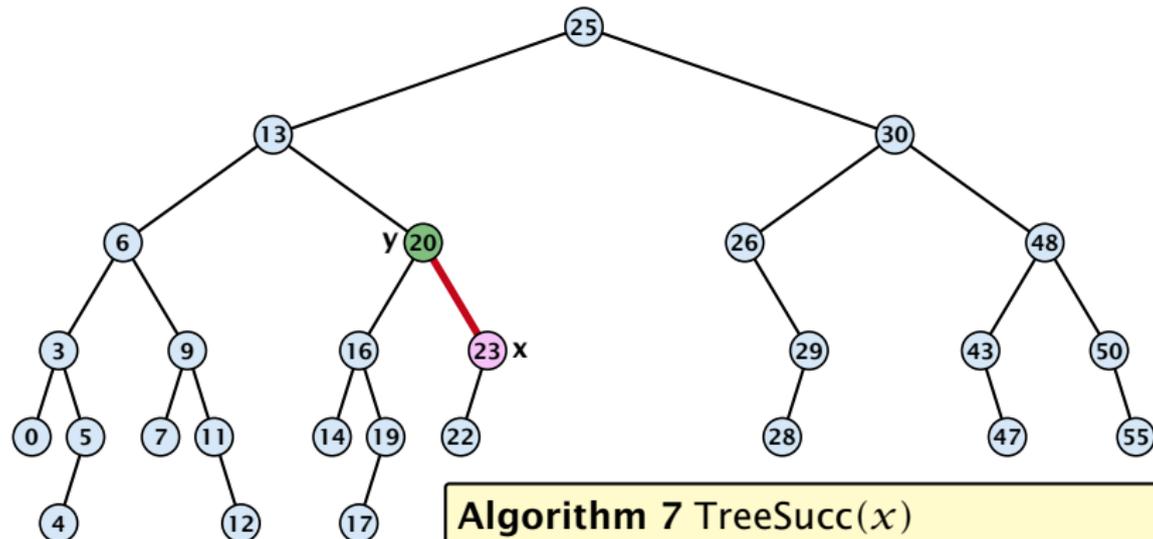
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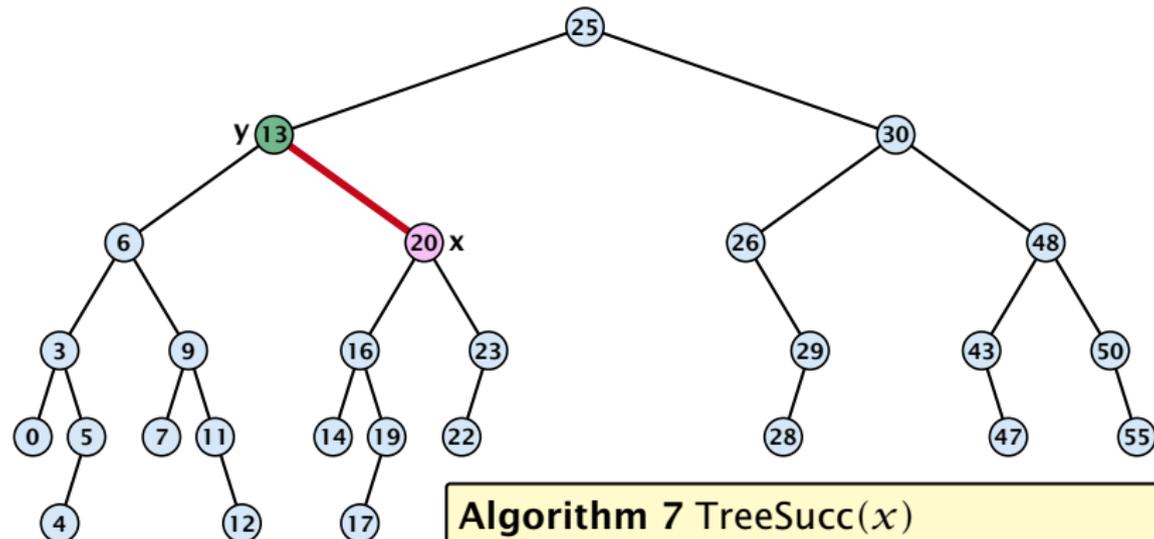
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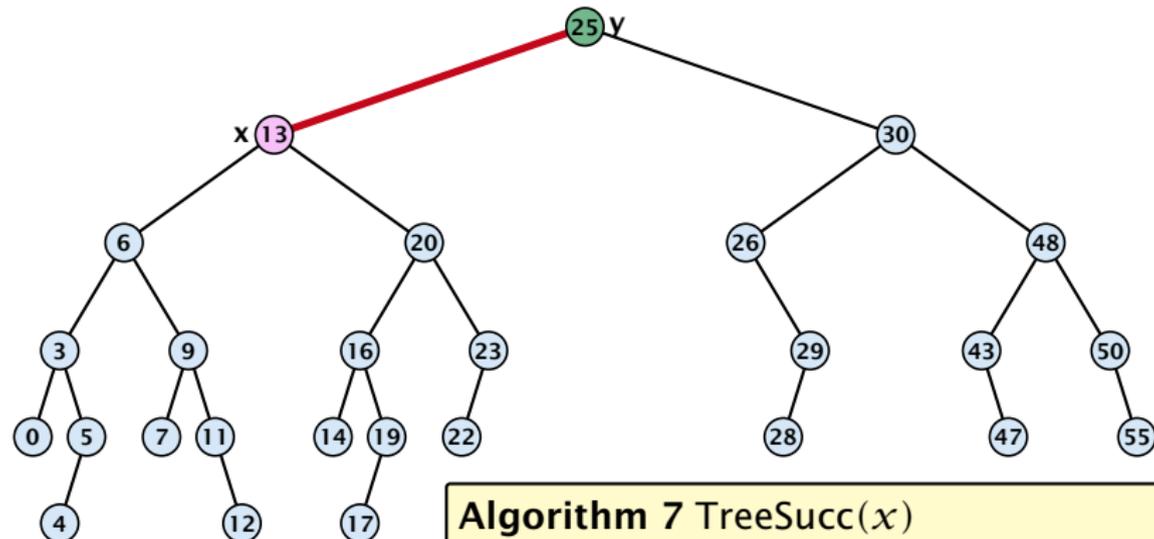
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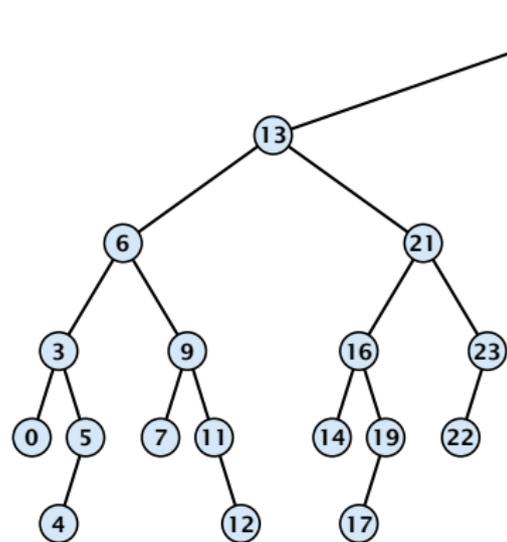
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Binary Search Trees: Insert

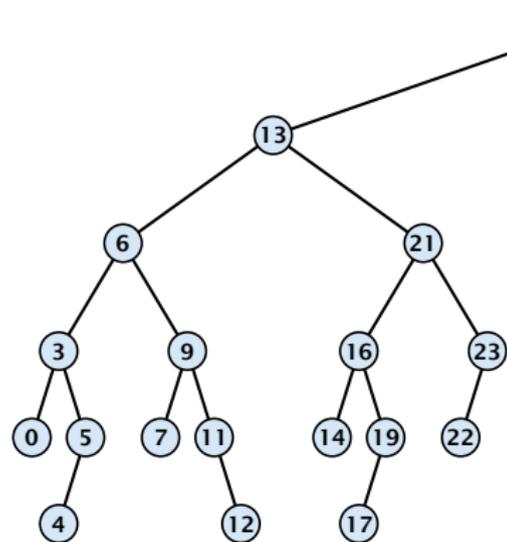


Algorithm 8 TreeInsert(x, z)

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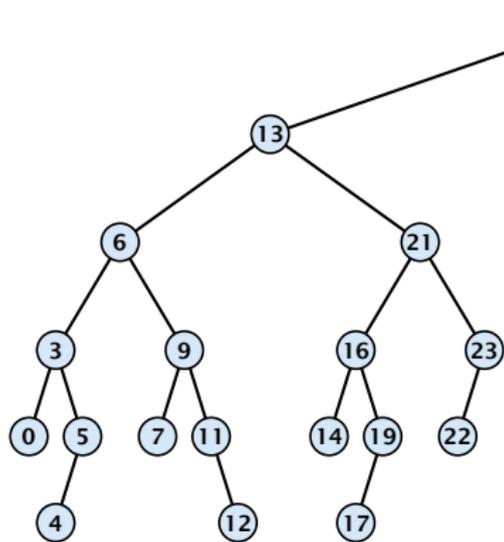


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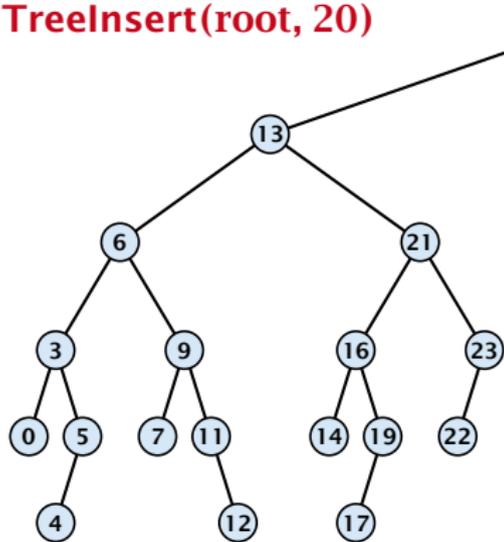
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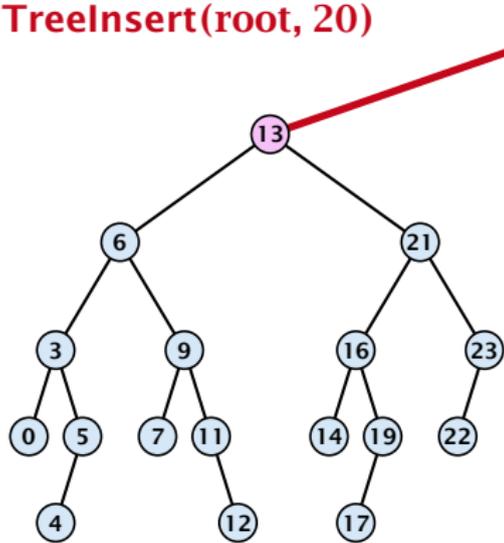
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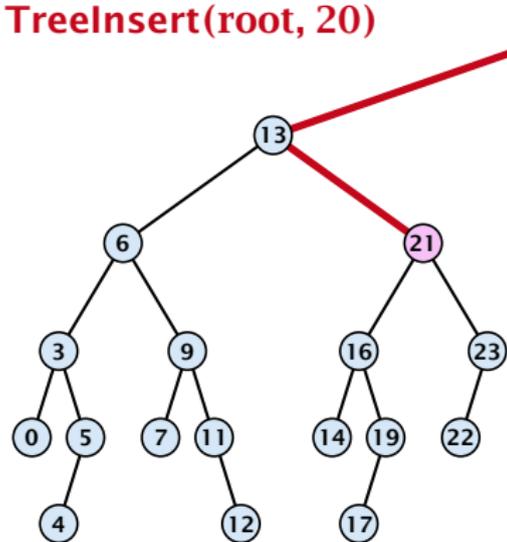
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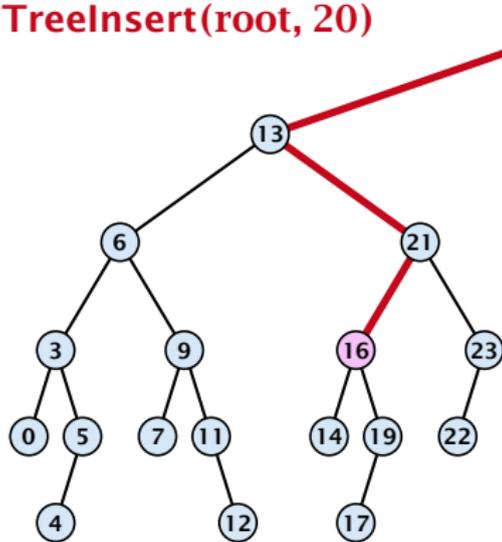
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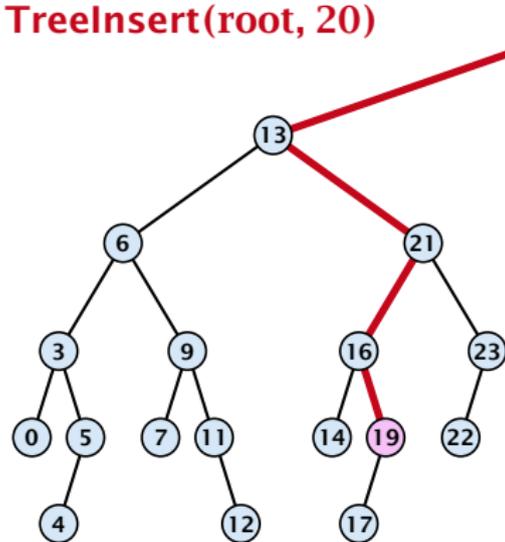
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10:    right[ $x$ ]  $\leftarrow z$ ; parent[ $z$ ]  $\leftarrow x$ ;
11:   else TreeInsert(right[ $x$ ],  $z$ );
```

Binary Search Trees: Insert

Insert element **not** in the tree.

TreeInsert(root, 20)



Search for z . At some point the search stops at a null-pointer. This is the place to insert z .

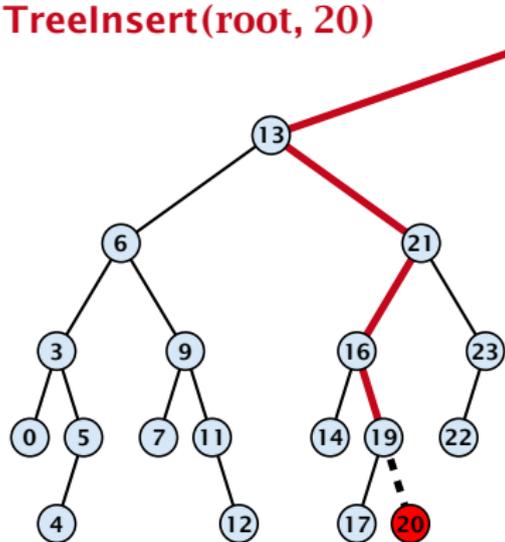
Algorithm 8 TreeInsert(x, z)

- 1: **if** $x = \text{null}$ **then**
- 2: $\text{root}[T] \leftarrow z$; $\text{parent}[z] \leftarrow \text{null}$;
- 3: **return**;
- 4: **if** $\text{key}[x] > \text{key}[z]$ **then**
- 5: **if** $\text{left}[x] = \text{null}$ **then**
- 6: $\text{left}[x] \leftarrow z$; $\text{parent}[z] \leftarrow x$;
- 7: **else** TreeInsert($\text{left}[x], z$);
- 8: **else**
- 9: **if** $\text{right}[x] = \text{null}$ **then**
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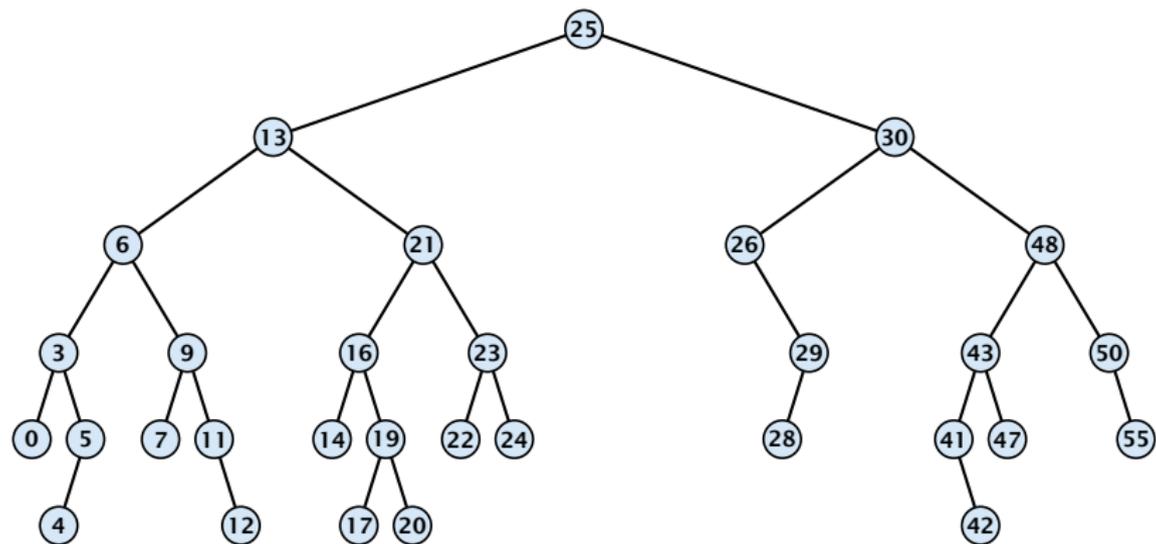


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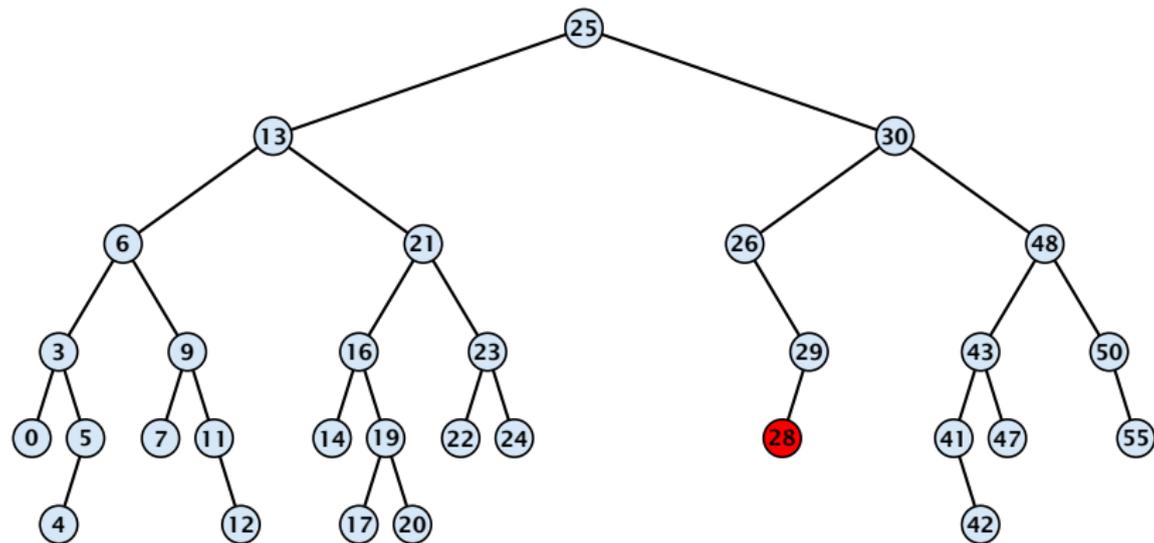
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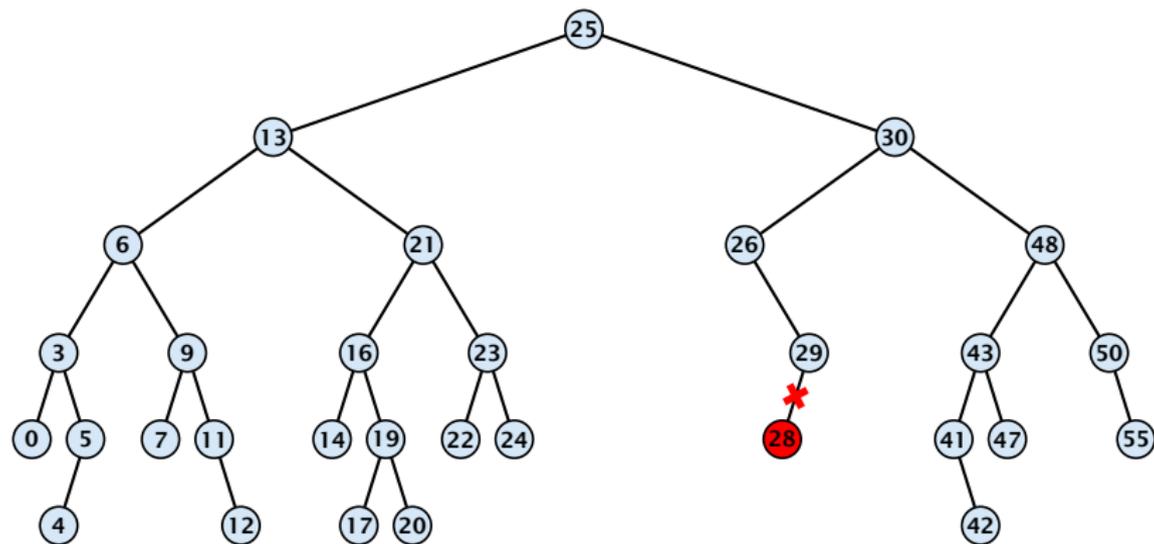


Case 1:

Element does not have any children

- ▶ Simply go to the parent and set the corresponding pointer to null.

Binary Search Trees: Delete

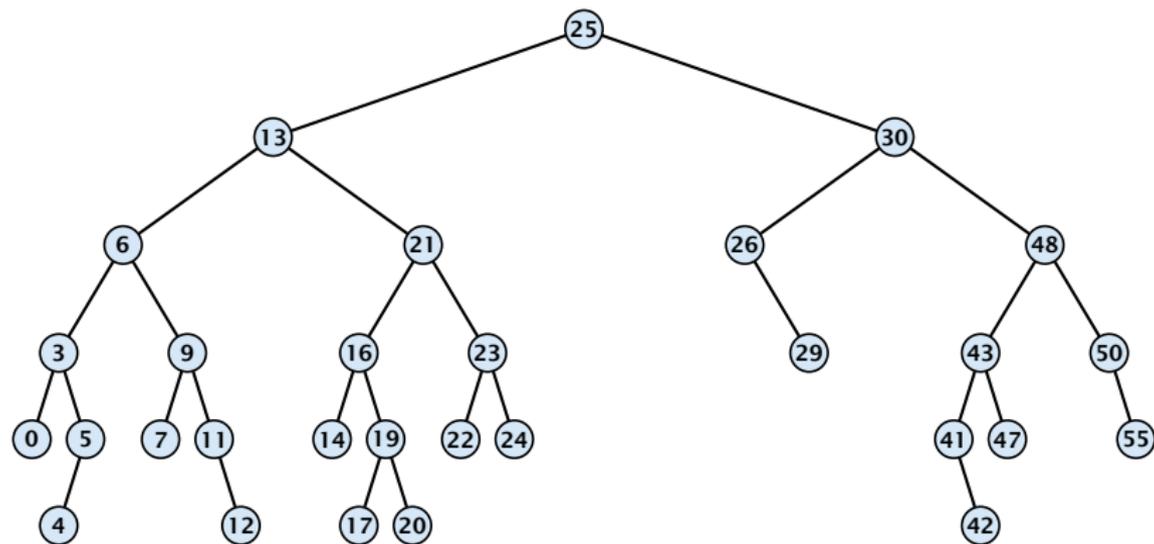


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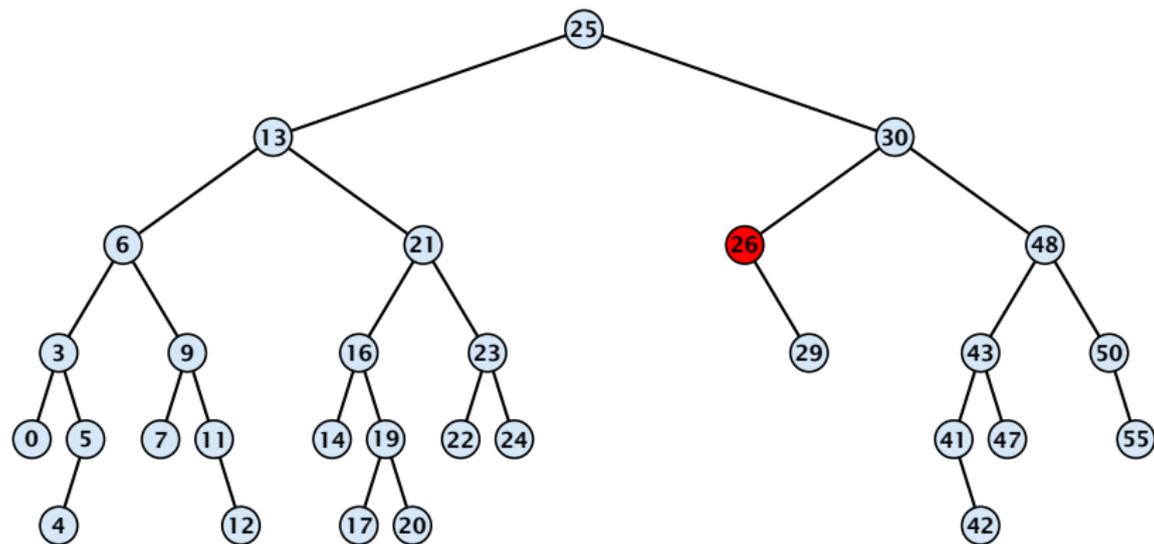


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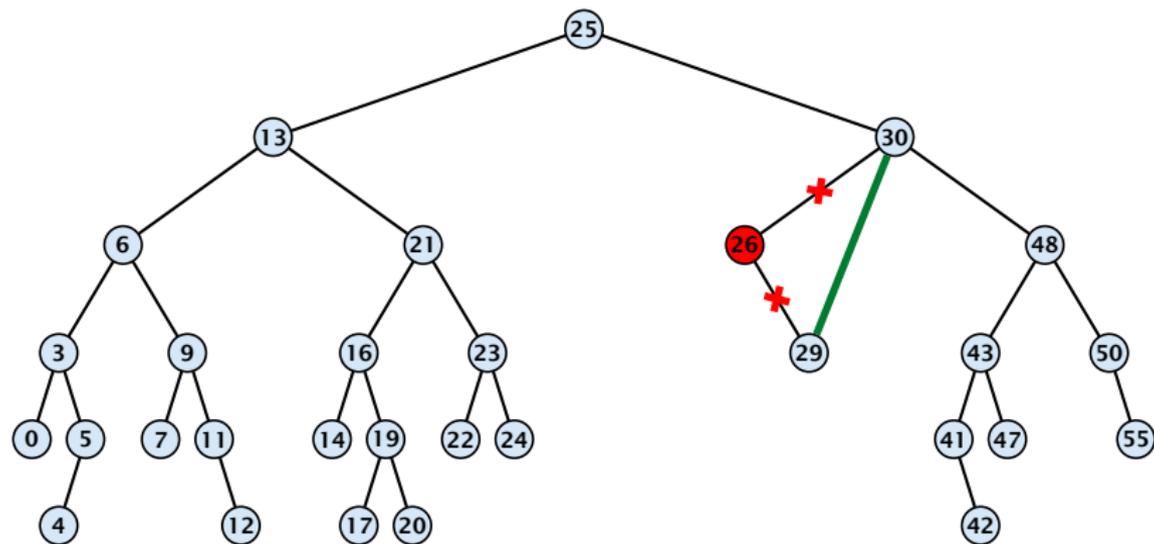


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Element has exactly one child

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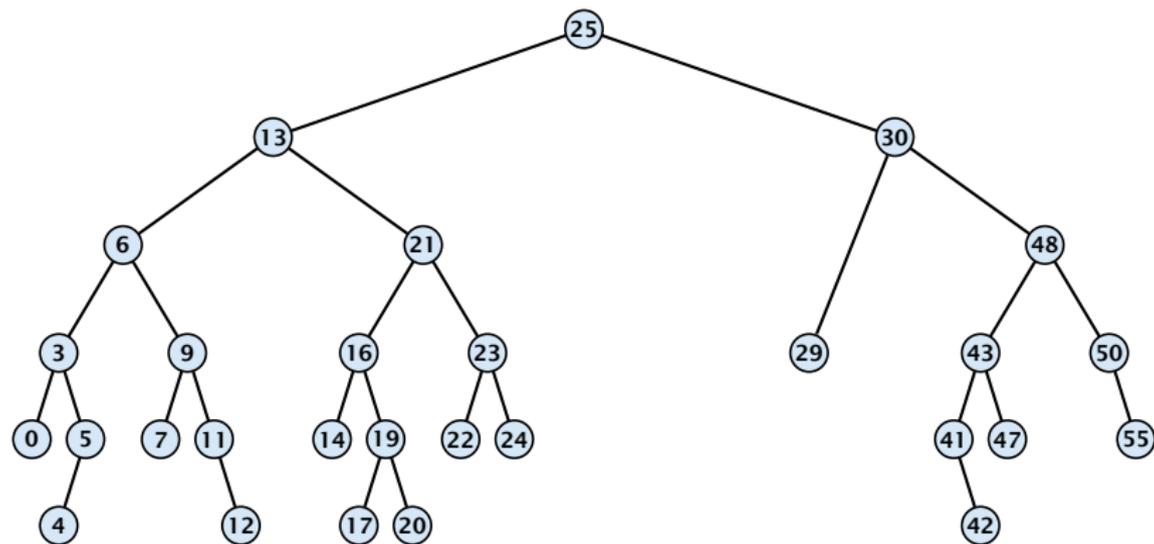


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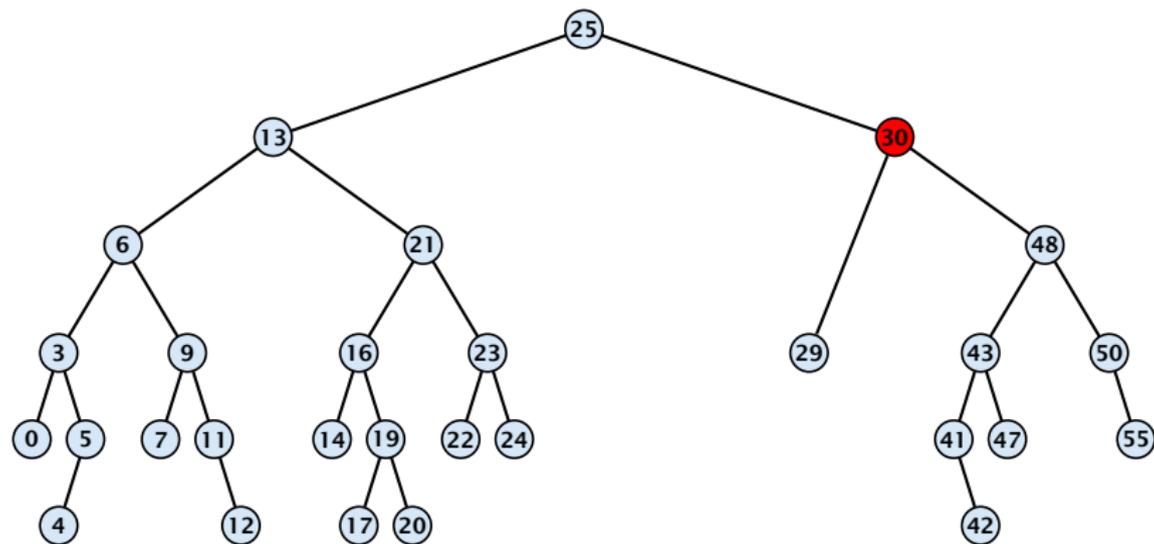


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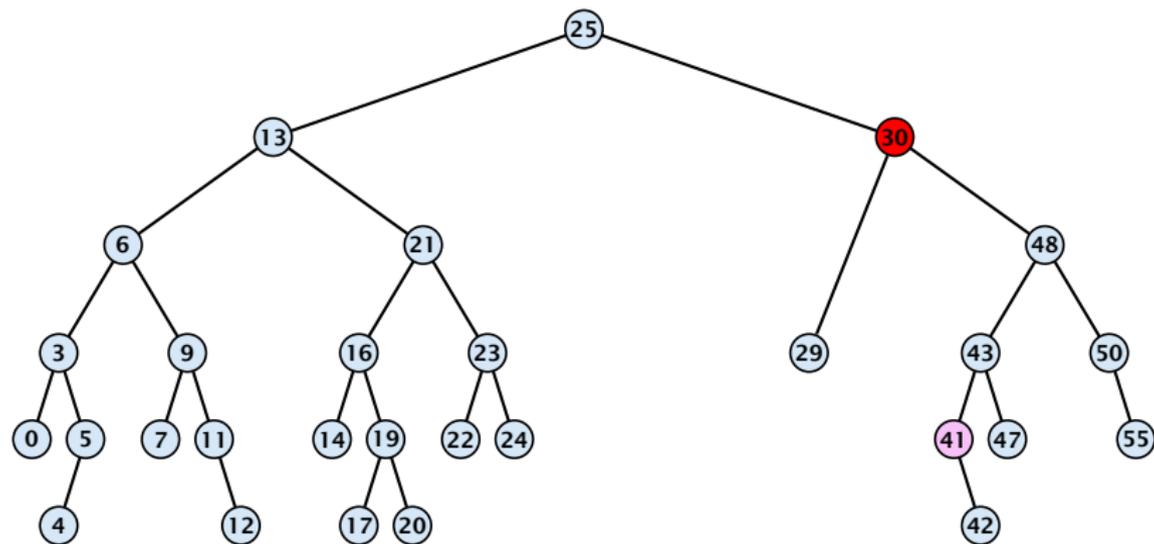


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Binary Search Trees: Delete

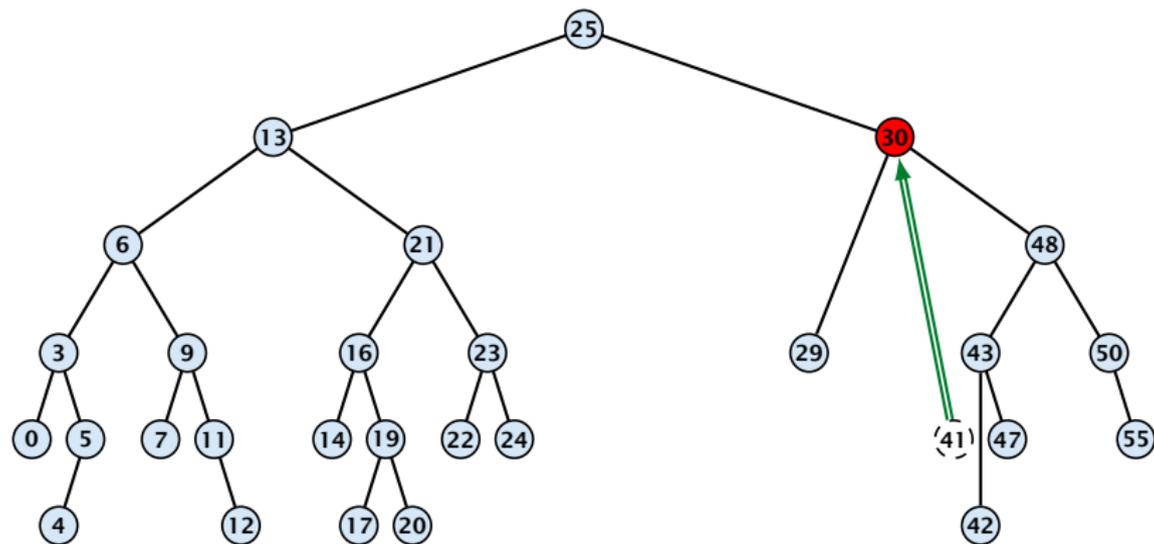


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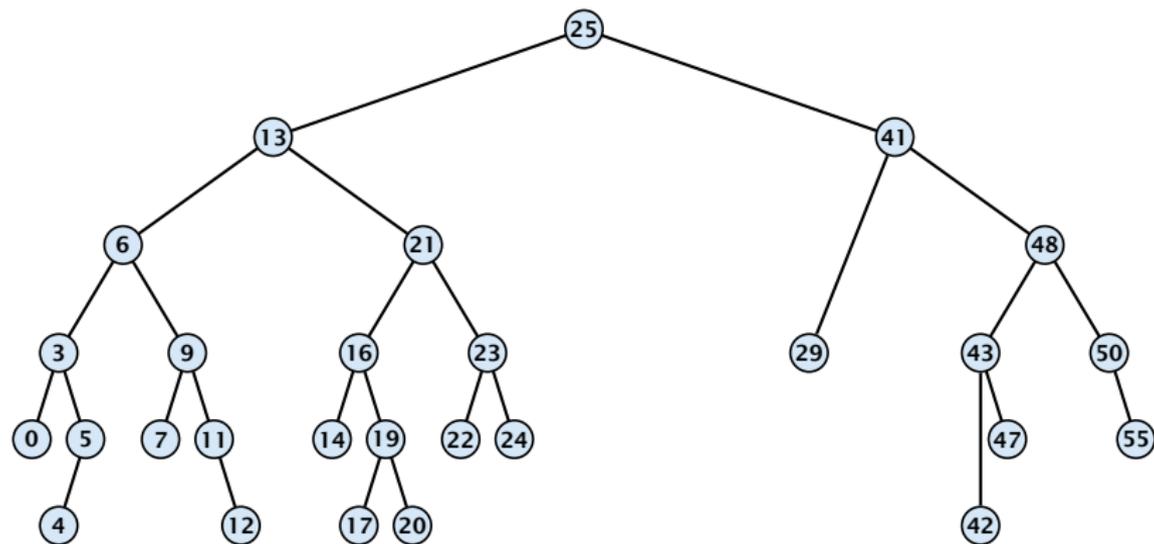


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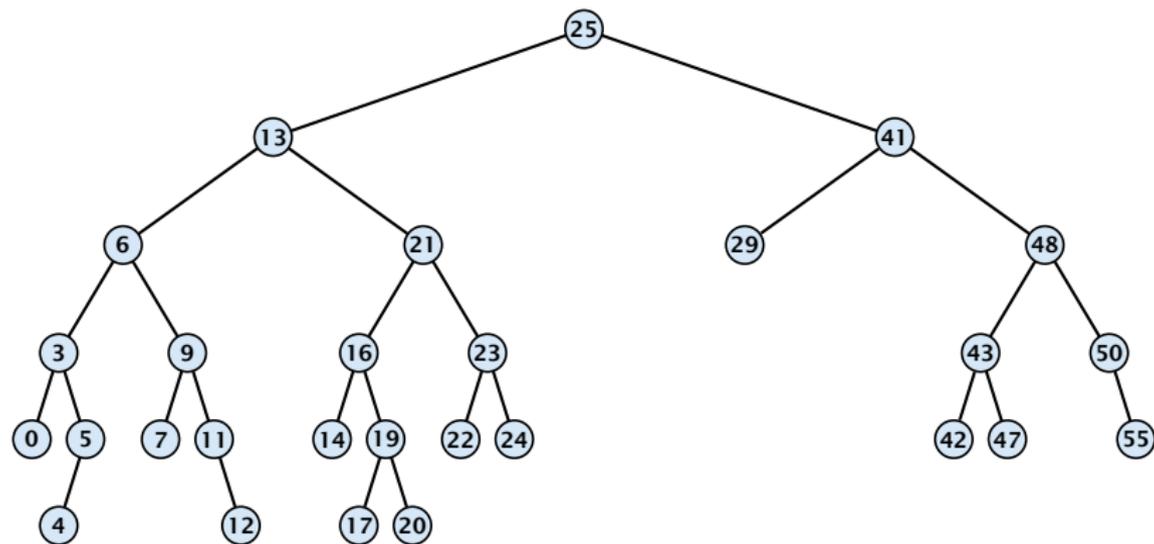


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Binary Search Trees: Delete

Algorithm 9 TreeDelete(z)

```
1: if left[ $z$ ] = null or right[ $z$ ] = null
2:   then  $y \leftarrow z$  else  $y \leftarrow \text{TreeSucc}(z)$ ;   select  $y$  to splice out
3:   if left[ $y$ ]  $\neq$  null
4:     then  $x \leftarrow \text{left}[y]$  else  $x \leftarrow \text{right}[y]$ ;  $x$  is child of  $y$  (or null)
5:   if  $x \neq \text{null}$  then parent[ $x$ ]  $\leftarrow$  parent[ $y$ ];   parent[ $x$ ] is correct
6:   if parent[ $y$ ] = null then
7:     root[ $T$ ]  $\leftarrow x$ 
8:   else
9:     if  $y = \text{left}[\text{parent}[y]]$  then
10:      left[parent[ $y$ ]]  $\leftarrow x$ 
11:     else
12:      right[parent[ $y$ ]]  $\leftarrow x$ 
13:   if  $y \neq z$  then copy  $y$ -data to  $z$ 
```

} fix pointer to x

Balanced Binary Search Trees

All operations on a binary search tree can be performed in time $\mathcal{O}(h)$, where h denotes the height of the tree.

However the height of the tree may become as large as $\Theta(n)$.

Balanced Binary Search Trees

With each insert- and delete-operation perform local adjustments to guarantee a height of $\mathcal{O}(\log n)$.

AVL-trees, Red-black trees, Scapegoat trees, 2-3 trees, B-trees, AA trees, Treaps

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A red black tree is a balanced binary search tree in which each internal node has two children. Each internal node has a color, such that

1. The root is black.
2. All leaf nodes are black.
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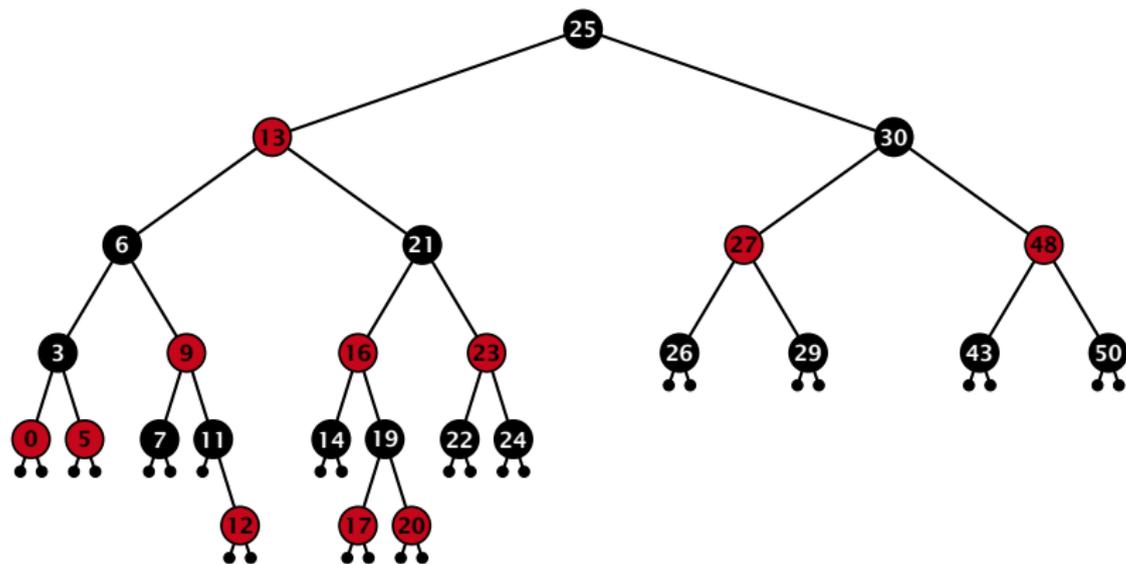
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Red Black Trees: Example



7.2 Red Black Trees

Lemma 2

A red-black tree with n internal nodes has height at most $\mathcal{O}(\log n)$.

Definition 3

The **black height** $\text{bh}(v)$ of a node v in a red black tree is the number of black nodes on a path from v to a leaf vertex (not counting v).

We first show:

Lemma 4

A sub-tree of black height $\text{bh}(v)$ in a red black tree contains at least $2^{\text{bh}(v)} - 1$ internal vertices.

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Proof of Lemma 4.

Induction on the height of v .

base case ($\text{height}(v) = 0$)

if $\text{height}(v)$ (maximum distance from v and a node in the subtree rooted at v) is 0 then v is a leaf.

The black height of v is 0.

The subtree rooted at v contains $0 = 2^0 - 1 = 1$ leaves.

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Proof (cont.)

induction step

Suppose v is a node with height $|v| > 0$.

v has two children with strictly smaller height.

These children (c_1, c_2) either have $h(c_1) = h(c_2) = |v| - 1$ or $h(c_1) = |v| - 1$.

By induction hypothesis both sub-trees contain at least $2^{h(c_1)}$ and $2^{h(c_2)}$ internal vertices.

The T_v itself is at least $2^{h(c_1)} + 2^{h(c_2)} + 1$.



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Proof (cont.)

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- ▶ Suppose v is a node with $\text{height}(v) > 0$.
- ▶ v has two children with strictly smaller height.
- ▶ These children (c_1, c_2) either have $\text{bh}(c_i) = \text{bh}(v)$ or $\text{bh}(c_i) = \text{bh}(v) - 1$.
- ▶ By induction hypothesis both sub-trees contain at least $2^{\text{bh}(v)-1} - 1$ internal vertices.
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Let h denote the height of the red-black tree, and let P denote a path from the root to the furthest leaf.

At least half of the nodes on P must be black, since a red node must be followed by a black node.

Hence, the black height of the root is at least $h/2$.

The tree contains at least $2^{h/2} - 1$ internal vertices. Hence,
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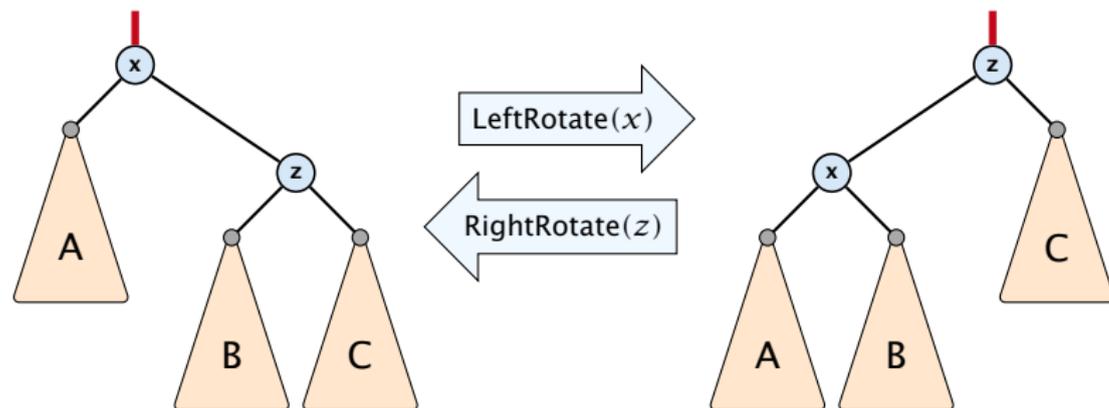
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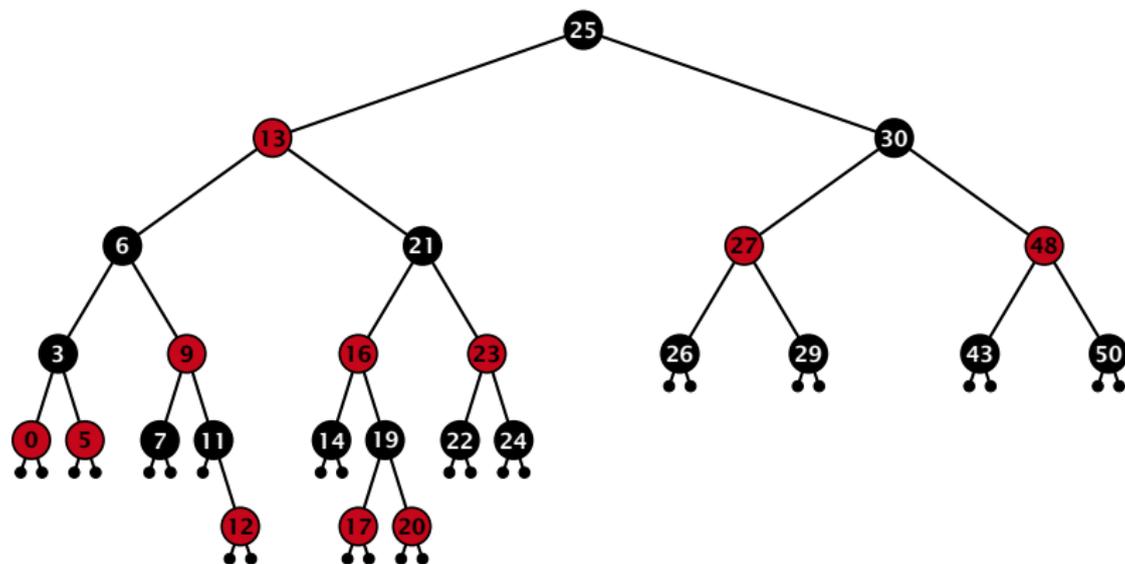
We need to adapt the insert and delete operations so that the red black properties are maintained.

Rotations

The properties will be maintained through rotations:



Red Black Trees: Insert

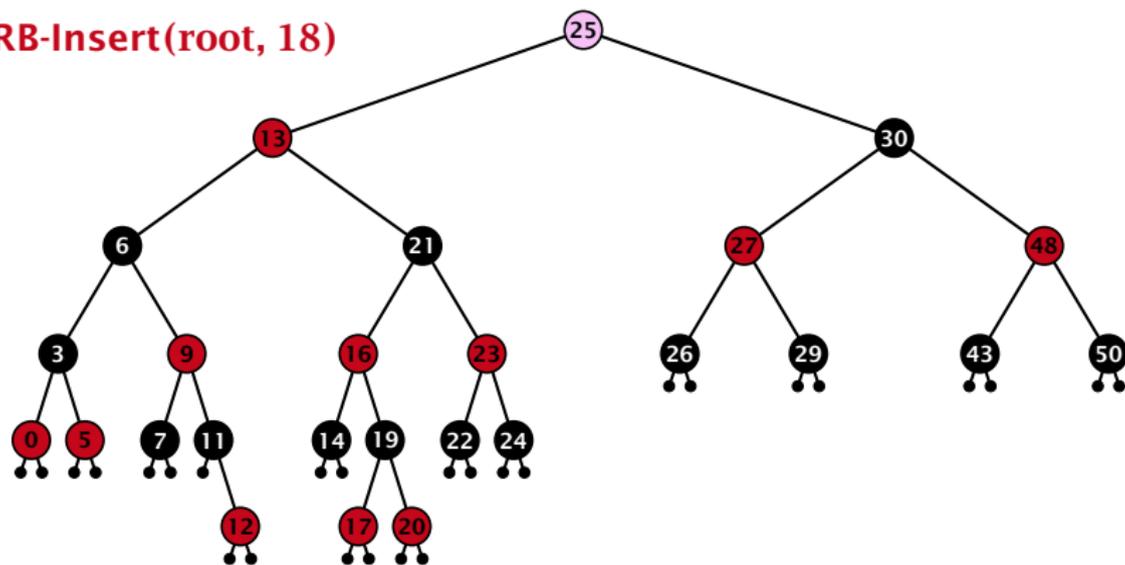


Insert:

- ▶ first make a normal insert into a binary search tree
- ▶ then fix red-black properties

Red Black Trees: Insert

RB-Insert(root, 18)

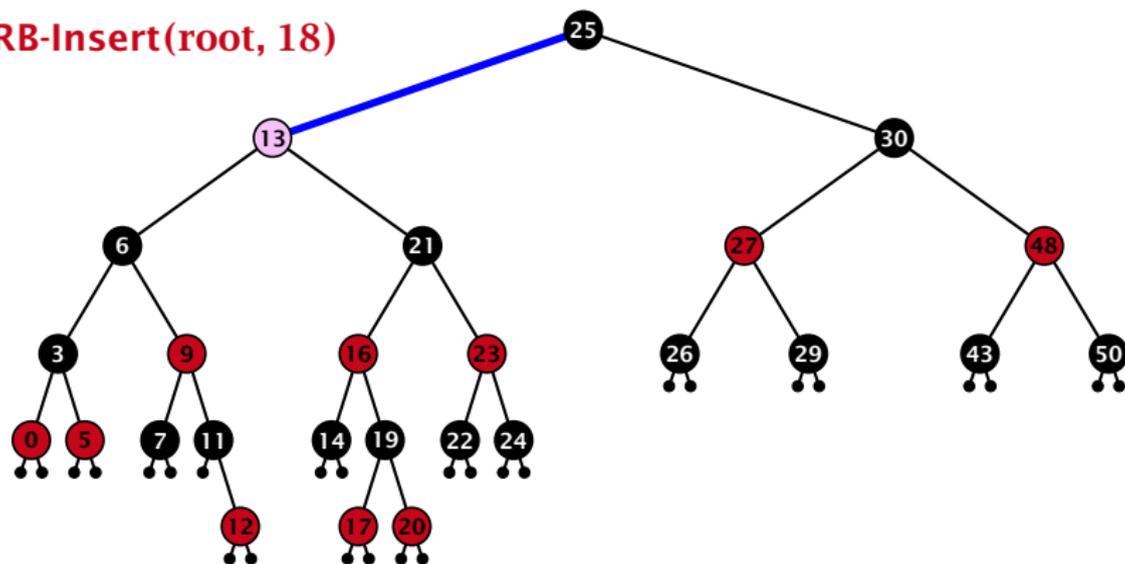


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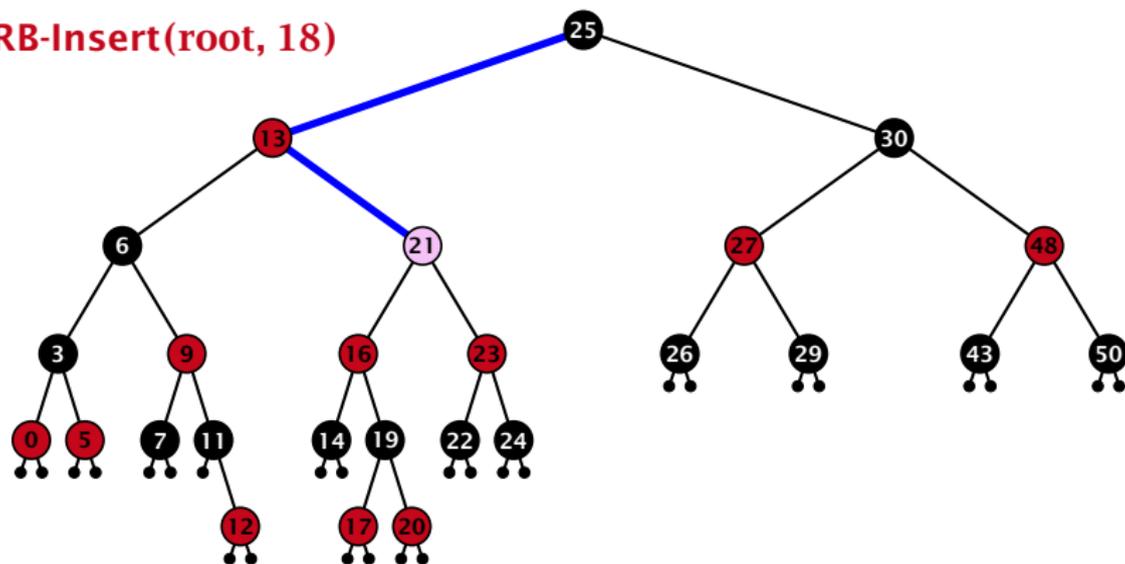


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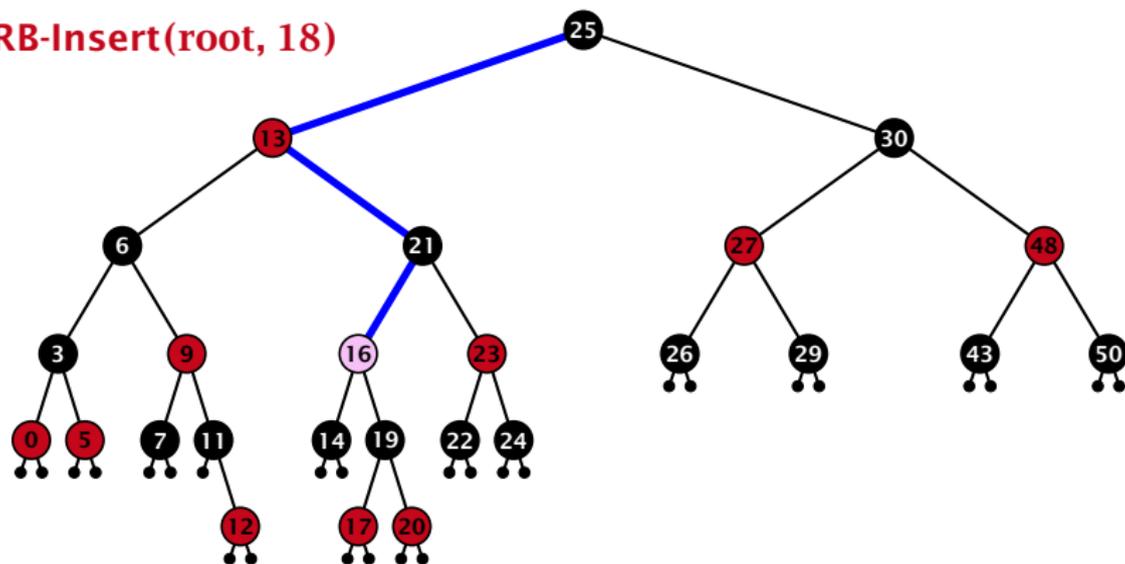


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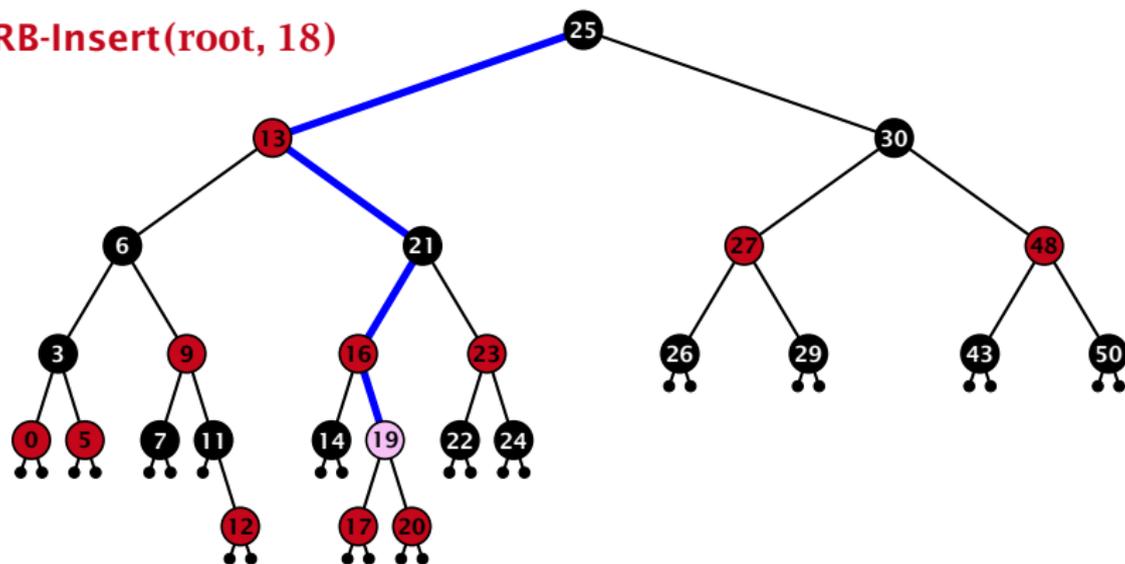


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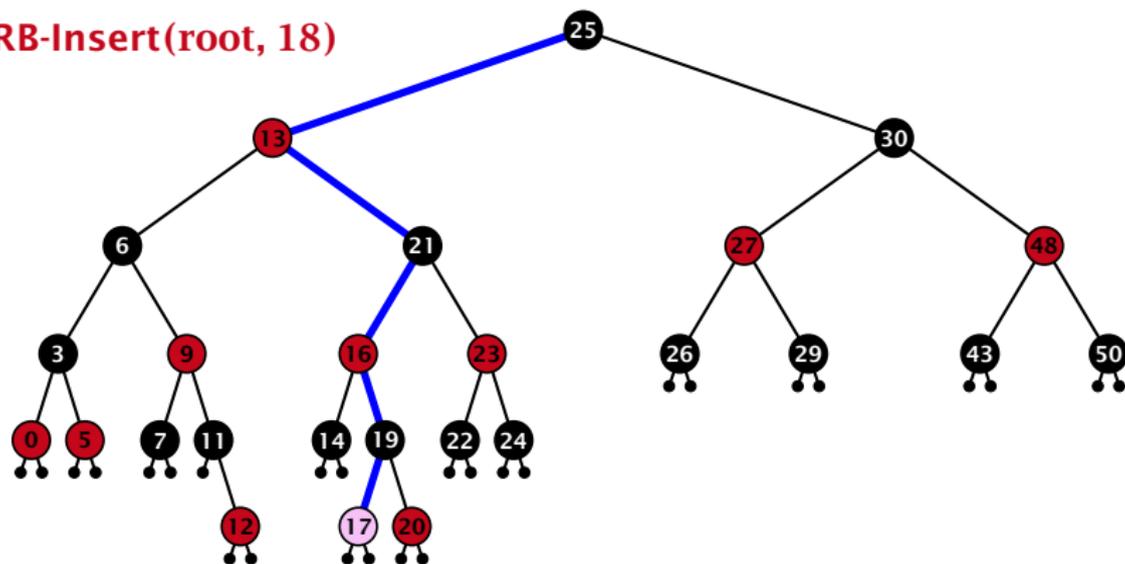


Insert:

- ▶ first make a normal insert into a binary search tree
- ▶ then fix red-black properties

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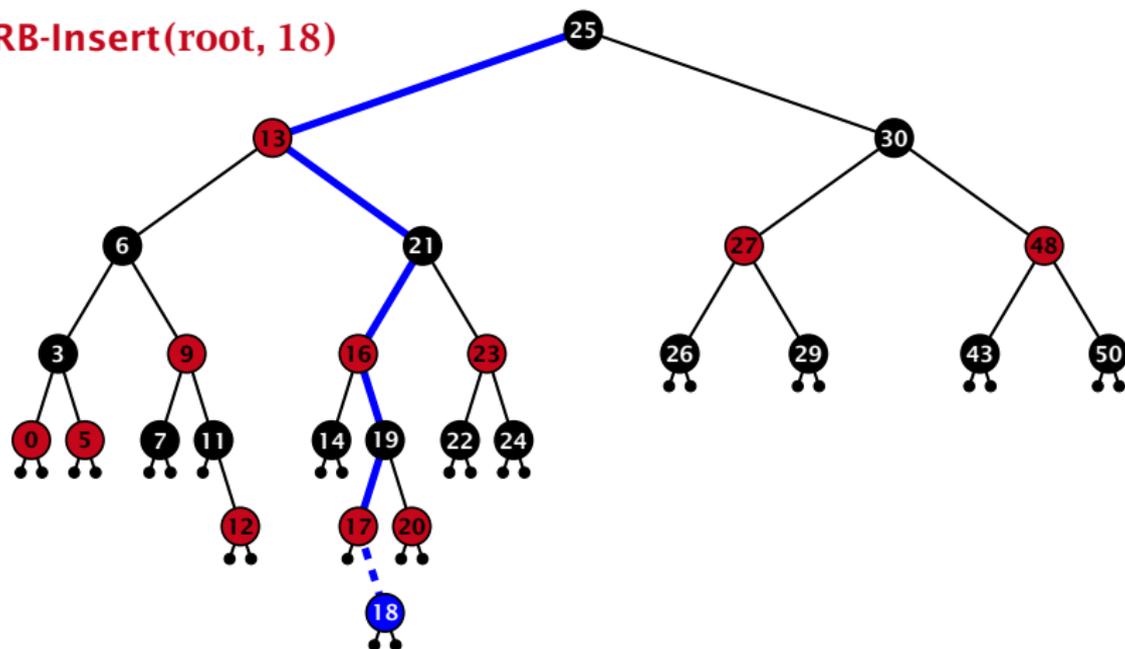


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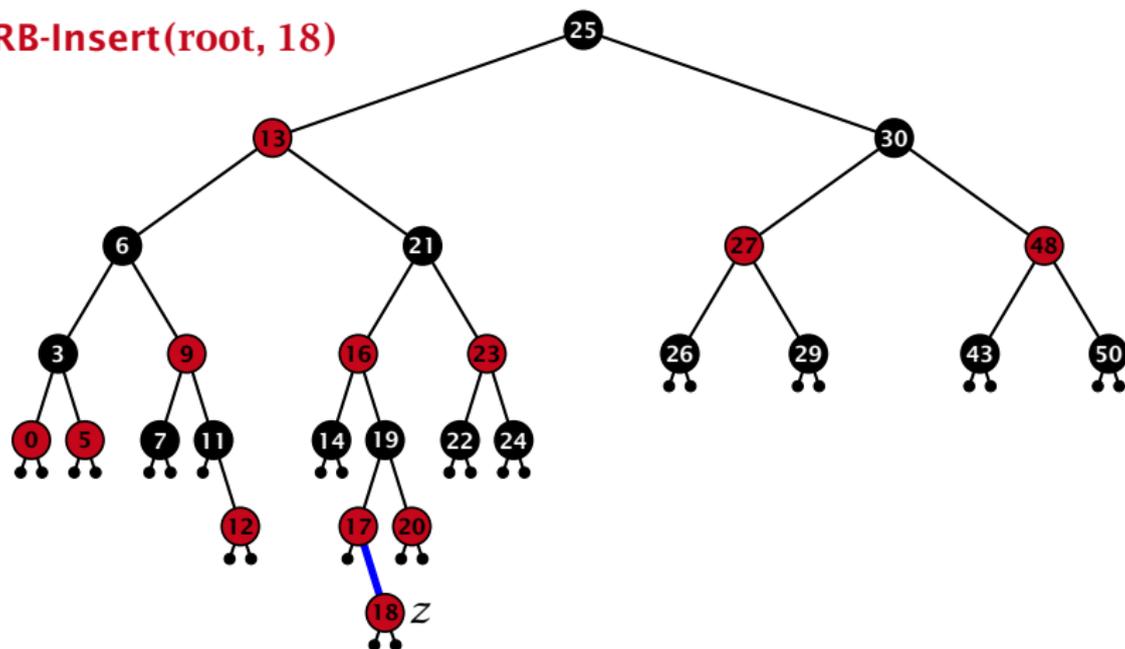


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Invariant of the fix-up algorithm:

- ▶ z is a red node
- ▶ the black-height property is fulfilled at every node
- ▶ the only violation of red-black properties occurs at z and $\text{parent}[z]$
 - either both of them are red (most important case)
 - or the parent does not exist (violation since root must be black)

If z has a parent but no grand-parent we could simply color the parent/root black; however this case never happens.

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Algorithm 10 InsertFix(z)

```
1: while parent[ $z$ ]  $\neq$  null and col[parent[ $z$ ]] = red do
2:   if parent[ $z$ ] = left[gp[ $z$ ]] then
3:      $uncle \leftarrow$  right[grandparent[ $z$ ]]
4:     if col[ $uncle$ ] = red then
5:       col[p[ $z$ ]]  $\leftarrow$  black; col[ $u$ ]  $\leftarrow$  black;
6:       col[gp[ $z$ ]]  $\leftarrow$  red;  $z \leftarrow$  grandparent[ $z$ ];
7:     else
8:       if  $z$  = right[parent[ $z$ ]] then
9:          $z \leftarrow$  p[ $z$ ]; LeftRotate( $z$ );
10:      col[p[ $z$ ]]  $\leftarrow$  black; col[gp[ $z$ ]]  $\leftarrow$  red;
11:      RightRotate(gp[ $z$ ]);
12:     else same as then-clause but right and left exchanged
13: col(root[ $T$ ])  $\leftarrow$  black;
```

Red Black Trees: Insert

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```
1: while parent[ $z$ ]  $\neq$  null and col[parent[ $z$ ]] = red do
2:   if parent[ $z$ ] = left[gp[ $z$ ]] then  $z$  in left subtree of grandparent
3:      $uncle \leftarrow$  right[grandparent[ $z$ ]]
4:     if col[ $uncle$ ] = red then
5:       col[p[ $z$ ]]  $\leftarrow$  black; col[ $u$ ]  $\leftarrow$  black;
6:       col[gp[ $z$ ]]  $\leftarrow$  red;  $z \leftarrow$  grandparent[ $z$ ];
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4:     if col[ $uncle$ ] = red then Case 1: uncle red
5:       col[p[ $z$ ]]  $\leftarrow$  black; col[ $u$ ]  $\leftarrow$  black;
6:       col[gp[ $z$ ]]  $\leftarrow$  red;  $z \leftarrow$  grandparent[ $z$ ];
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Red Black Trees: Insert

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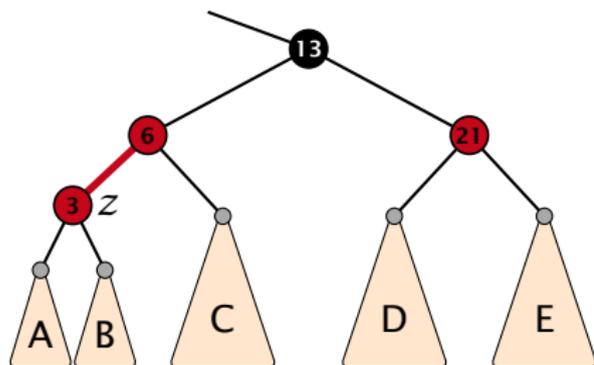
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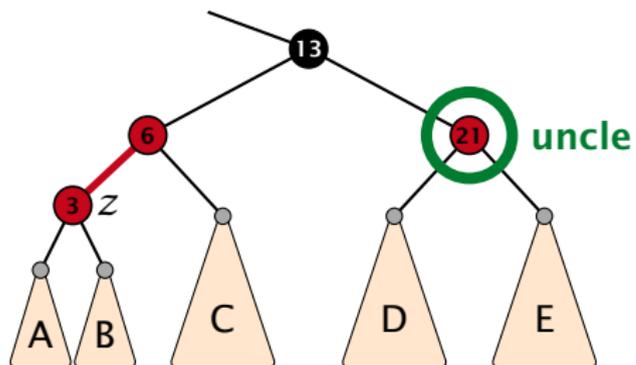
Algorithm 10 InsertFix(z)

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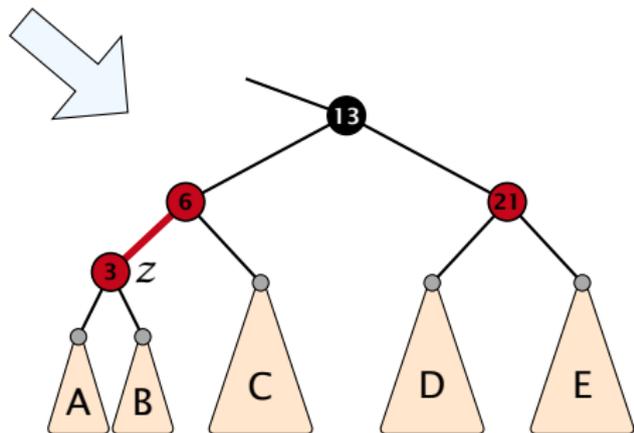
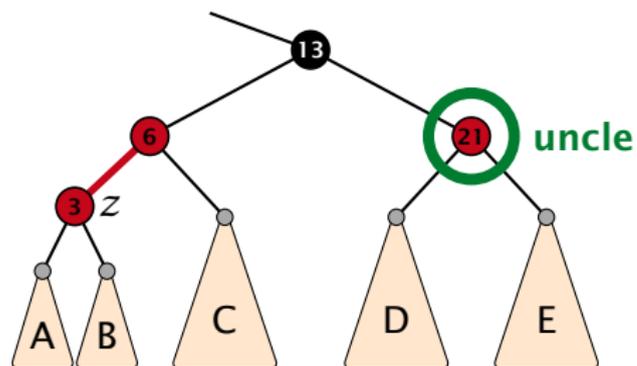
Case 1: Red Uncle



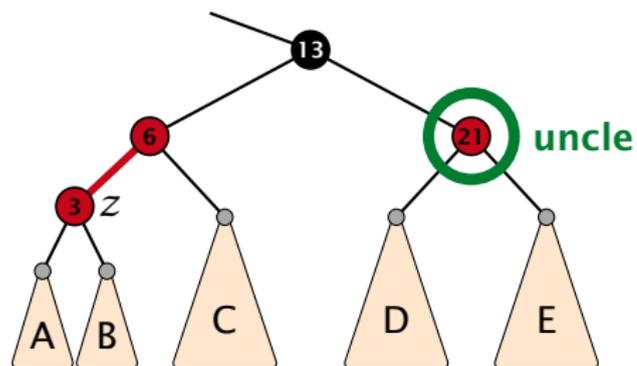
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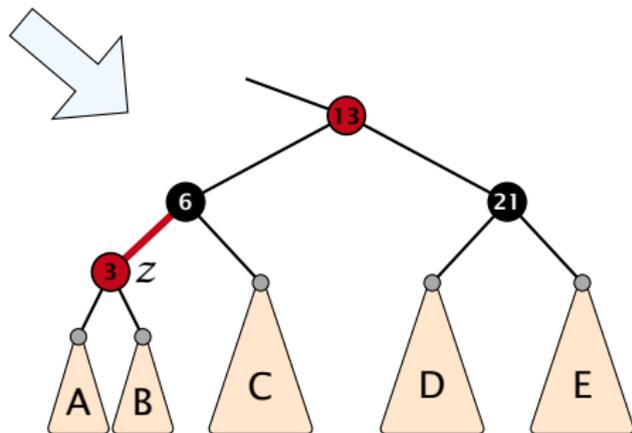
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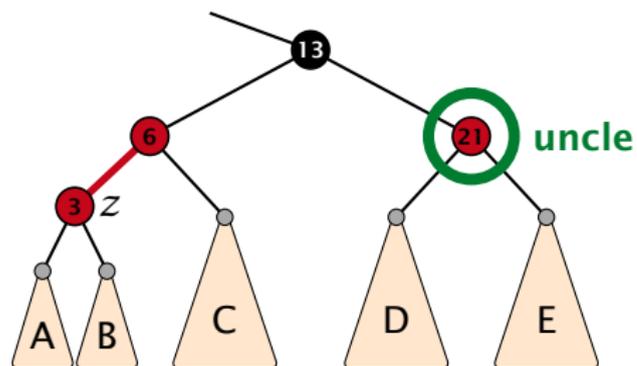
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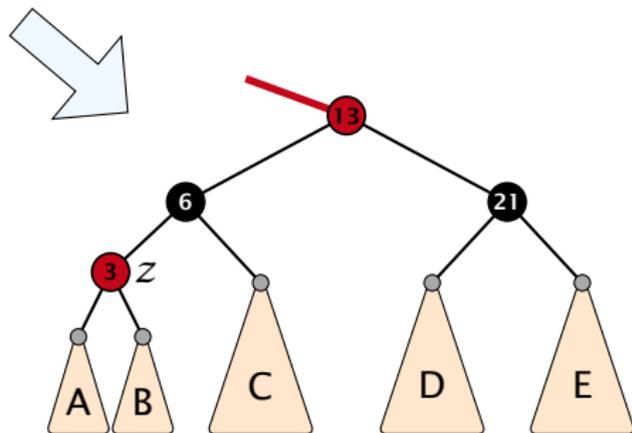
1. recolour



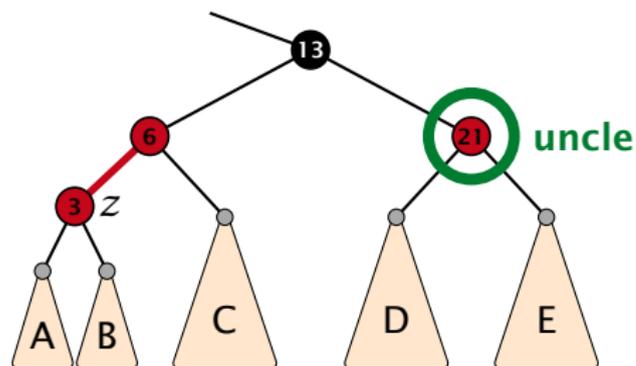
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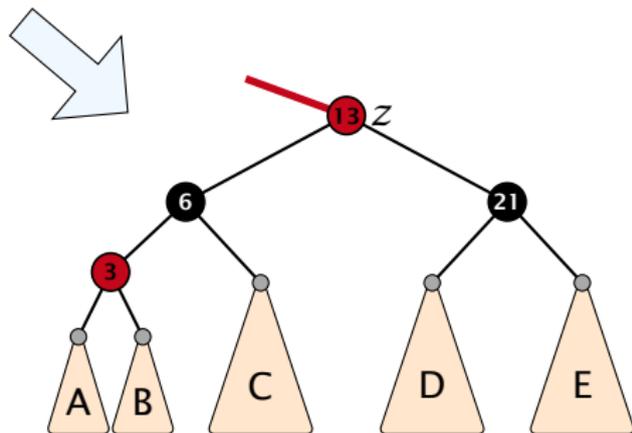
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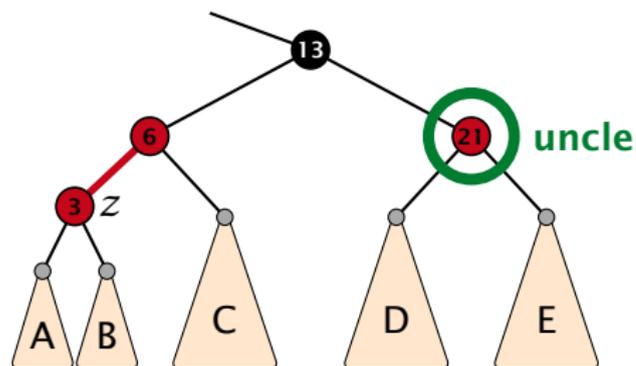
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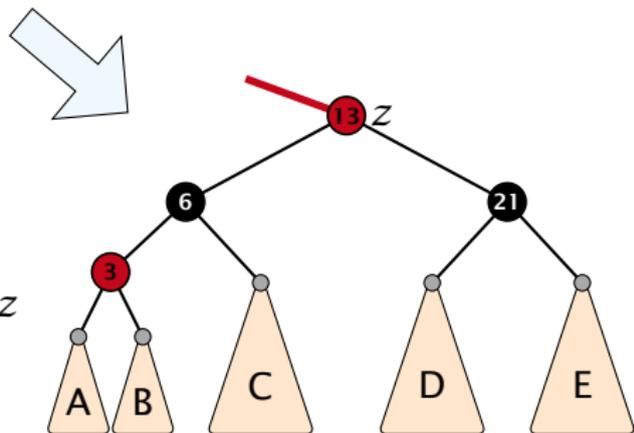
1. recolour
2. move z to grand-parent



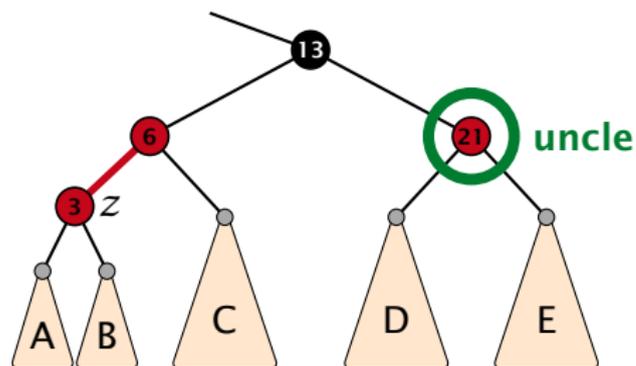
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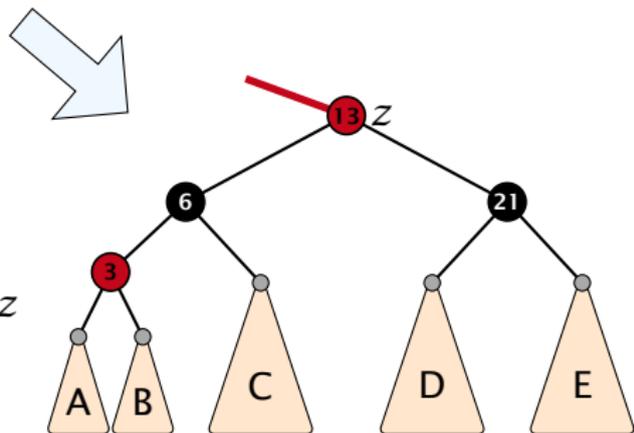
1. recolour
2. move z to grand-parent
3. invariant is fulfilled for new z



Case 1: Red Uncle

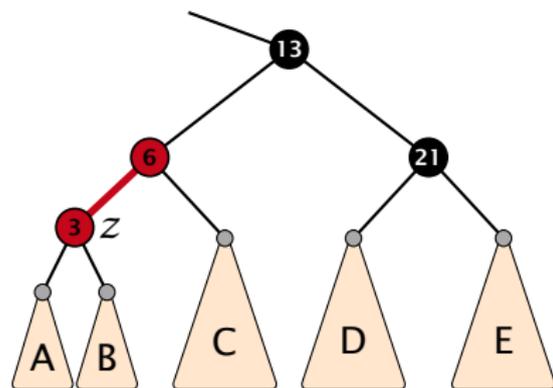


1. recolour
2. move z to grand-parent
3. invariant is fulfilled for new z
4. you made progress



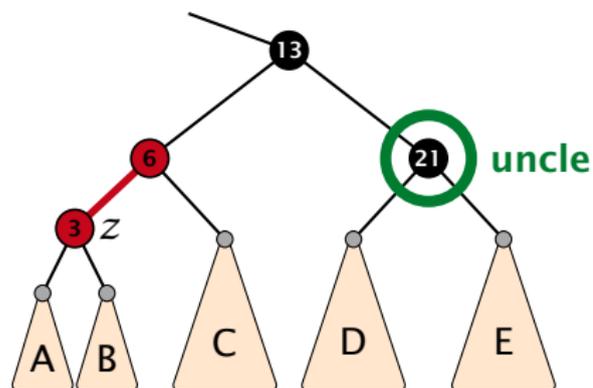
Case 2b: Black uncle and z is left child

1. rotate around grandparent
2. re-colour to ensure that black height property holds
3. you have a red black tree



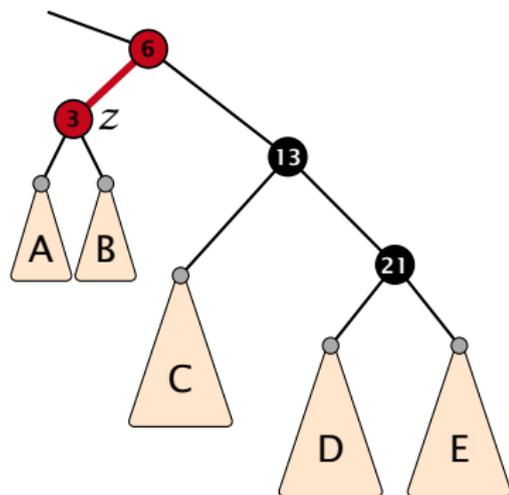
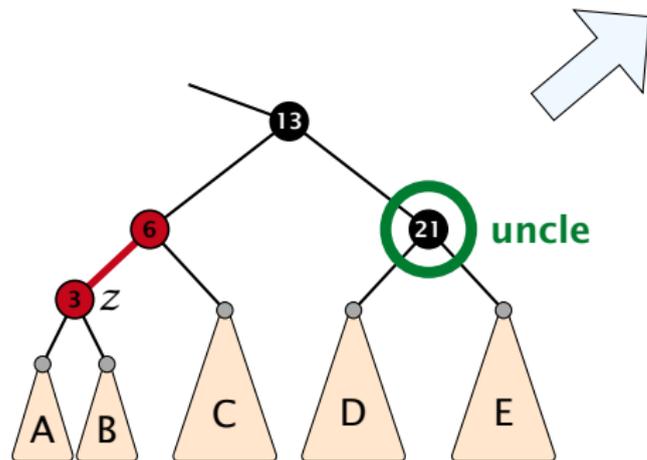
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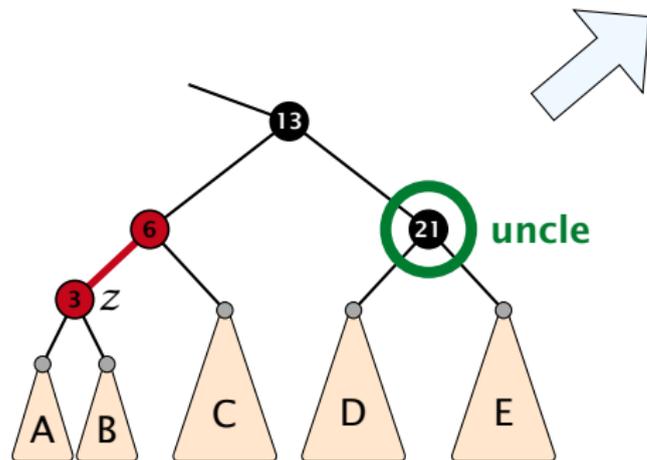
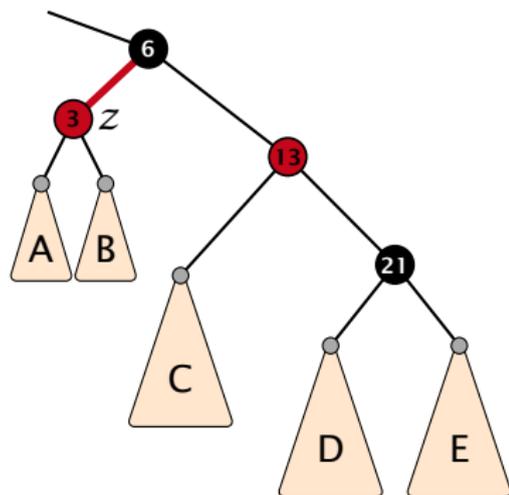
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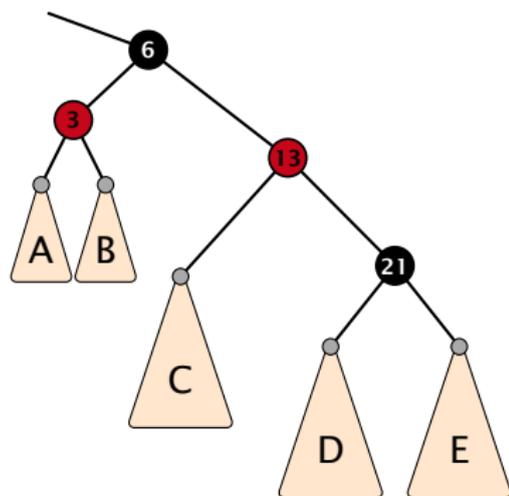
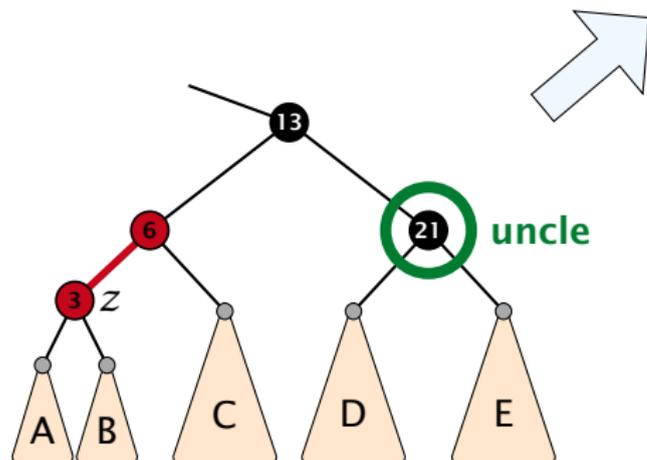
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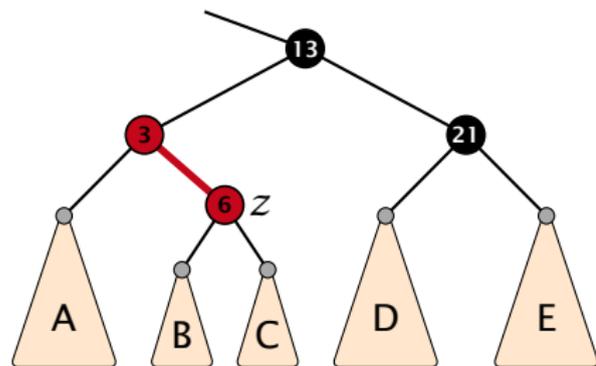
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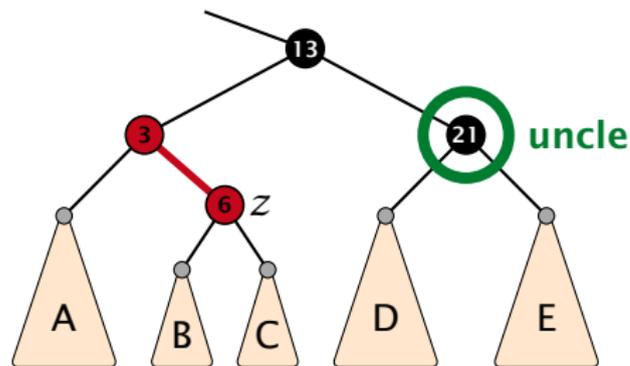
Case 2a: Black uncle and z is right child

1. rotate around parent
2. move z downwards
3. you have Case 2b.



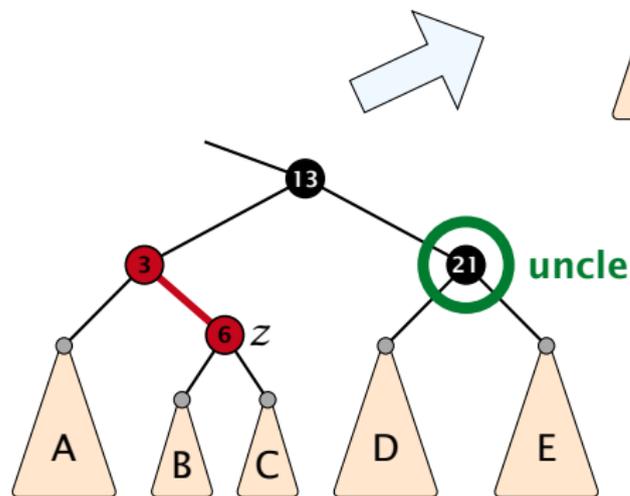
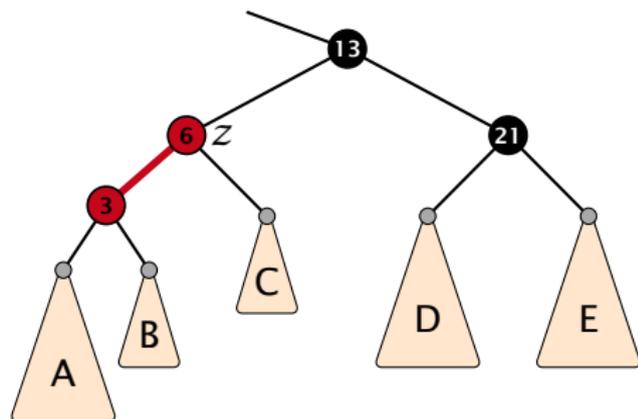
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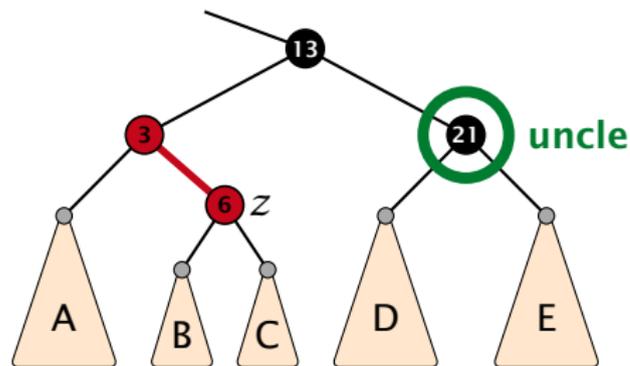
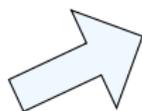
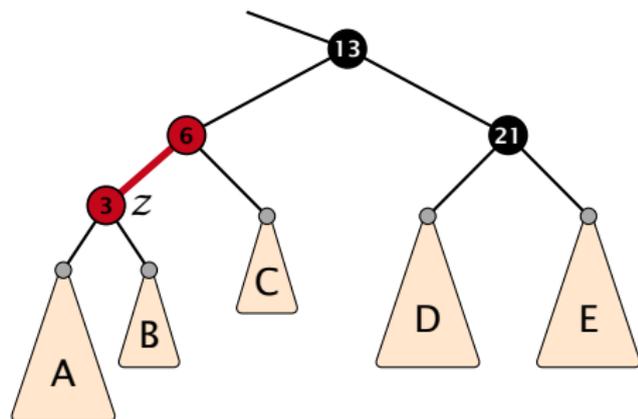
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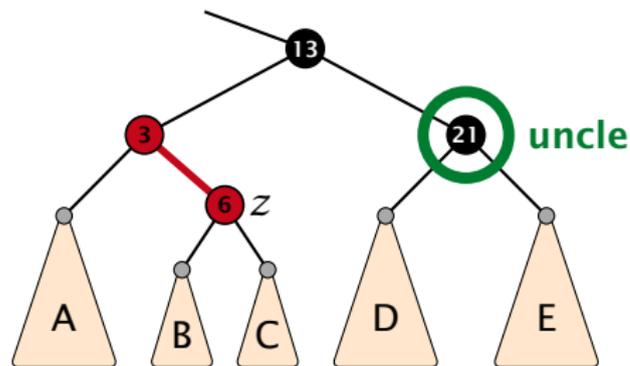
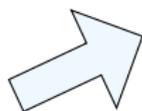
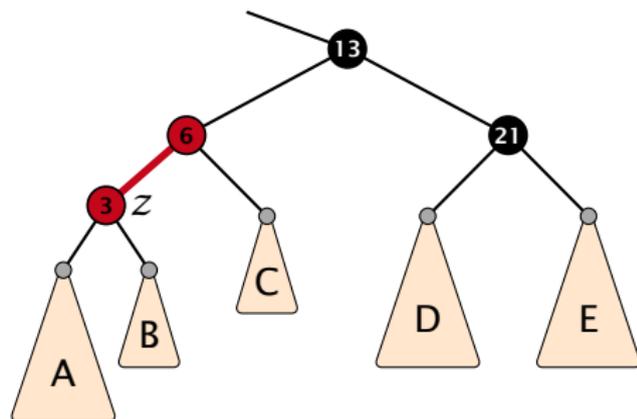
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Red Black Trees: Insert

Running time:

- ▶ Only Case 1 may repeat; but only $h/2$ many steps, where h is the height of the tree.
- ▶ Case 2a → Case 2b → red-black tree
- ▶ Case 2b → red-black tree

Performing Case 1 at most $\mathcal{O}(\log n)$ times and every other case at most once, we get a red-black tree. Hence $\mathcal{O}(\log n)$ re-colorings and at most 2 rotations.

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Red Black Trees: Delete

First do a standard delete.

If the spliced out node x was red everything is fine.

If it was black there may be the following problems.

• Parent and child of x were red; two adjacent red vertices.

• If you delete the root, the root may now be red.

• Every path from an ancestor of x to a descendant leaf of x changes the number of black nodes. Black height property might be violated.

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• x was the root, the root may now be red.

• x was the root, an ancestor of x is a double black node.

• x was the root, the number of black nodes (Black Height) property

is not violated.

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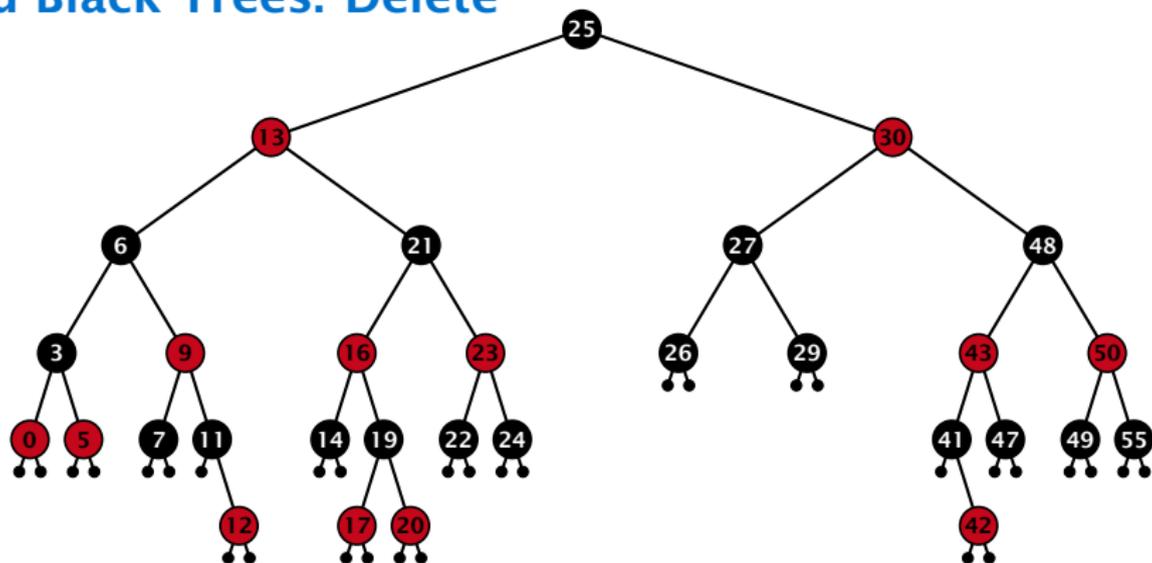
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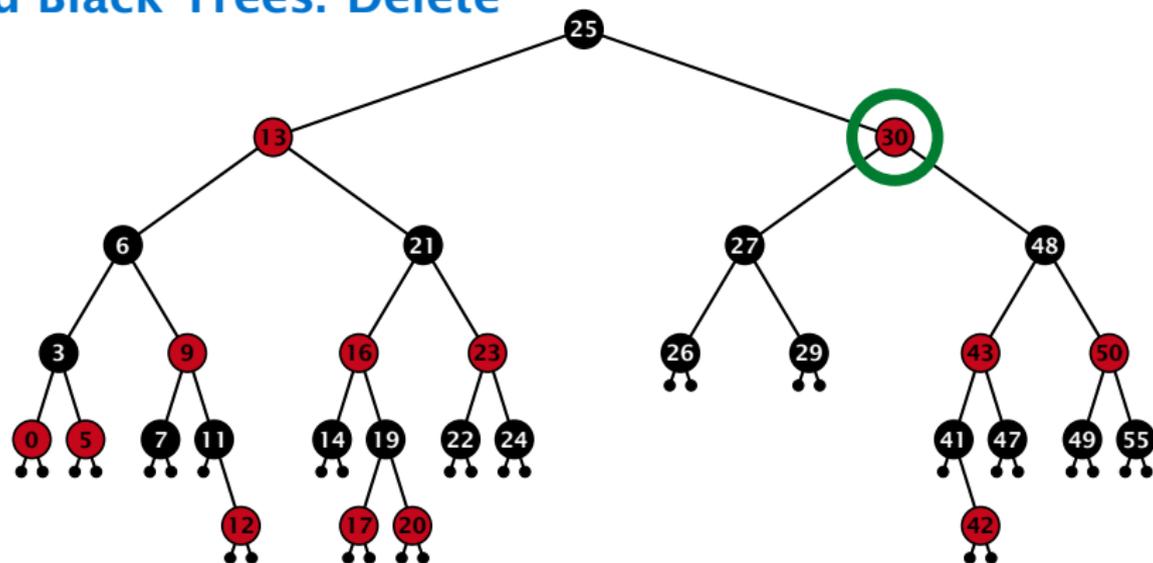
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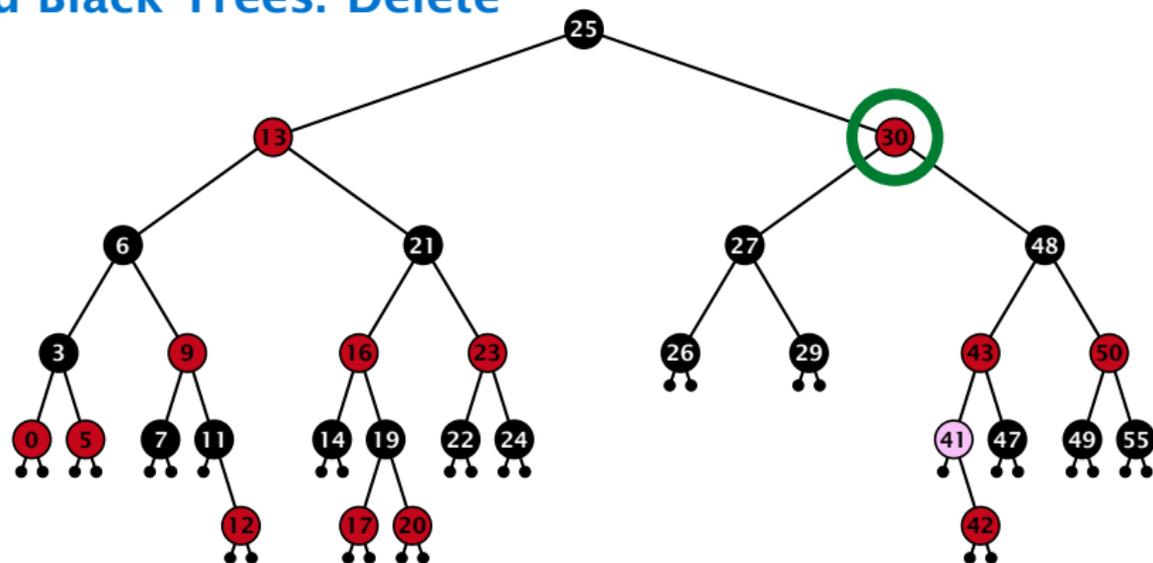


Case 3:

Element has two children

- ▶ do normal delete
- ▶ when replacing content by content of successor, don't change color of node

Red Black Trees: Delete

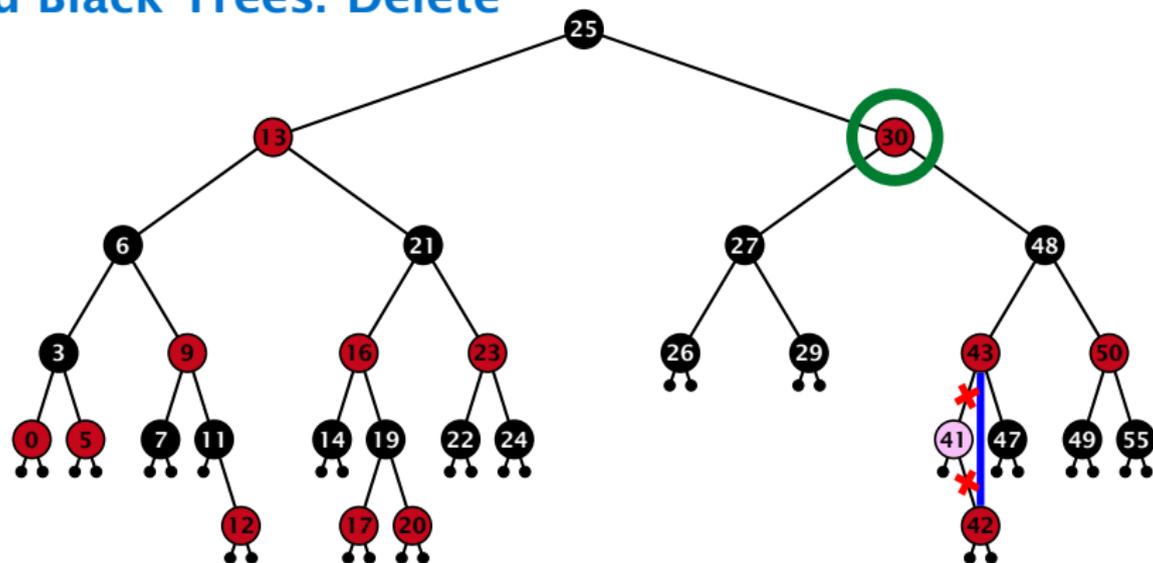


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Element has two children

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- ▶ when replacing content by content of successor, don't change color of node

Red Black Trees: Delete

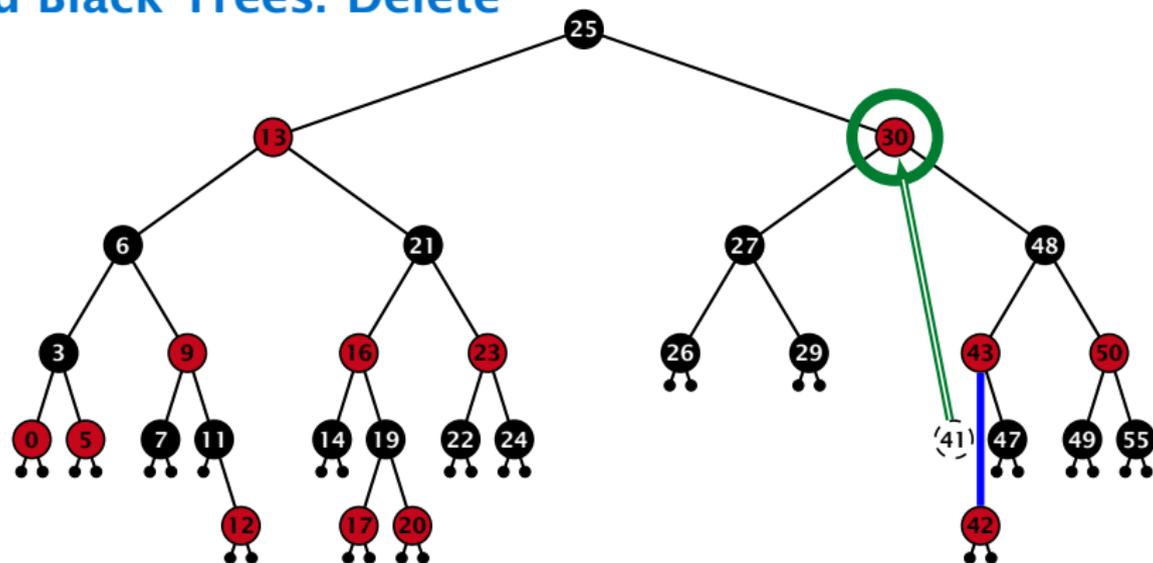


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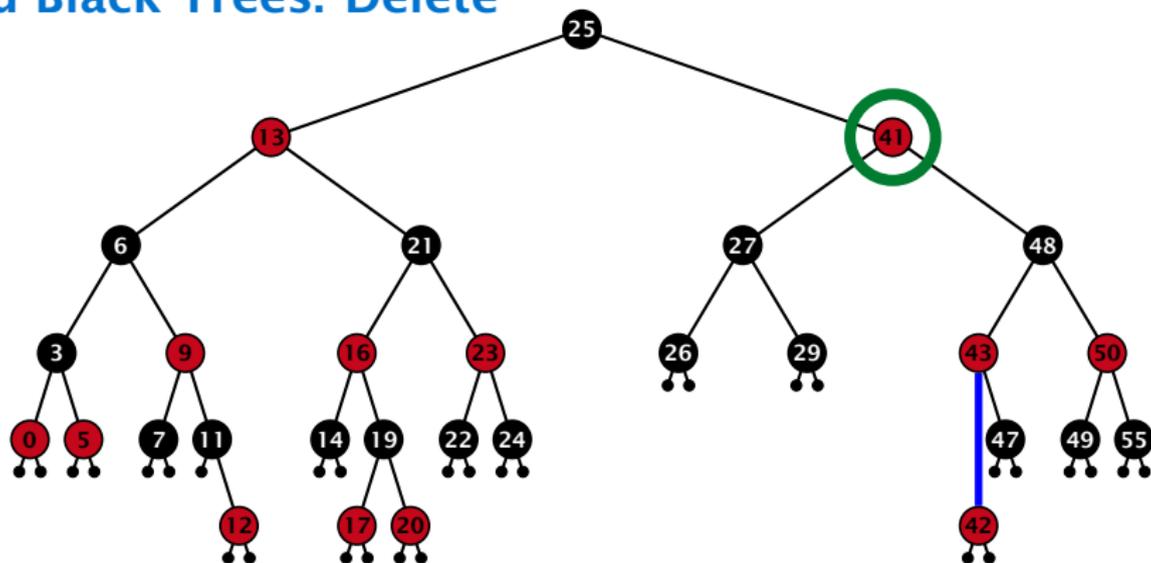


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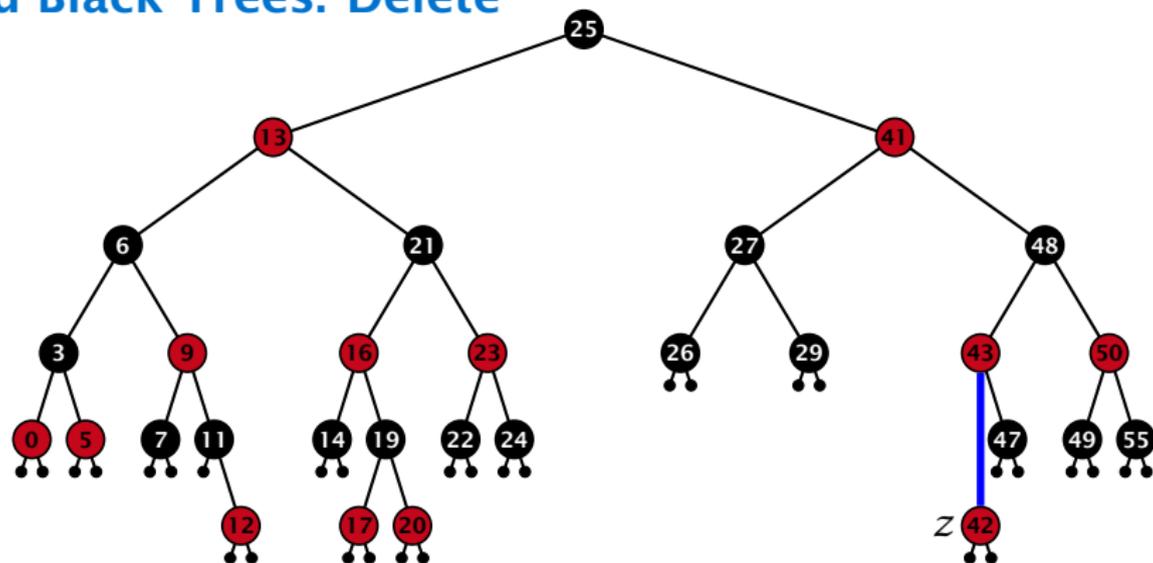


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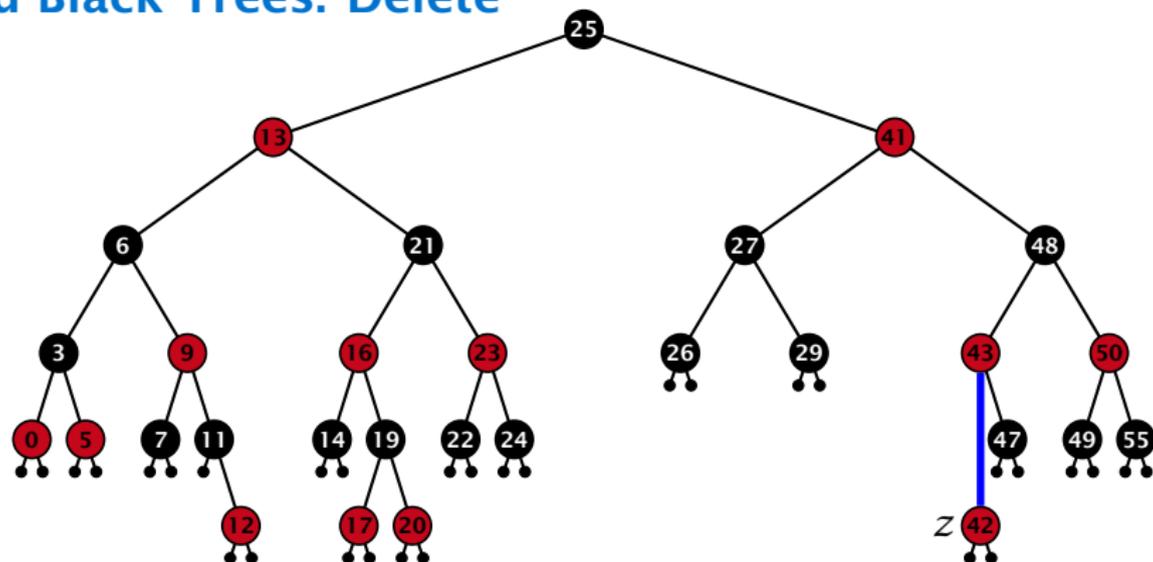
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Delete:

- ▶ deleting black node messes up black-height property
- ▶ if z is red, we can simply color it black and everything is fine
- ▶ the problem is if z is black (e.g. a dummy-leaf); we call a fix-up procedure to fix the problem.

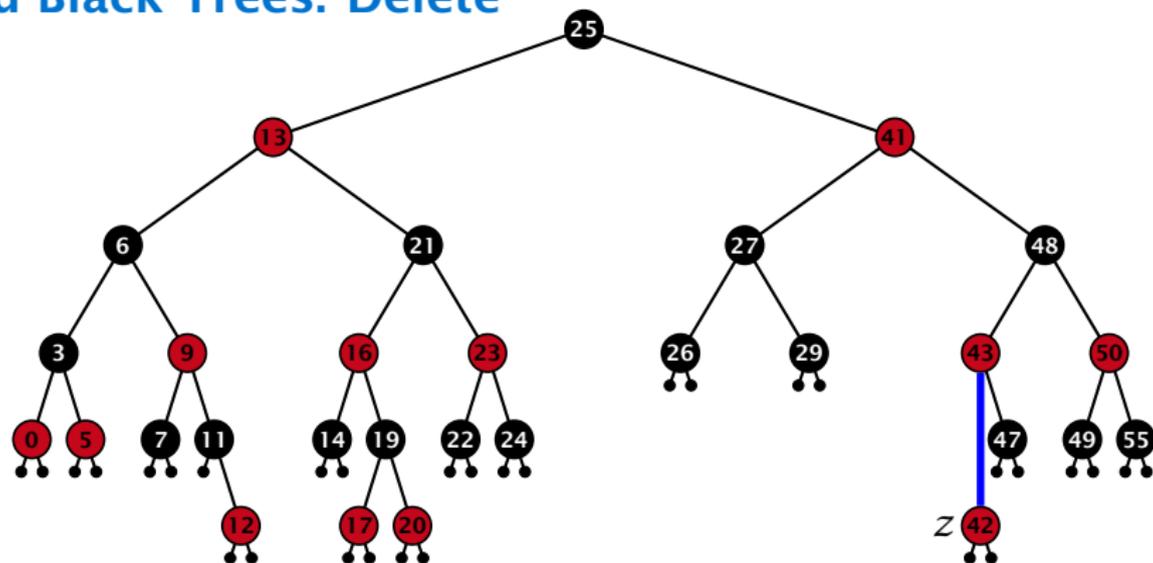
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Invariant of the fix-up algorithm

- ▶ the node z is black
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Goal: make rotations in such a way that you at some point can remove the fake black unit from the edge.

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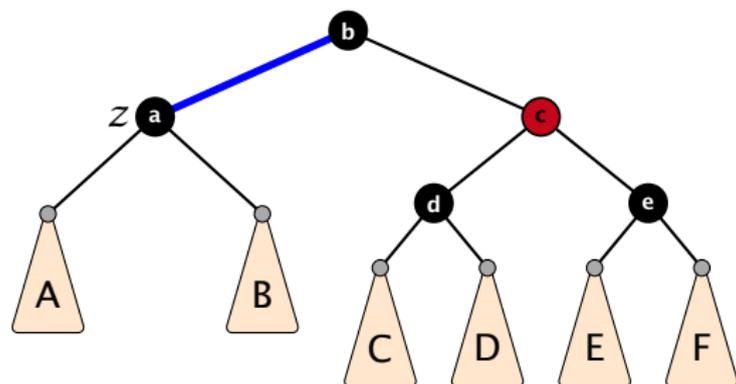
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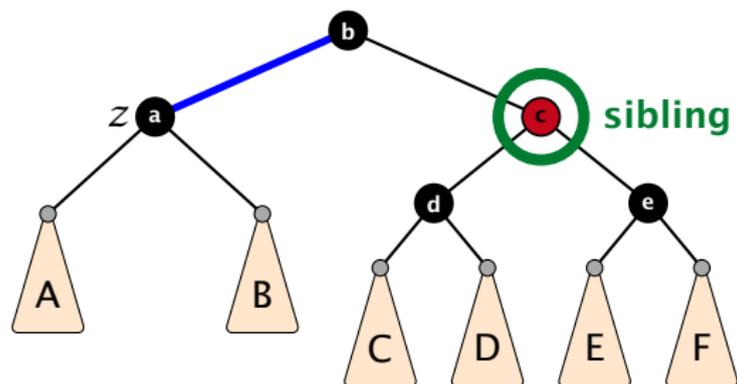
Case 1: Sibling of z is red



1. left-rotate around parent of z
2. recolor nodes b and c
3. the new sibling is black
(and parent of z is red)
4. Case 2 (special),
or Case 3, or Case 4



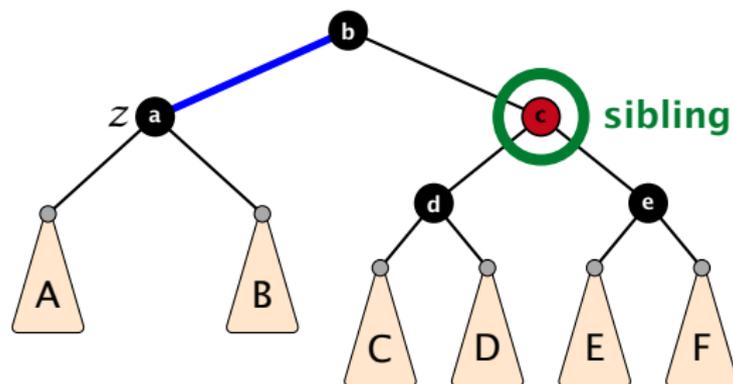
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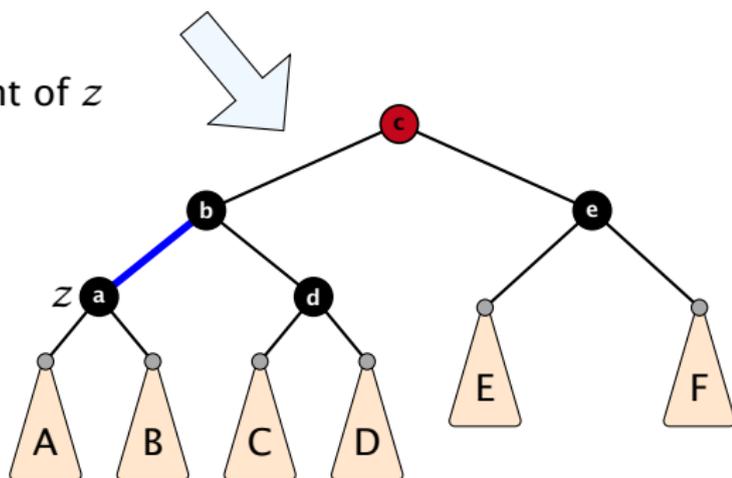


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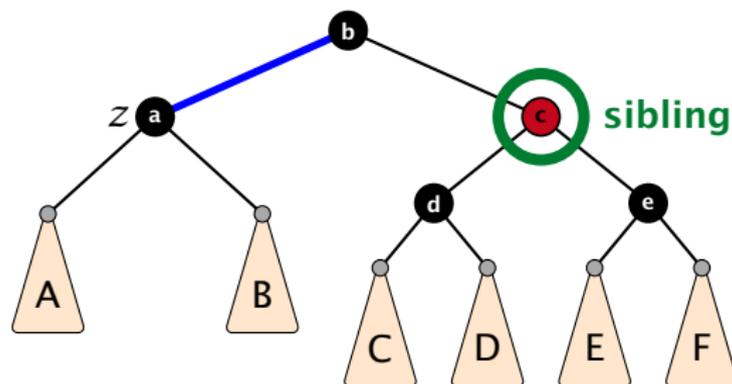
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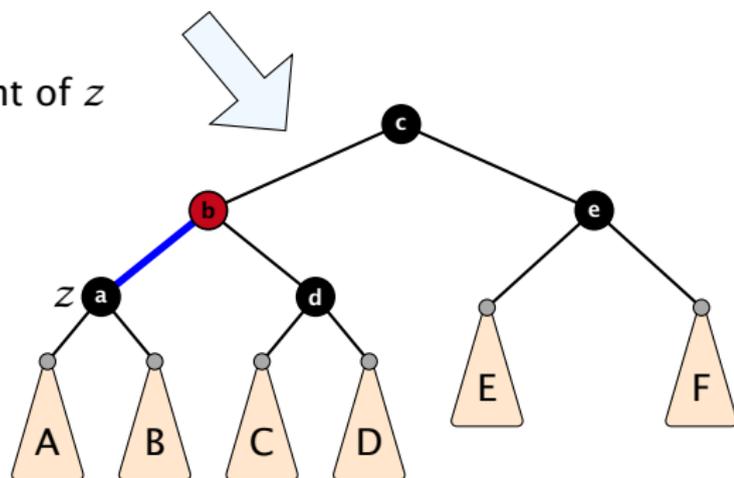
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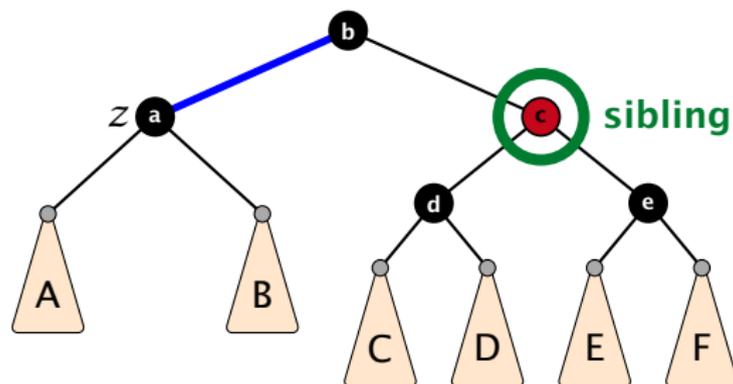
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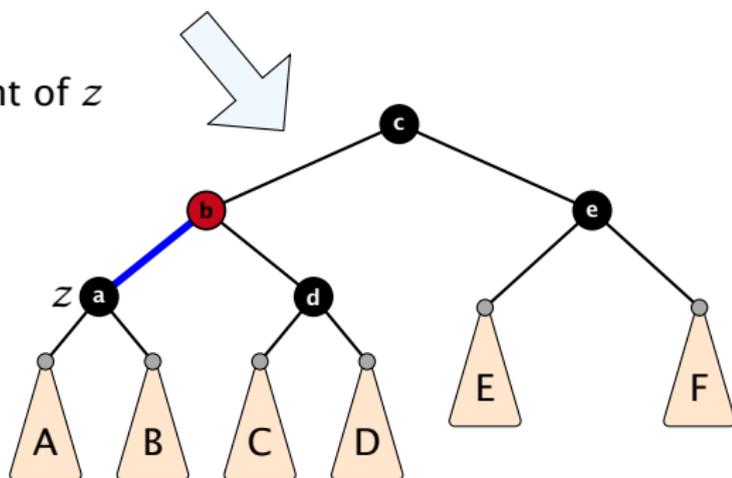
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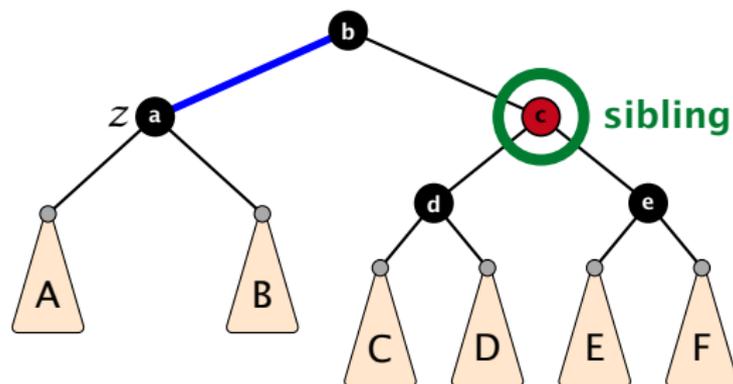
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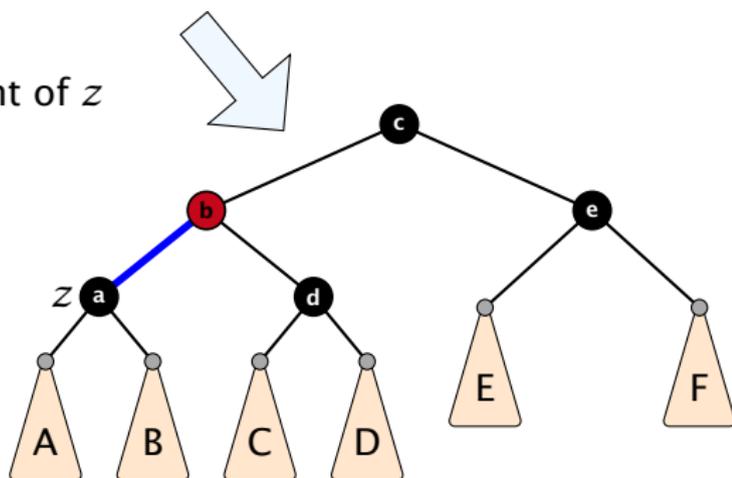
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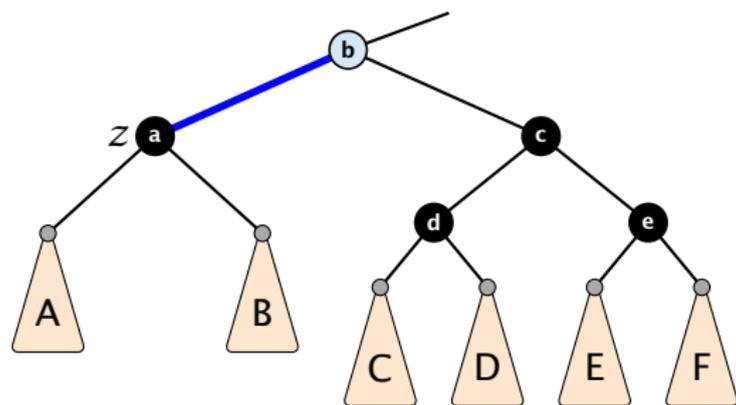
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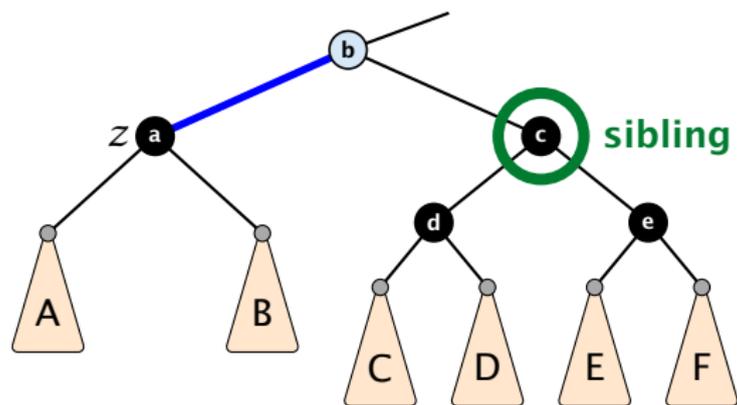
Case 2: Sibling is black with two black children



1. re-color node c
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5. if b is red we color it black and are done



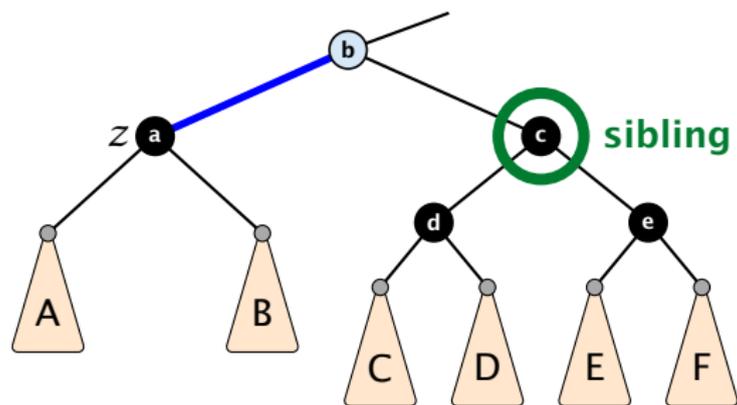
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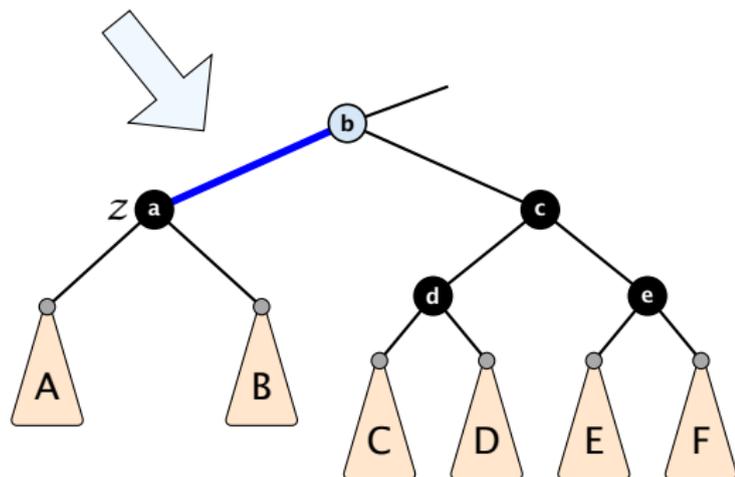
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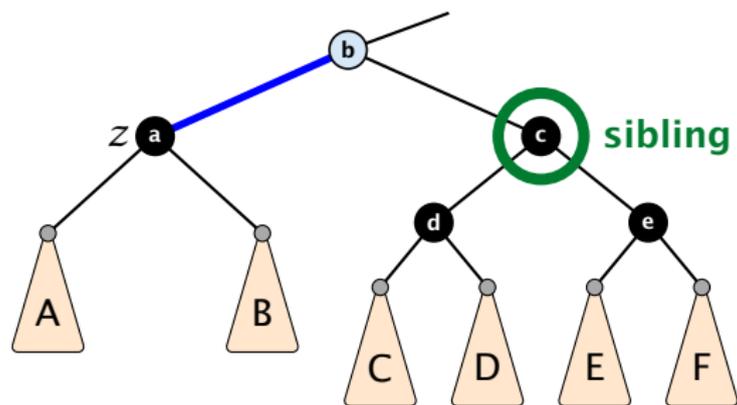
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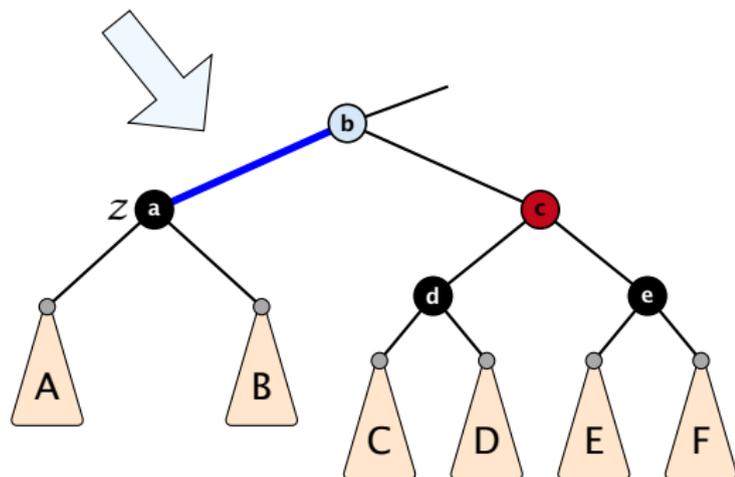
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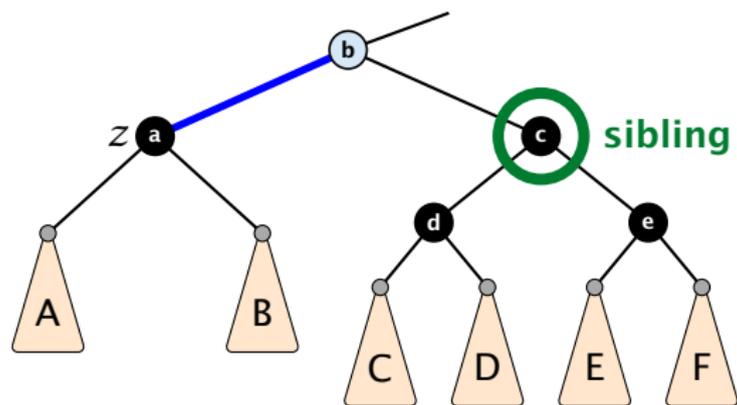
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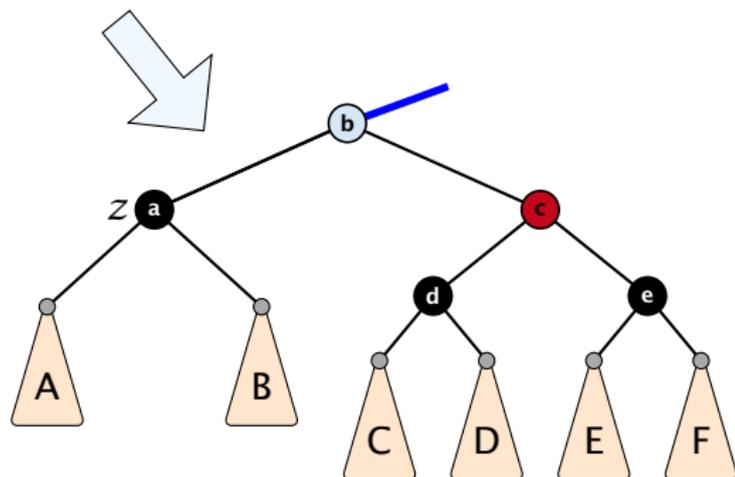
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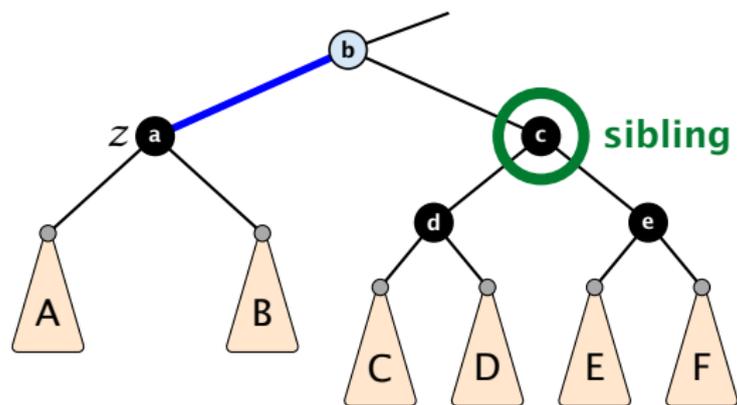
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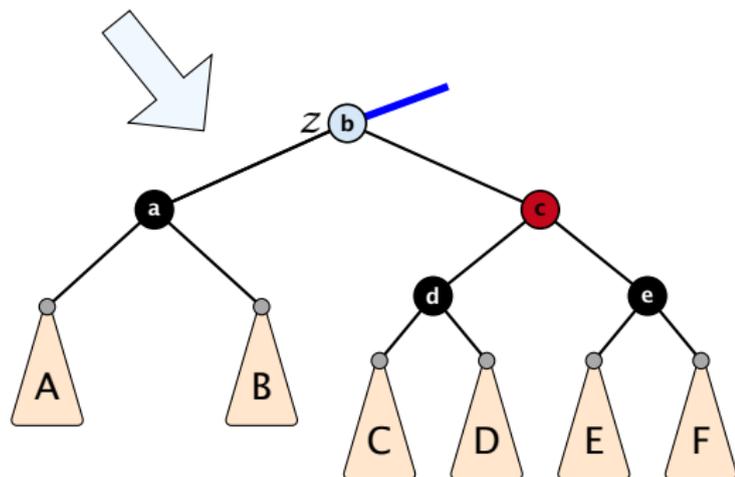
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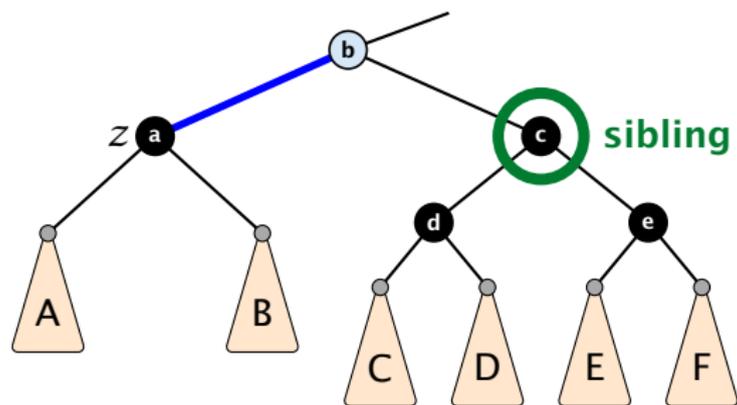
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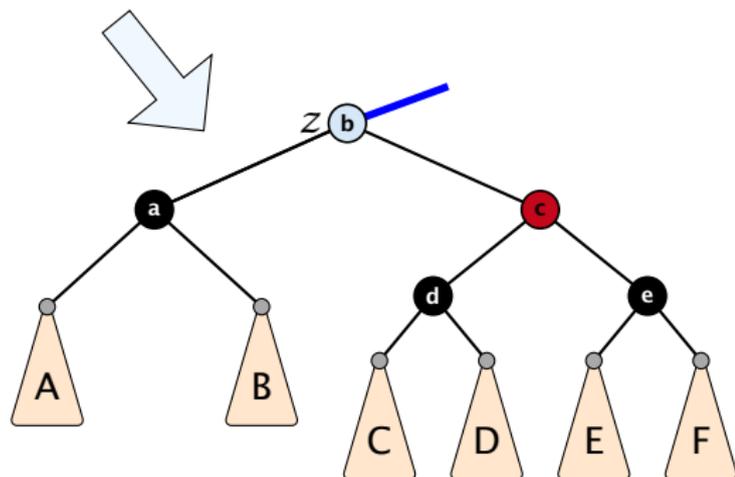
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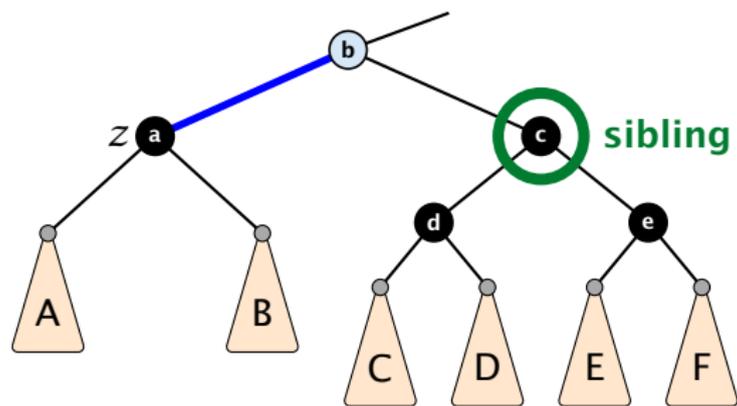
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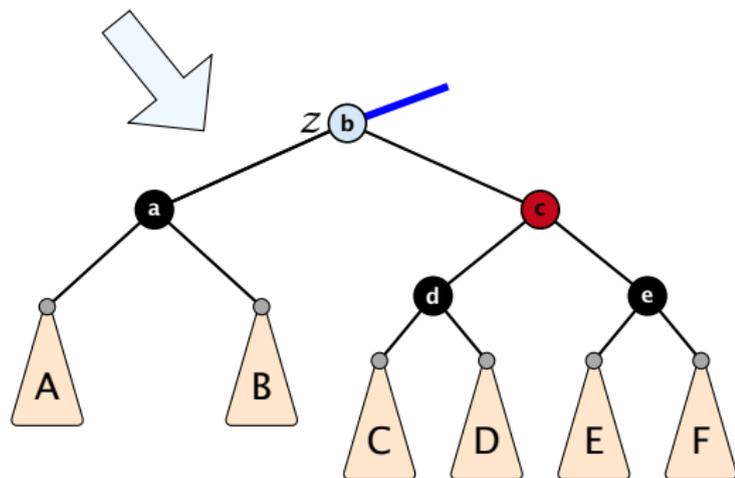
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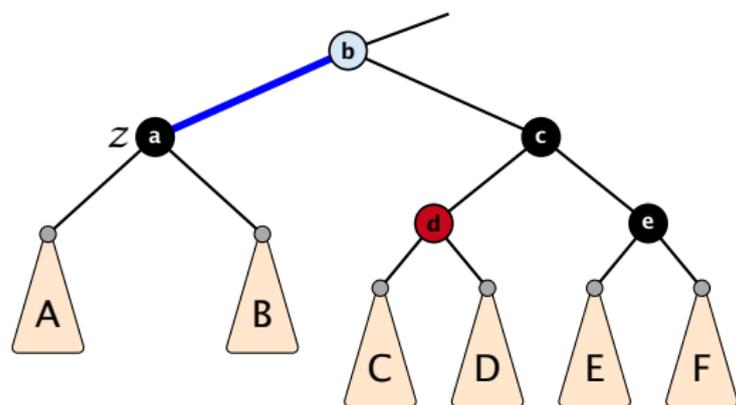


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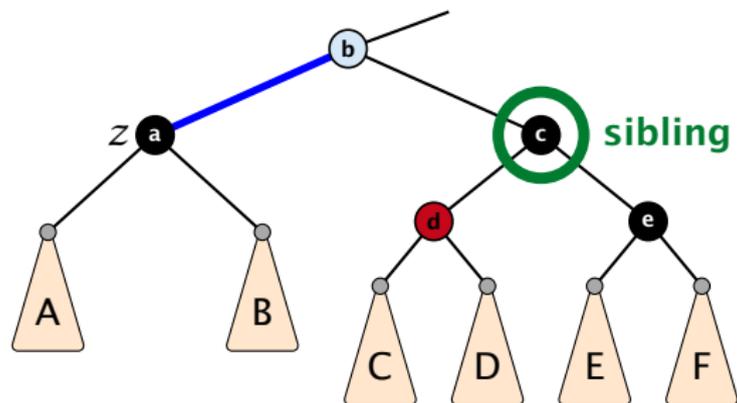
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1. do a right-rotation at sibling
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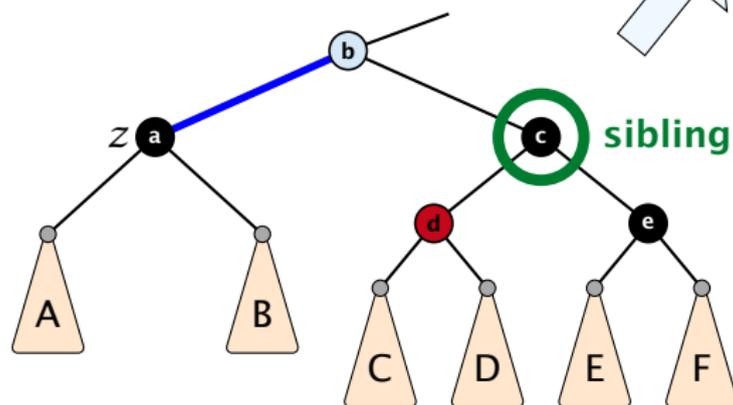
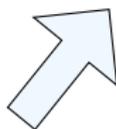
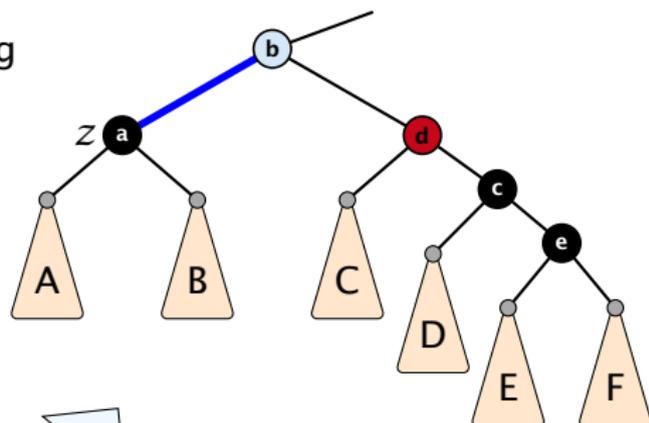
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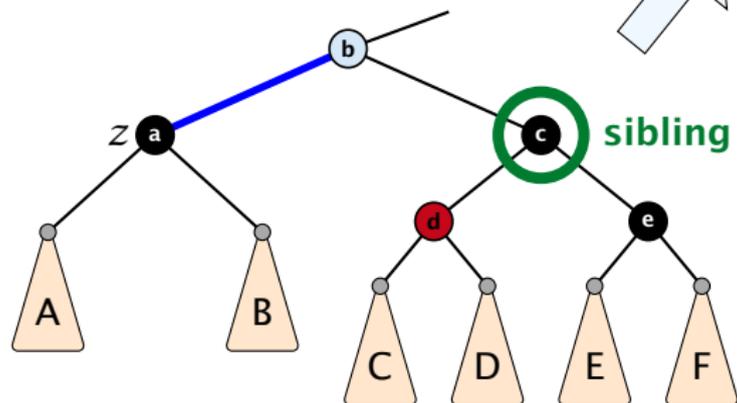
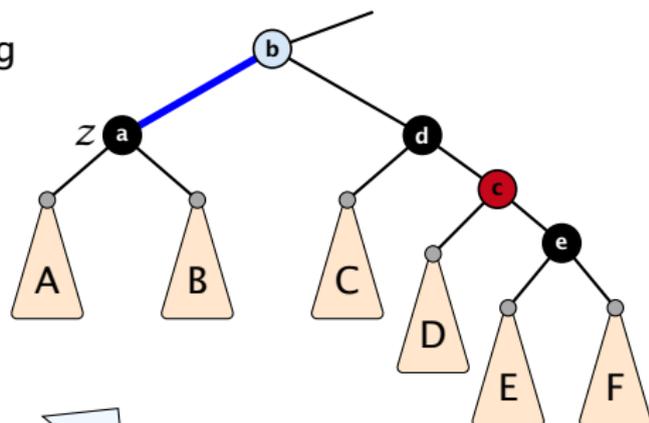
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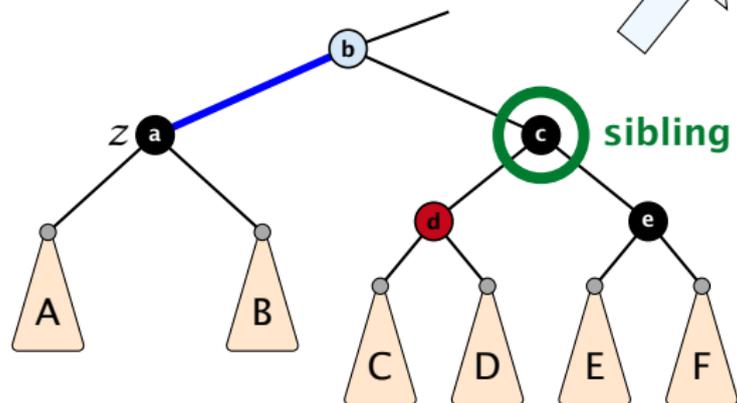
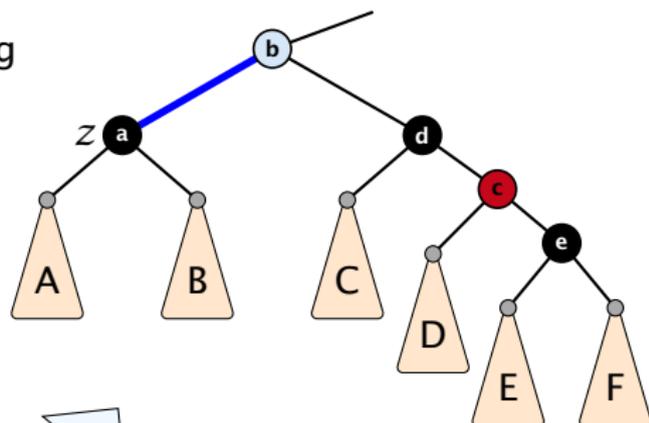
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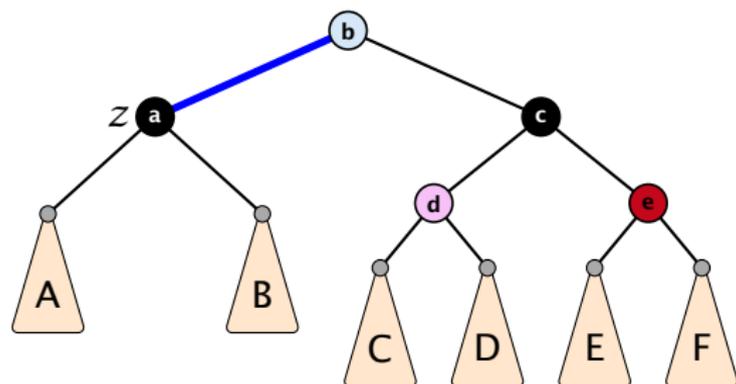


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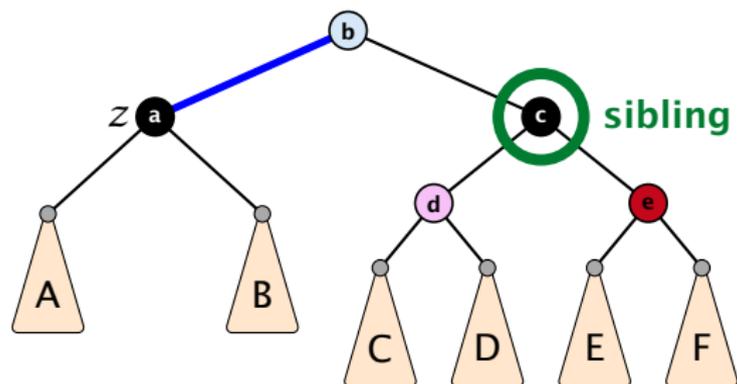
Case 4: Sibling is black with red right child



1. left-rotate around b
2. recolor nodes b , c , and e
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4. you have a valid red black tree



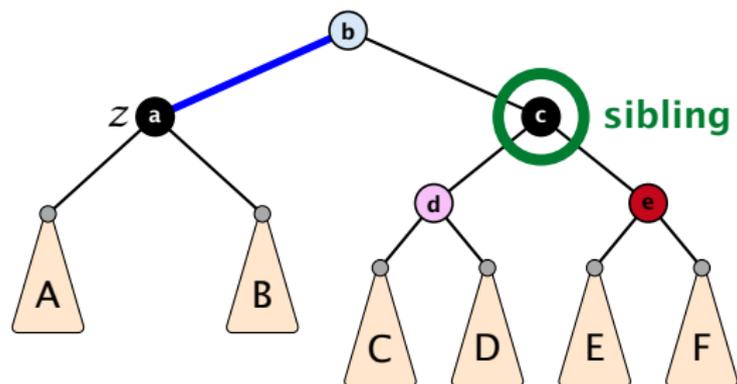
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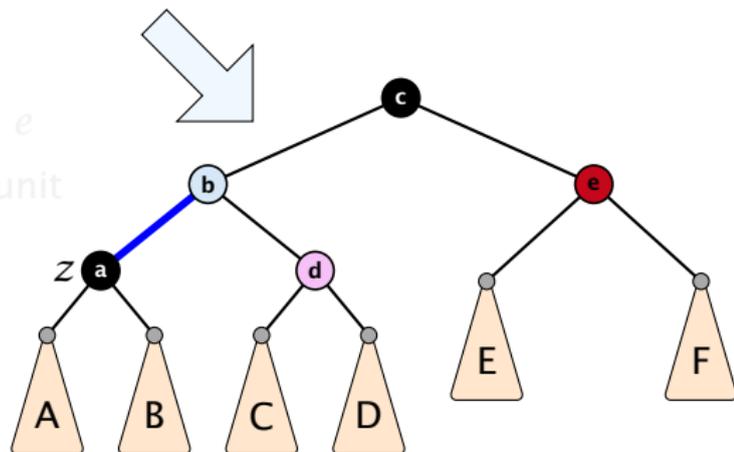
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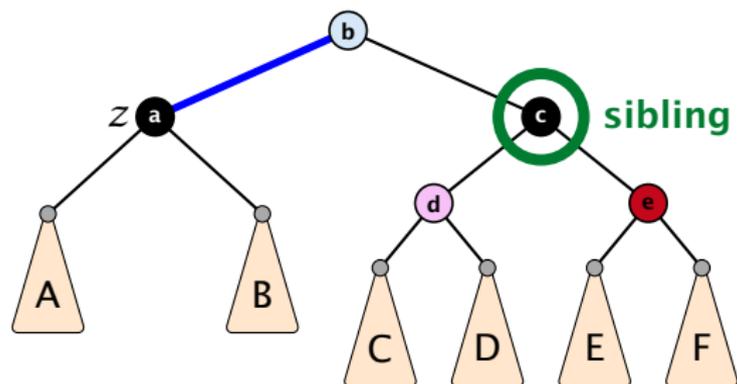
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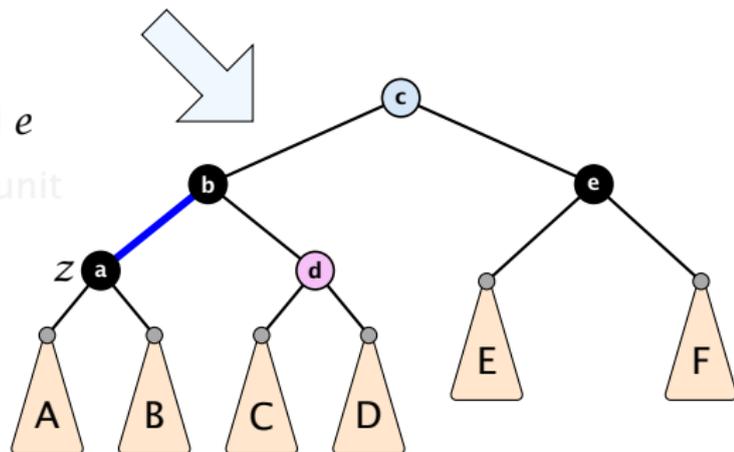
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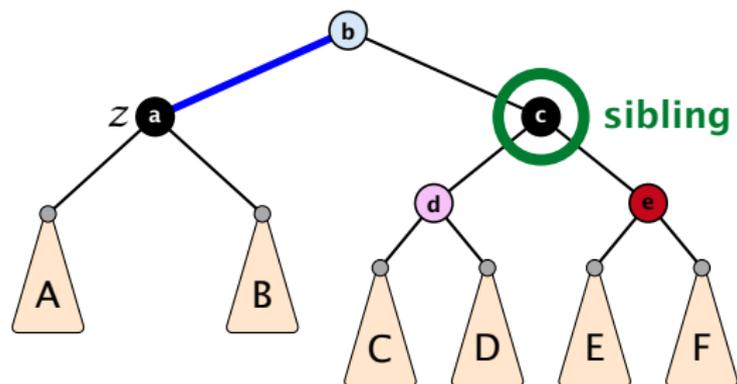
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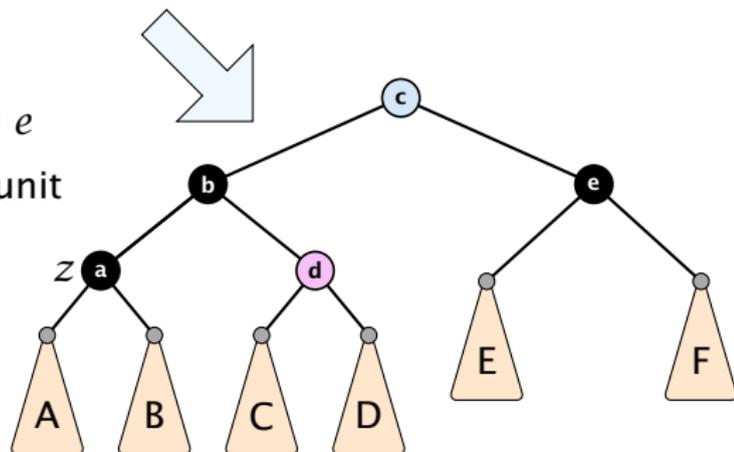
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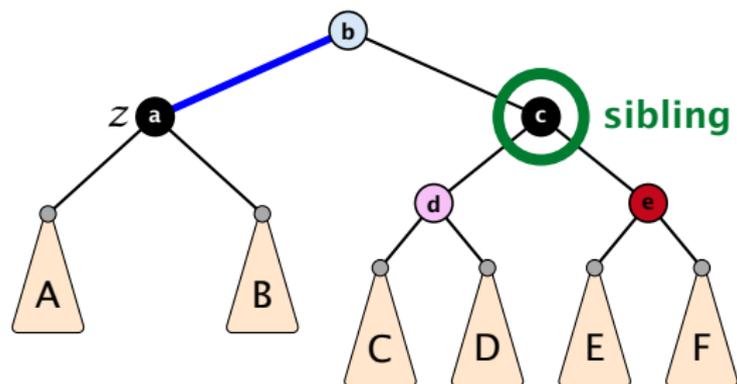
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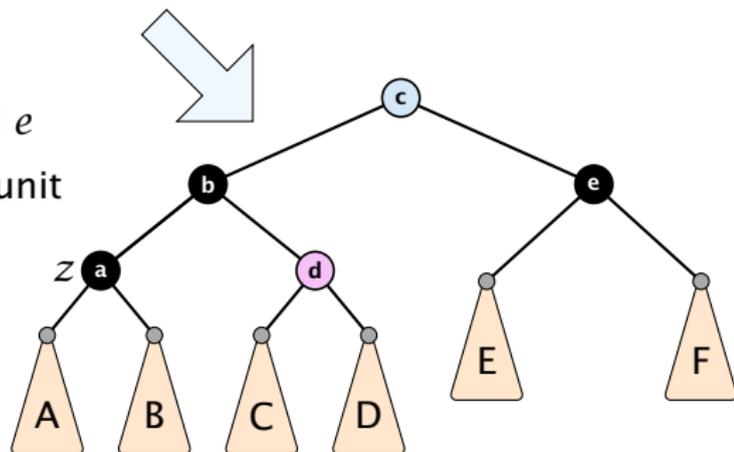
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Running time:

- ▶ only Case 2 can repeat; but only h many steps, where h is the height of the tree
- ▶ Case 1 → Case 2 (special) → red black tree
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Performing Case 2 at most $\mathcal{O}(\log n)$ times and every other step at most once, we get a red black tree. Hence, $\mathcal{O}(\log n)$ re-colorings and at most 3 rotations.

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7.3 AVL-Trees

Definition 5

AVL-trees are binary search trees that fulfill the following balance condition. For every node v

$$|\text{height}(\text{left sub-tree}(v)) - \text{height}(\text{right sub-tree}(v))| \leq 1 .$$

Lemma 6

An AVL-tree of height h contains at least $F_{h+2} - 1$ and at most $2^h - 1$ internal nodes, where F_n is the n -th Fibonacci number ($F_0 = 0, F_1 = 1$), and the height is the maximal number of edges from the root to an (empty) dummy leaf.

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Proof.

The upper bound is clear, as a binary tree of height h can only contain

$$\sum_{j=0}^{h-1} 2^j = 2^h - 1$$

internal nodes.

AVL trees

Proof (cont.)

Induction (base cases):

1. an AVL-tree of height $h = 1$ contains at least one internal node, $1 \geq F_3 - 1 = 2 - 1 = 1$.
2. an AVL tree of height $h = 2$ contains at least two internal nodes, $2 \geq F_4 - 1 = 3 - 1 = 2$



AVL trees

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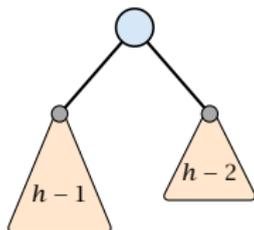


Induction step:

An AVL-tree of height $h \geq 2$ of minimal size has a root with sub-trees of height $h - 1$ and $h - 2$, respectively. Both, sub-trees have minimal node number.

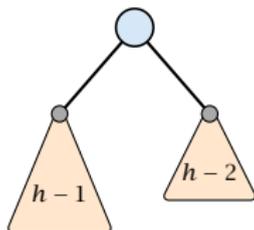
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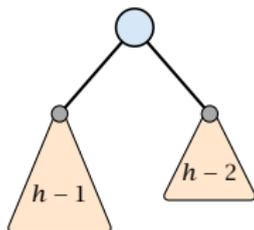


Let

$$g_h := 1 + \text{minimal size of AVL-tree of height } h .$$

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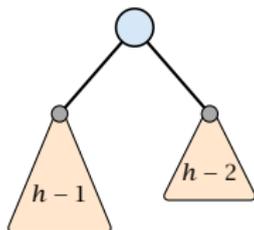
Let

$$g_h := 1 + \text{minimal size of AVL-tree of height } h .$$

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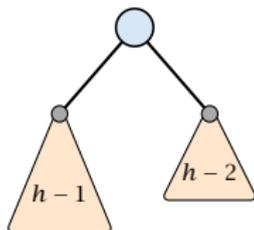
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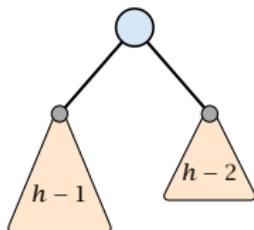
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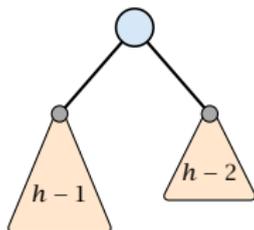
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$$g_h = g_{h-1} + g_{h-2} \qquad = F_{h+2}$$

7.3 AVL-Trees

An AVL-tree of height h contains at least $F_{h+2} - 1$ internal nodes.

Since

$$n + 1 \geq F_{h+2} = \Omega \left(\left(\frac{1 + \sqrt{5}}{2} \right)^h \right),$$

we get

$$n \geq \Omega \left(\left(\frac{1 + \sqrt{5}}{2} \right)^h \right),$$

and, hence, $h = \mathcal{O}(\log n)$.

7.3 AVL-Trees

We need to maintain the balance condition through rotations.

For this we store in every internal tree-node v the **balance** of the node. Let v denote a tree node with left child c_ℓ and right child c_r .

$$\text{balance}[v] := \text{height}(T_{c_\ell}) - \text{height}(T_{c_r}) ,$$

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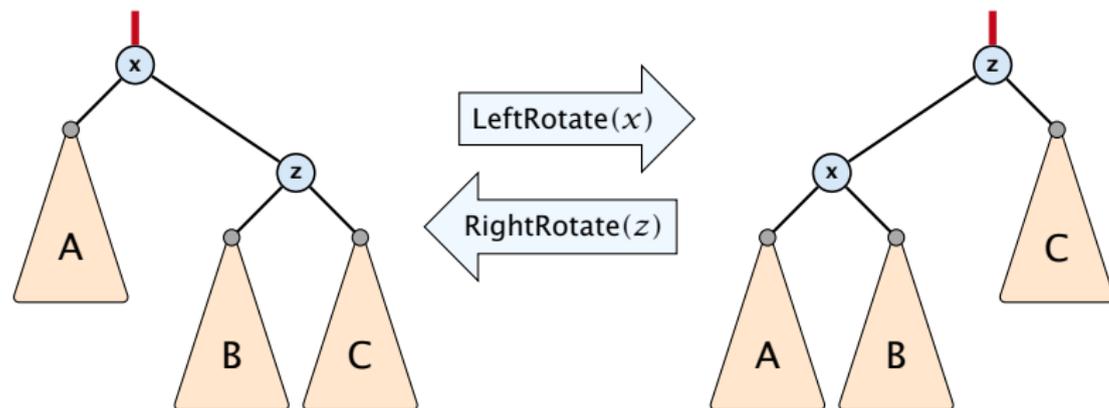
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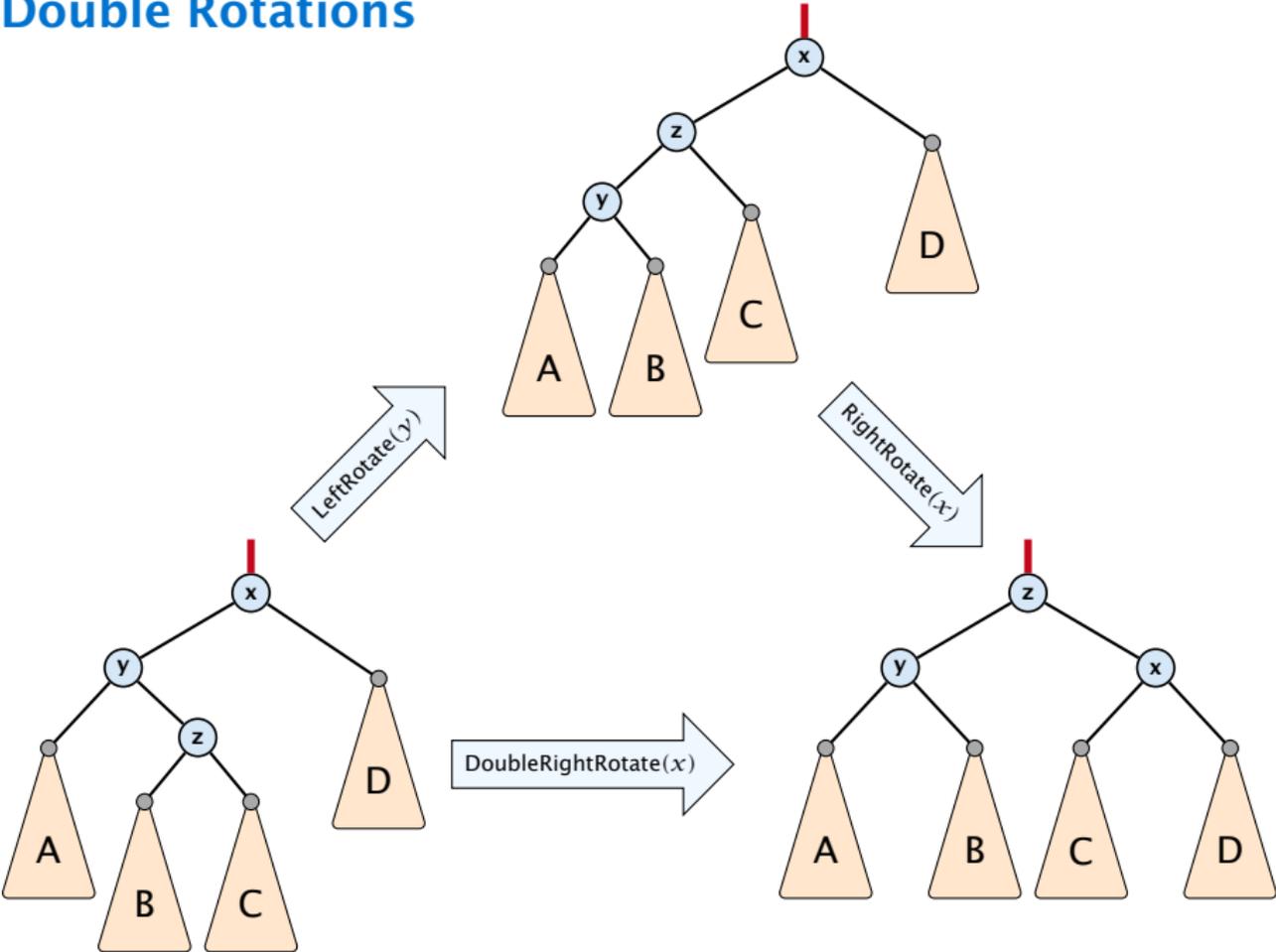
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Rotations

The properties will be maintained through rotations:



Double Rotations



AVL-trees: Insert

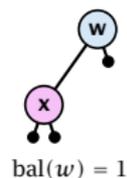
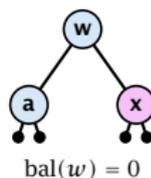
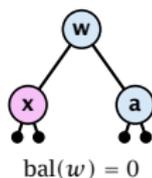
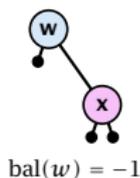
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AVL-trees: Insert

- ▶ Insert like in a binary search tree.
- ▶ Let w denote the parent of the newly inserted node x .

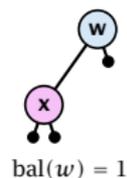
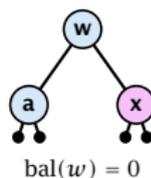
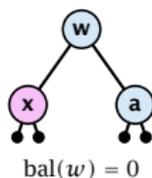
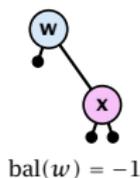
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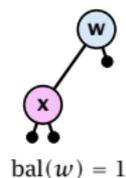
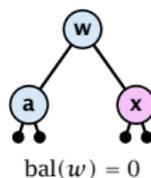
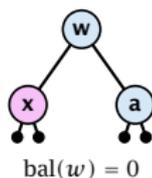
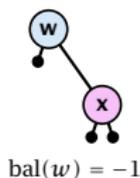
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- ▶ One of the following cases holds:



- ▶ If $\text{bal}[w] \neq 0$, T_w has changed height; the balance-constraint may be violated at ancestors of w .
- ▶ Call $\text{AVL-fix-up-insert}(\text{parent}[w])$ to restore the balance-condition.

Invariant at the beginning of AVL-fix-up-insert(ν):

1. The balance constraints hold at all descendants of ν .
2. A node has been inserted into T_c , where c is either the right or left child of ν .
3. T_c has increased its height by one (otw. we would already have aborted the fix-up procedure).
4. The balance at node c fulfills $\text{balance}[c] \in \{-1, 1\}$. This holds because if the balance of c is 0, then T_c did not change its height, and the whole procedure would have been aborted in the previous step.

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AVL-trees: Insert

Algorithm 11 AVL-fix-up-insert(v)

- 1: **if** $\text{balance}[v] \in \{-2, 2\}$ **then** DoRotationInsert(v);
- 2: **if** $\text{balance}[v] \in \{0\}$ **return**;
- 3: AVL-fix-up-insert(parent(v));

We will show that the above procedure is correct, and that it will do at most one rotation.

Algorithm 12 DoRotationInsert(v)

```
1: if balance[ $v$ ] = -2 then // insert in right sub-tree
2:     if balance[right[ $v$ ]] = -1 then
3:         LeftRotate( $v$ );
4:     else
5:         DoubleLeftRotate( $v$ );
6: else // insert in left sub-tree
7:     if balance[left[ $v$ ]] = 1 then
8:         RightRotate( $v$ );
9:     else
10:        DoubleRightRotate( $v$ );
```

AVL-trees: Insert

It is clear that the invariant for the fix-up routine holds as long as no rotations have been done.

We have to show that after doing one rotation all balance constraints are fulfilled.

We show that after doing a rotation at v :

- ▶ v fulfills balance condition.
- ▶ All children of v still fulfill the balance condition.
- ▶ The height of T_v is the same as before the insert-operation took place.

We only look at the case where the insert happened into the right sub-tree of v . The other case is symmetric.

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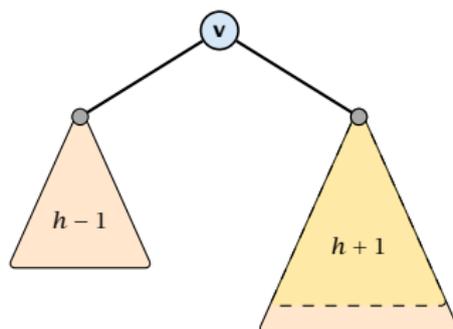
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We have the following situation:

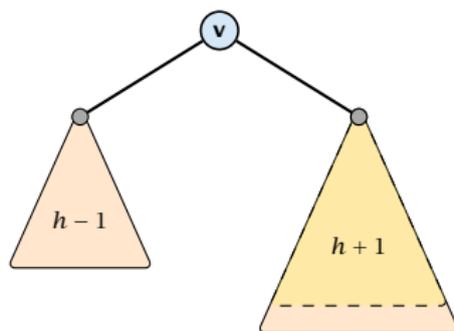


The right sub-tree of v has increased its height which results in a balance of -2 at v .

Before the insertion the height of T_v was $h + 1$.

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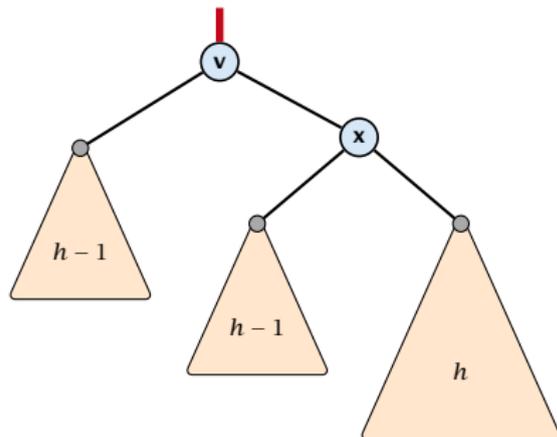
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Case 1: $\text{balance}[\text{right}[v]] = -1$

We do a left rotation at v

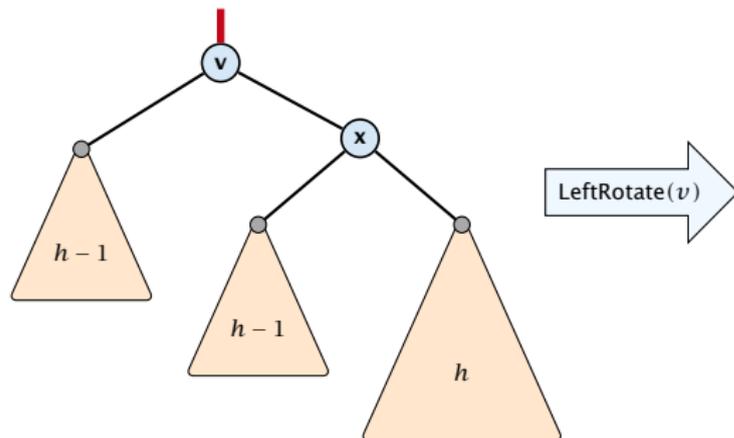
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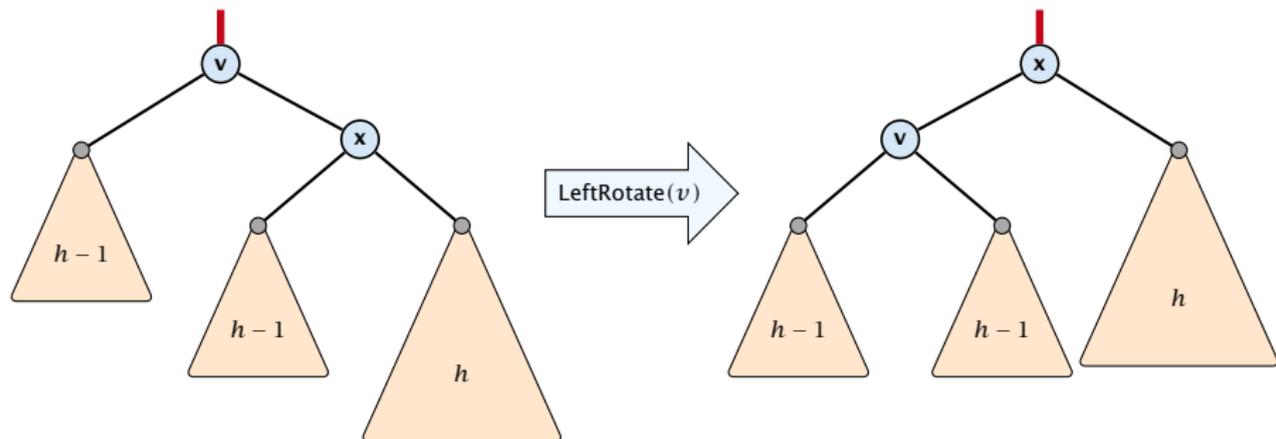
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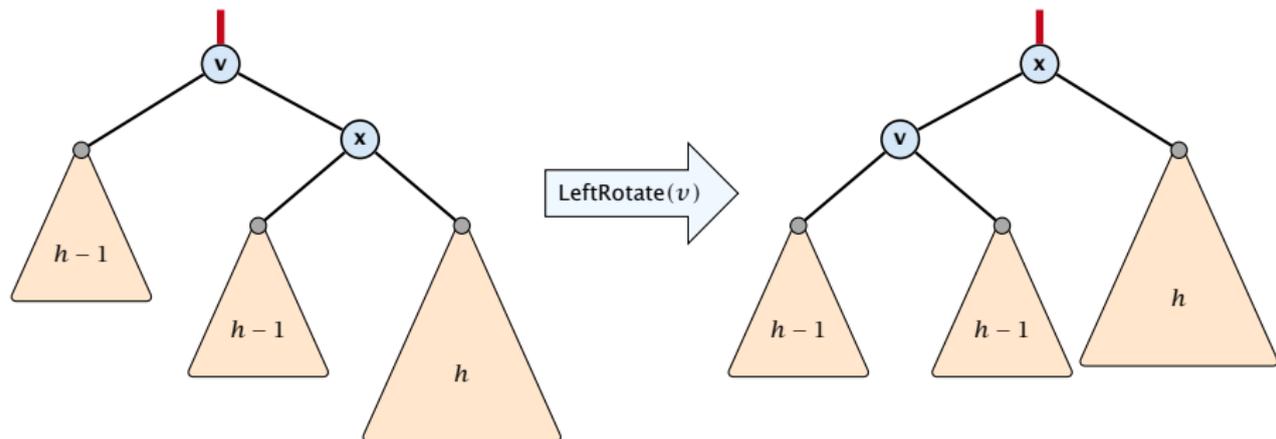
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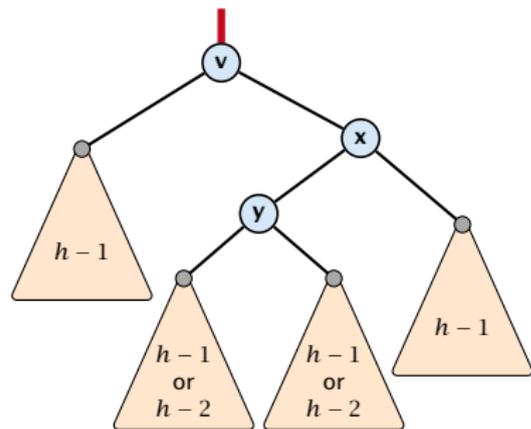
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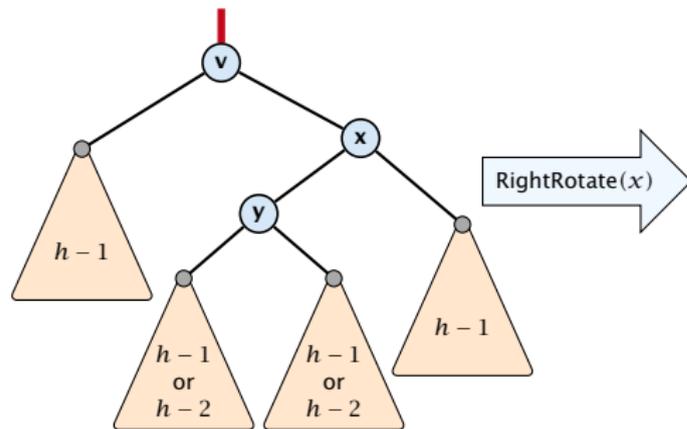
Now, the subtree has height $h + 1$ as before the insertion.
Hence, we do not need to continue.

Case 2: $\text{balance}[\text{right}[v]] = 1$

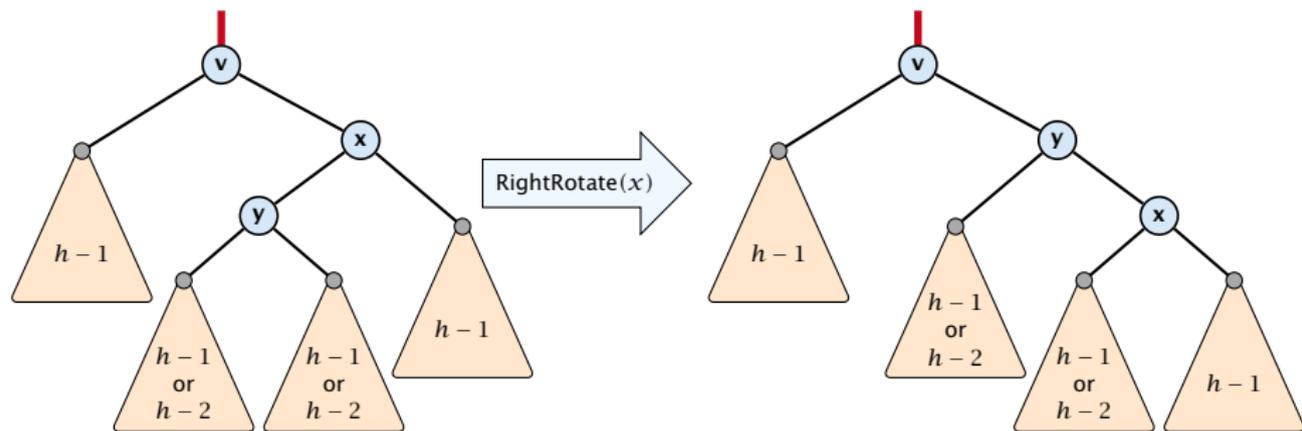
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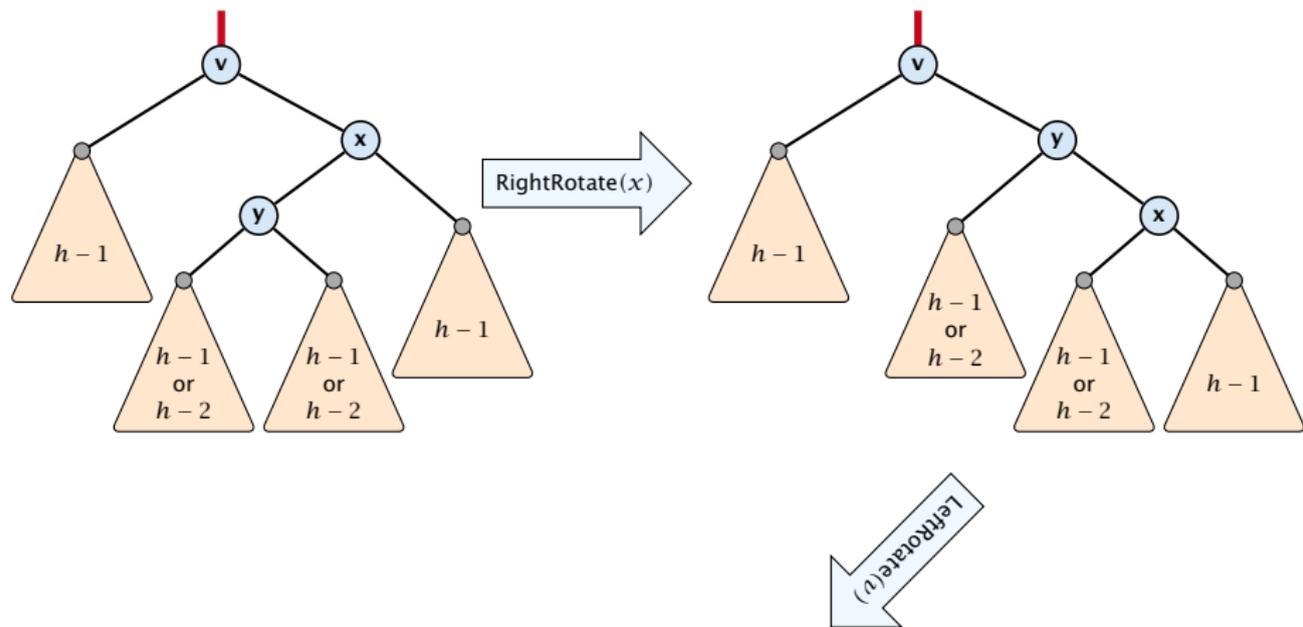
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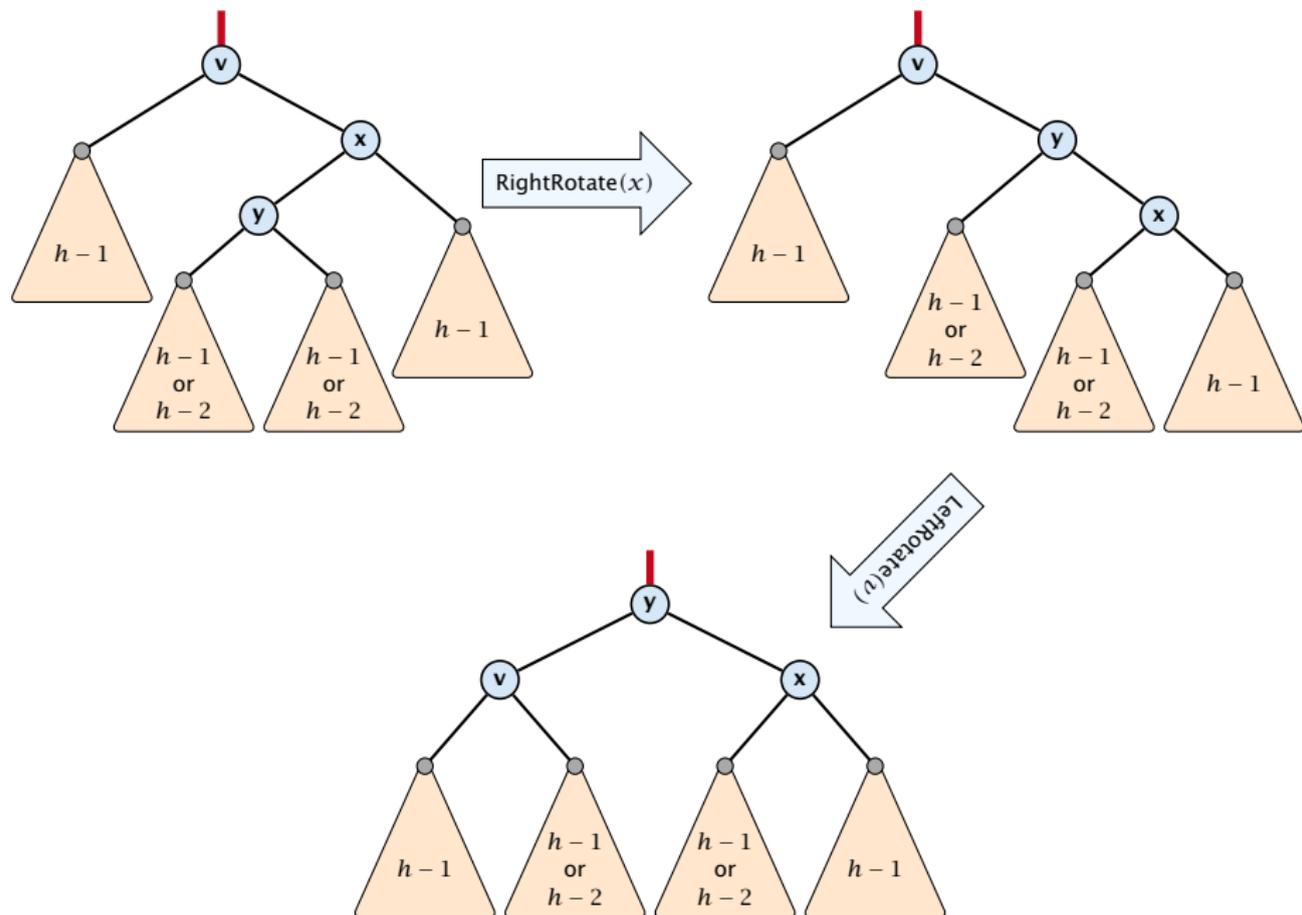
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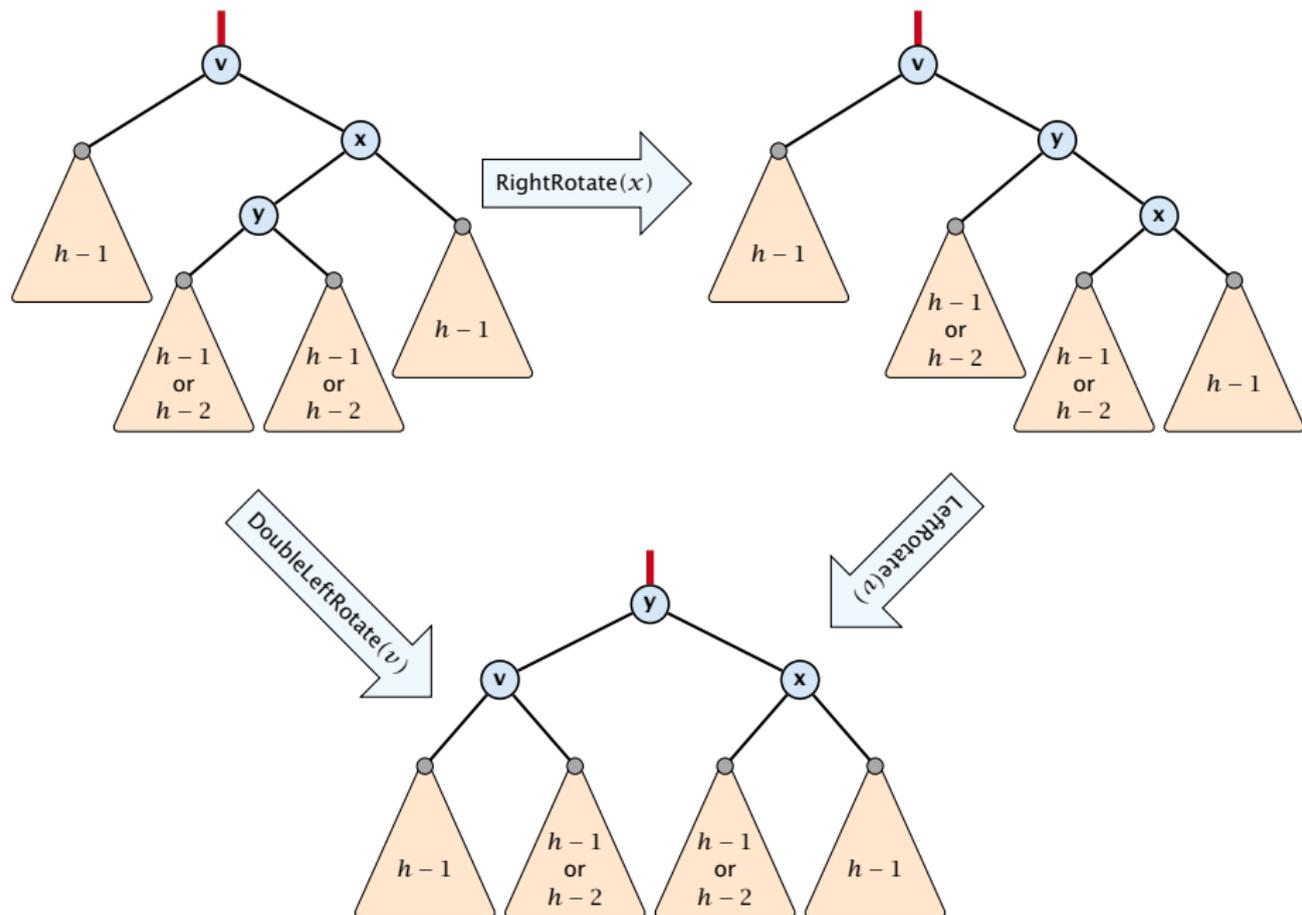
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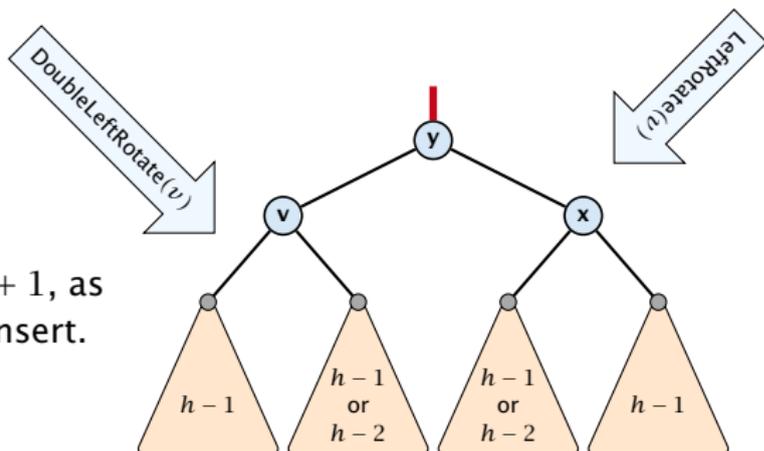
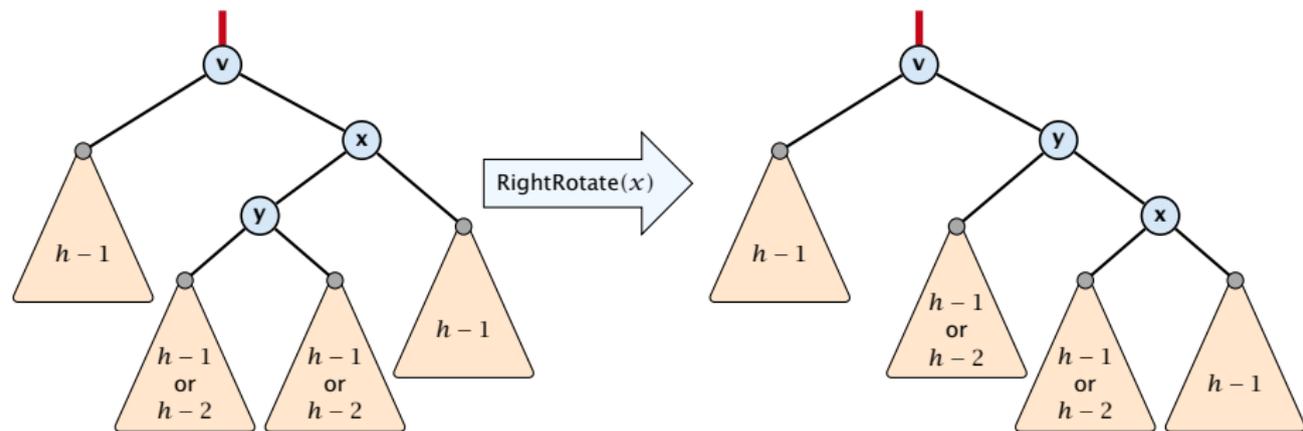
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Height is $h + 1$, as before the insert.

AVL-trees: Delete

- ▶ Delete like in a binary search tree.
- ▶ Let v denote the parent of the node that has been spliced out.
- ▶ The balance-constraint may be violated at v , or at ancestors of v , as a sub-tree of a child of v has reduced its height.
- ▶ Initially, the node c —the new root in the sub-tree that has changed—is either a dummy leaf or a node with two dummy leaves as children.



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Case 2

In both cases $\text{bal}[c] = 0$.

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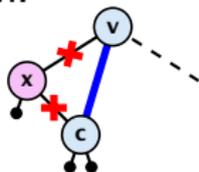
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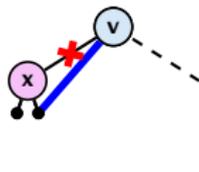
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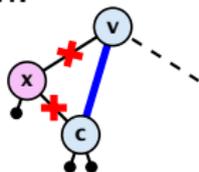
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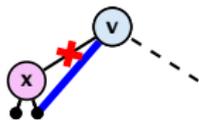
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4. The balance at the node c fulfills $\text{balance}[c] = 0$. This holds because if the balance of c is in $\{-1, 1\}$, then T_c did not change its height, and the whole procedure would have been aborted in the previous step.

AVL-trees: Delete

Algorithm 13 AVL-fix-up-delete(v)

- 1: **if** $\text{balance}[v] \in \{-2, 2\}$ **then** DoRotationDelete(v);
- 2: **if** $\text{balance}[v] \in \{-1, 1\}$ **return**;
- 3: AVL-fix-up-delete(parent[v]);

We will show that the above procedure is correct. However, for the case of a delete there may be a logarithmic number of rotations.

AVL-trees: Delete

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We will show that the above procedure is correct. However, for the case of a delete there may be a logarithmic number of rotations.

Algorithm 14 DoRotationDelete(v)

```
1: if balance[ $v$ ] = -2 then // deletion in left sub-tree
2:     if balance[right[ $v$ ]]  $\in$  {0, -1} then
3:         LeftRotate( $v$ );
4:     else
5:         DoubleLeftRotate( $v$ );
6: else // deletion in right sub-tree
7:     if balance[left[ $v$ ]] = {0, 1} then
8:         RightRotate( $v$ );
9:     else
10:        DoubleRightRotate( $v$ );
```

AVL-trees: Delete

It is clear that the invariant for the fix-up routine hold as long as no rotations have been done.

We show that after doing a rotation at v :

- ▶ v fulfills the balance condition.
- ▶ All children of v still fulfill the balance condition.
- ▶ If now $\text{balance}[v] \in \{-1, 1\}$ we can stop as the height of T_v is the same as before the deletion.

We only look at the case where the deleted node was in the right sub-tree of v . The other case is symmetric.

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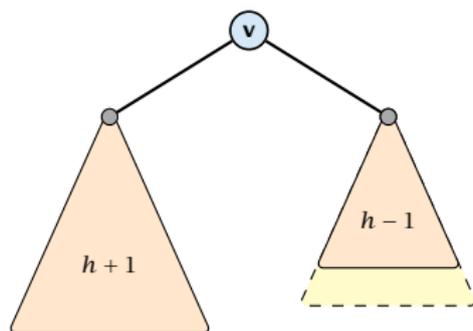
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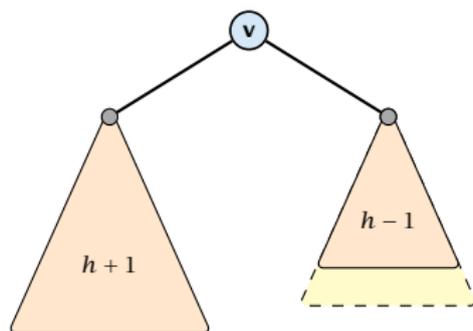


The right sub-tree of v has decreased its height which results in a balance of 2 at v .

Before the deletion the height of T_v was $h + 2$.

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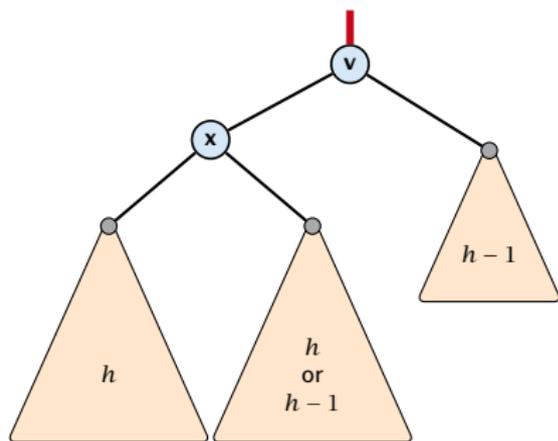


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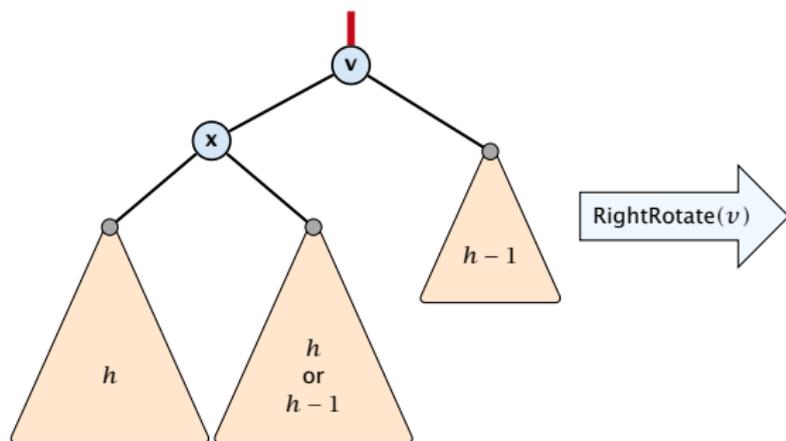
Before the deletion the height of T_v was $h + 2$.

Case 1: $\text{balance}[\text{left}[v]] \in \{0, 1\}$

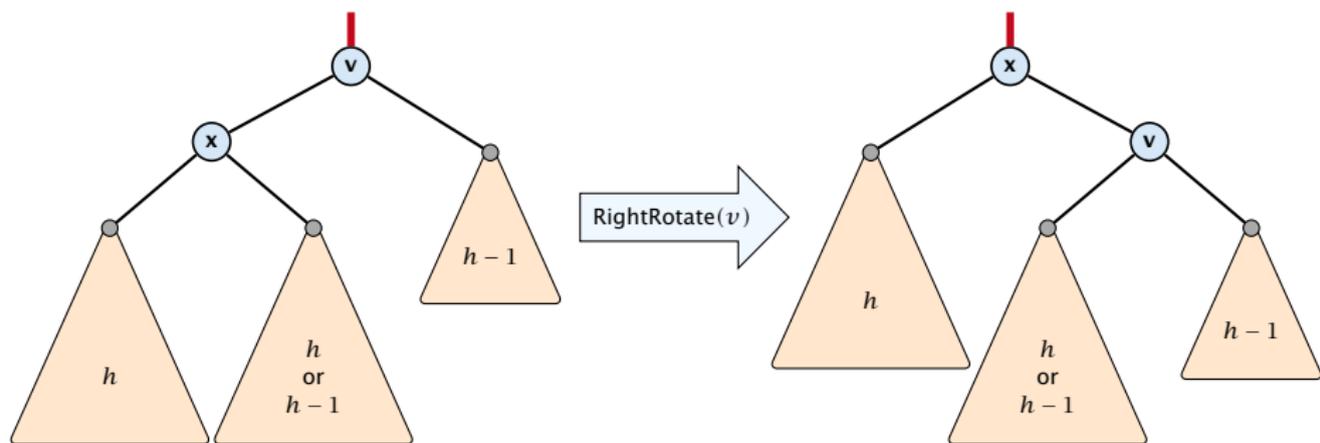
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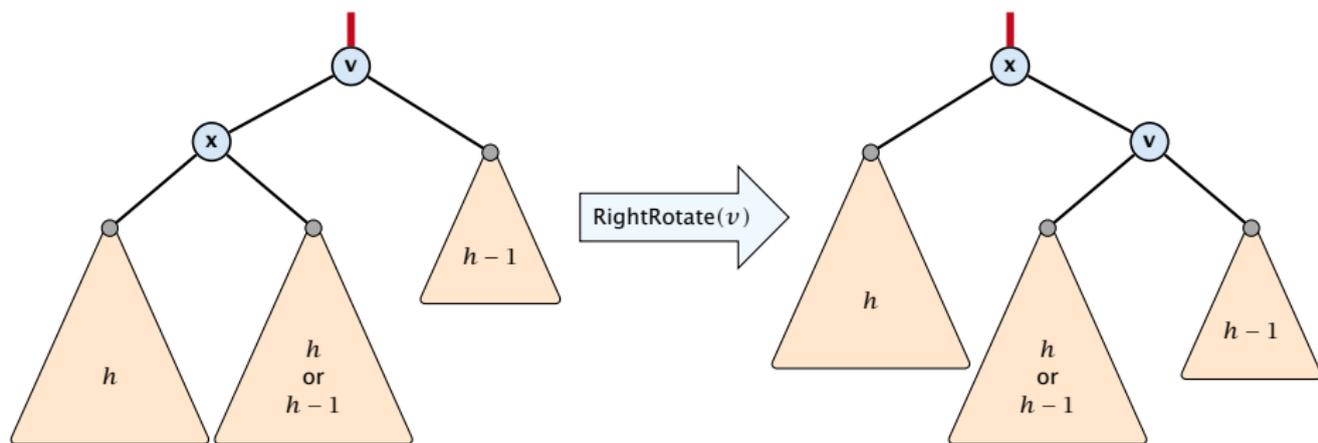
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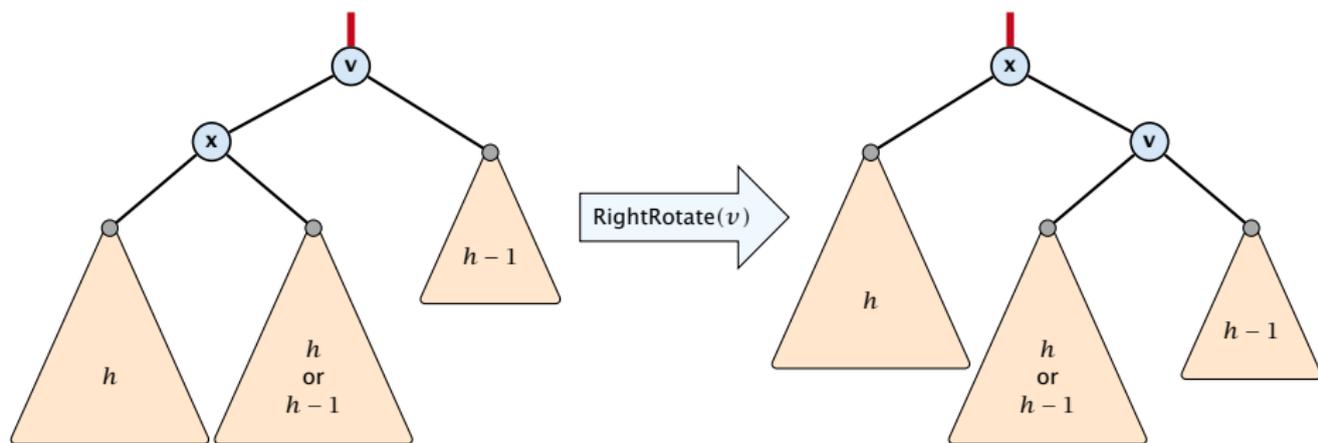


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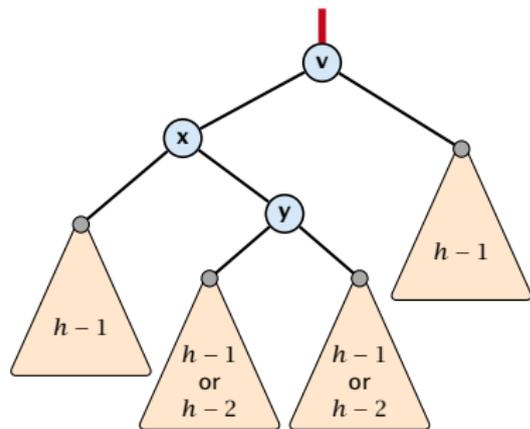


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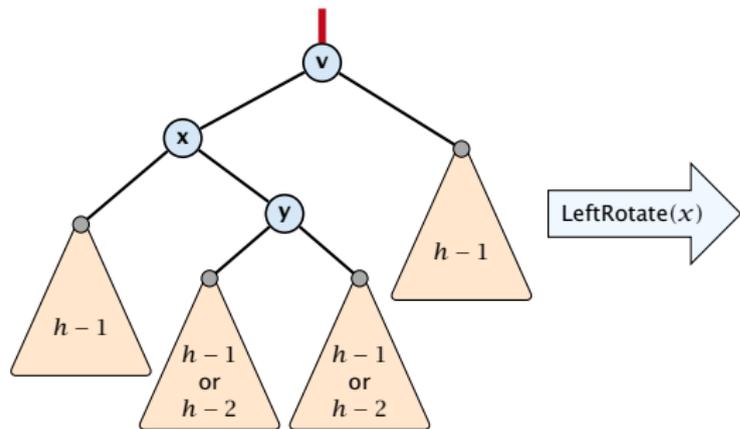
If the middle subtree has height $h - 1$ the whole tree has decreased its height from $h + 2$ to $h + 1$. We do continue the fix-up procedure as the balance at the root is zero.

Case 2: $\text{balance}[\text{left}[v]] = -1$

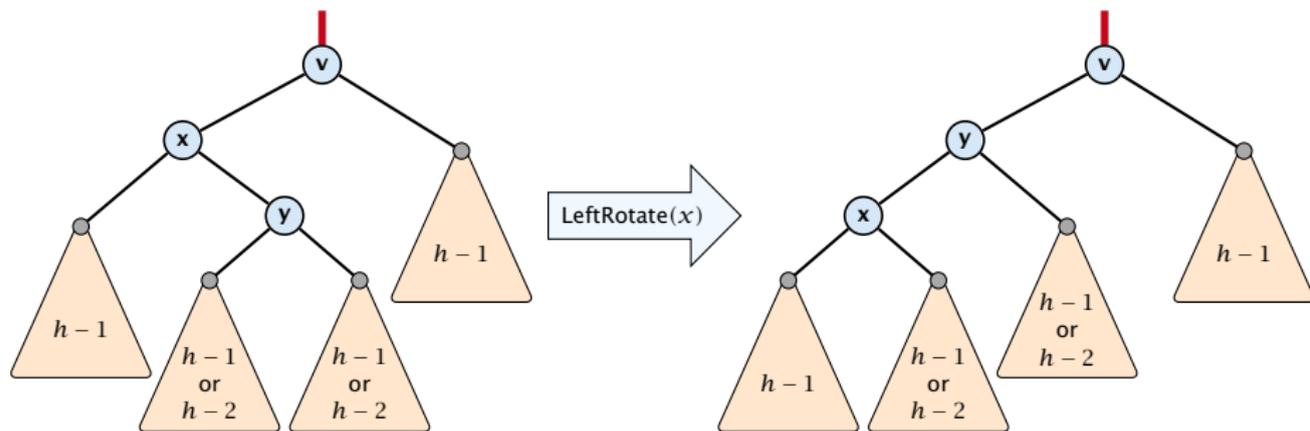
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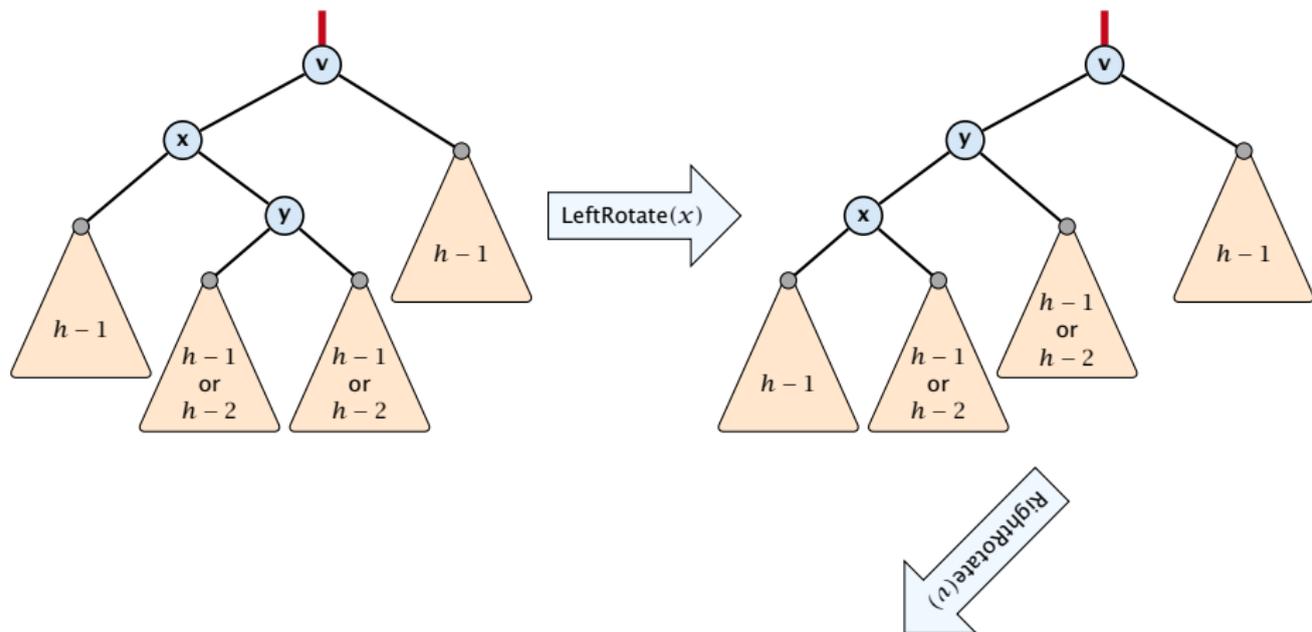
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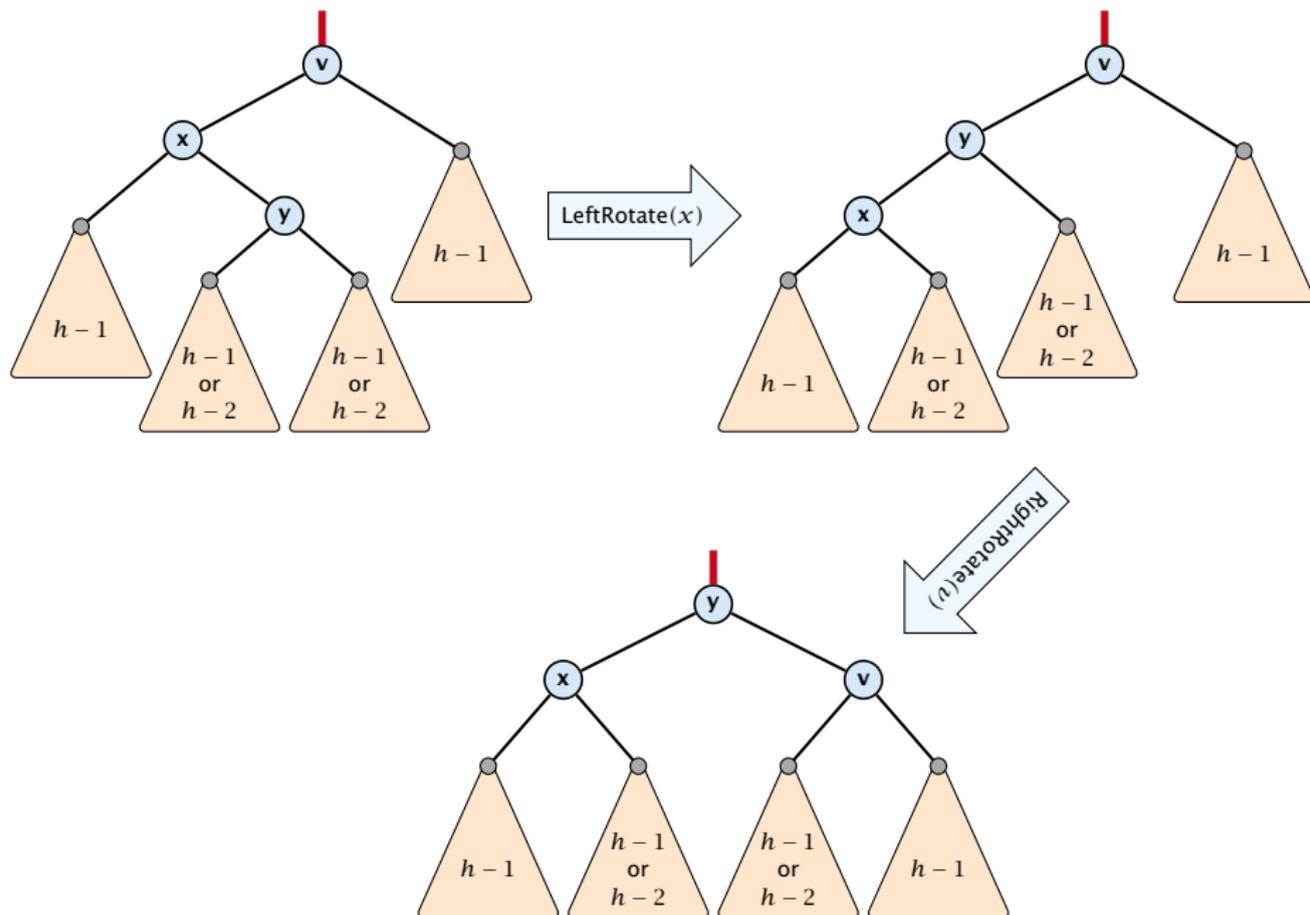
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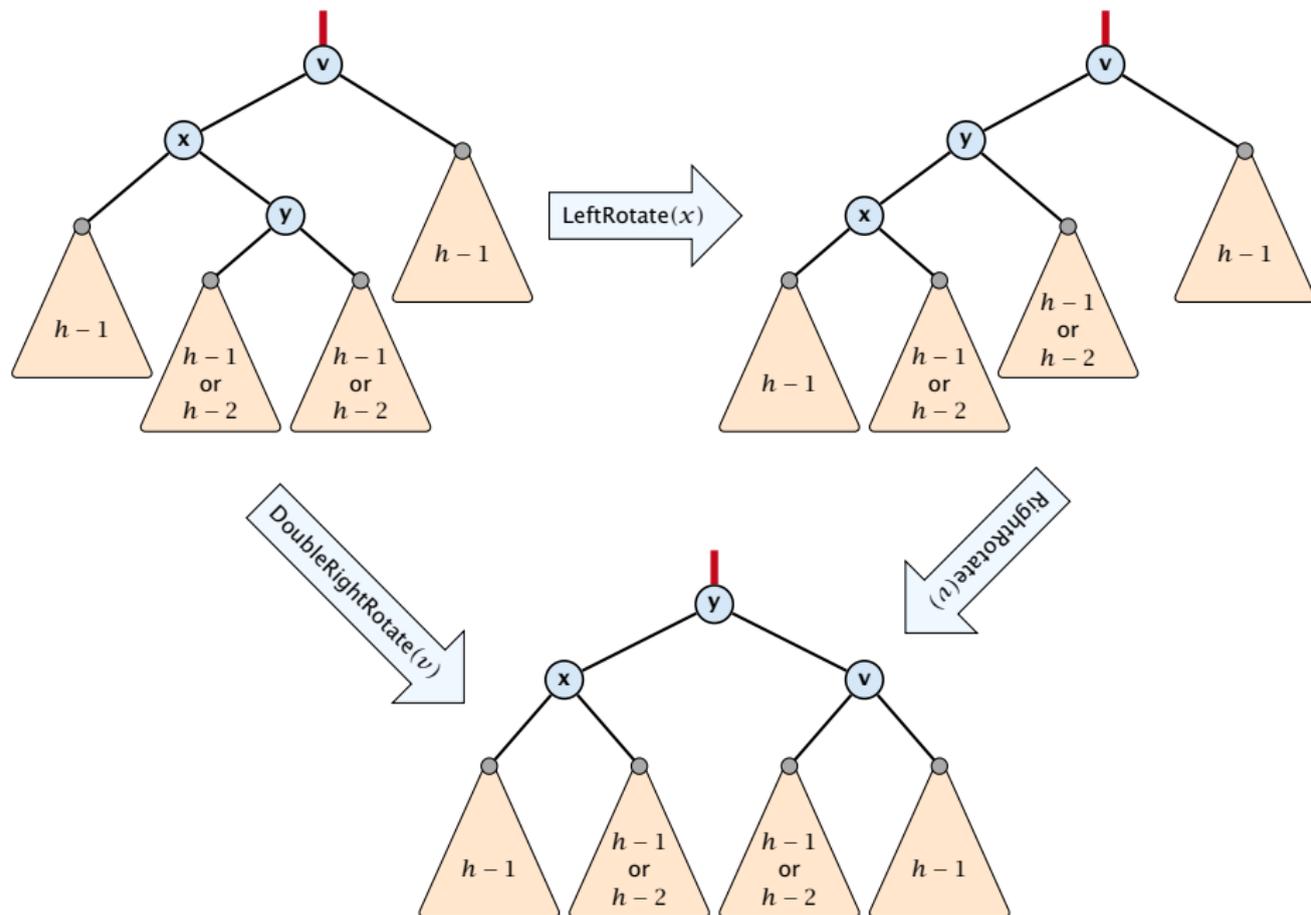
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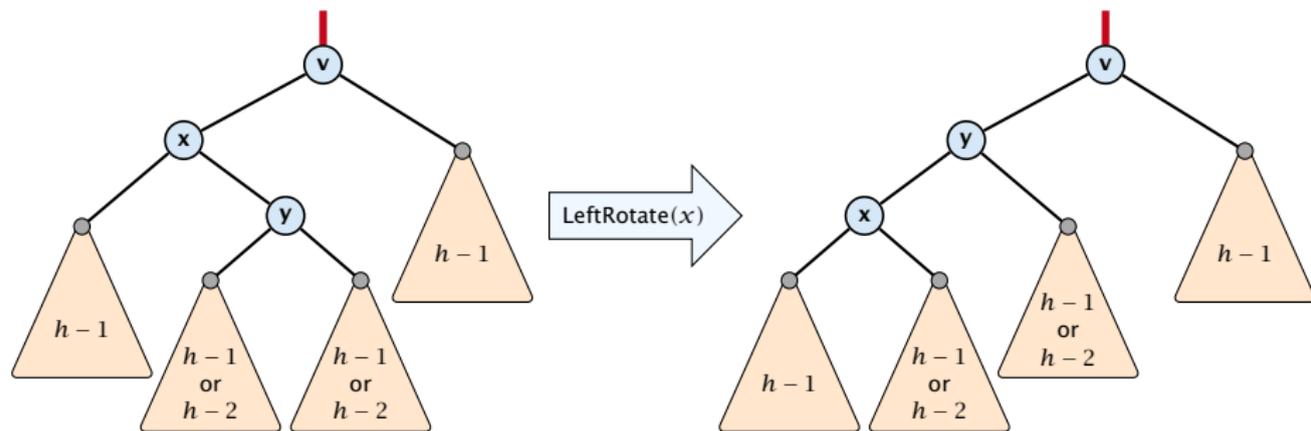
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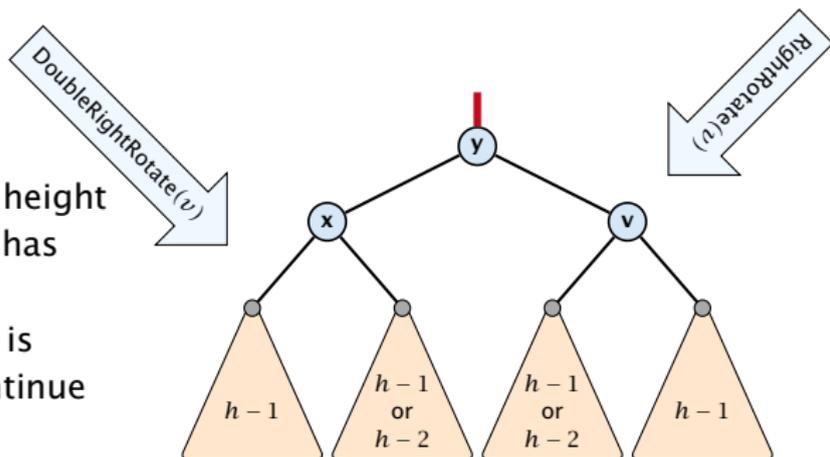
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Sub-tree has height $h + 1$, i.e., it has shrunk. The balance at y is zero. We continue the iteration.



7.4 Augmenting Data Structures

Suppose you want to develop a data structure with:

- ▶ **Insert(x):** insert element x .
- ▶ **Search(k):** search for element with key k .
- ▶ **Delete(x):** delete element referenced by pointer x .
- ▶ **find-by-rank(ℓ):** return the ℓ -th element; return “error” if the data-structure contains less than ℓ elements.

Augment an existing data-structure instead of developing a new one.

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How to augment a data-structure

1. choose an underlying data-structure
2. determine additional information to be stored in the underlying structure
3. verify/show how the additional information can be maintained for the basic modifying operations on the underlying structure.
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Goal: Design a data-structure that supports insert, delete, search, and find-by-rank in time $\mathcal{O}(\log n)$.

1. We choose a red-black tree as the underlying data-structure.
2. We store in each node v the size of the sub-tree rooted at v .
3. We need to be able to update the size-field in each node without asymptotically affecting the running time of insert, delete, and search. We come back to this step later...

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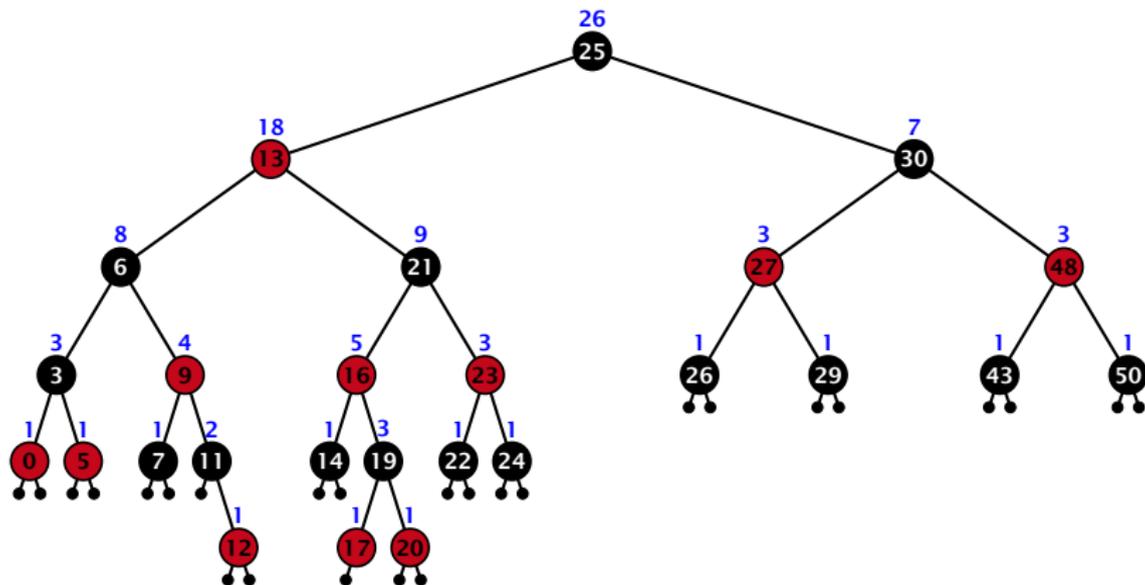
4. How does find-by-rank work?

Find-by-rank(k) := Select(root, k) with

Algorithm 15 Select(x, i)

```
1: if  $x = \text{null}$  then return error
2: if left[ $x$ ]  $\neq$  null then  $r \leftarrow$  left[ $x$ ].size + 1 else  $r \leftarrow 1$ 
3: if  $i = r$  then return  $x$ 
4: if  $i < r$  then
5:     return Select(left[ $x$ ],  $i$ )
6: else
7:     return Select(right[ $x$ ],  $i - r$ )
```

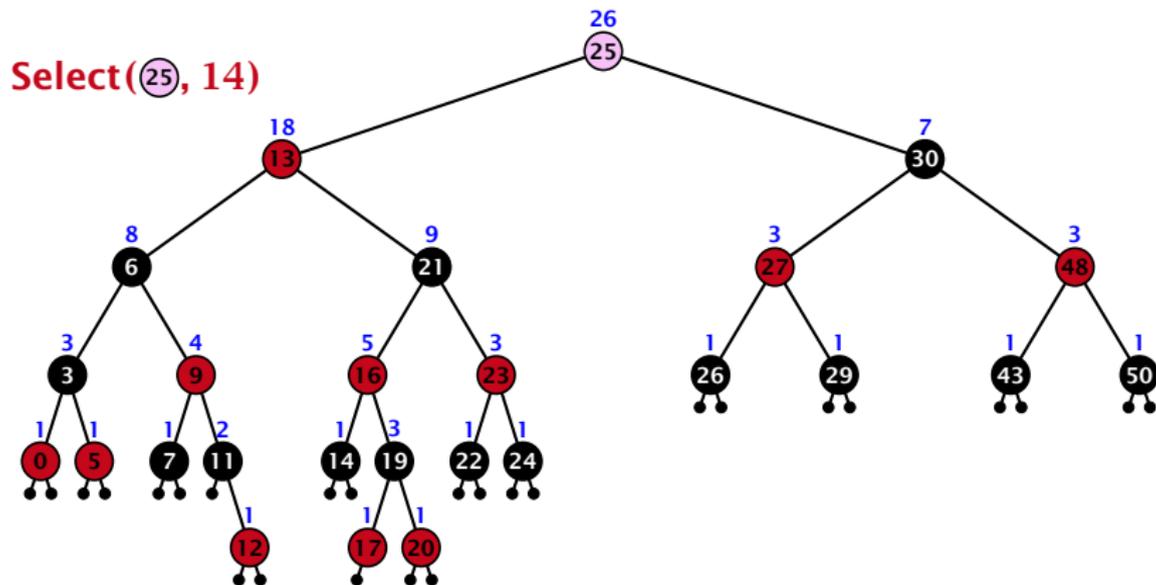
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Find-by-rank:

- ▶ decide whether you have to proceed into the left or right sub-tree
- ▶ adjust the rank that you are searching for if you go right

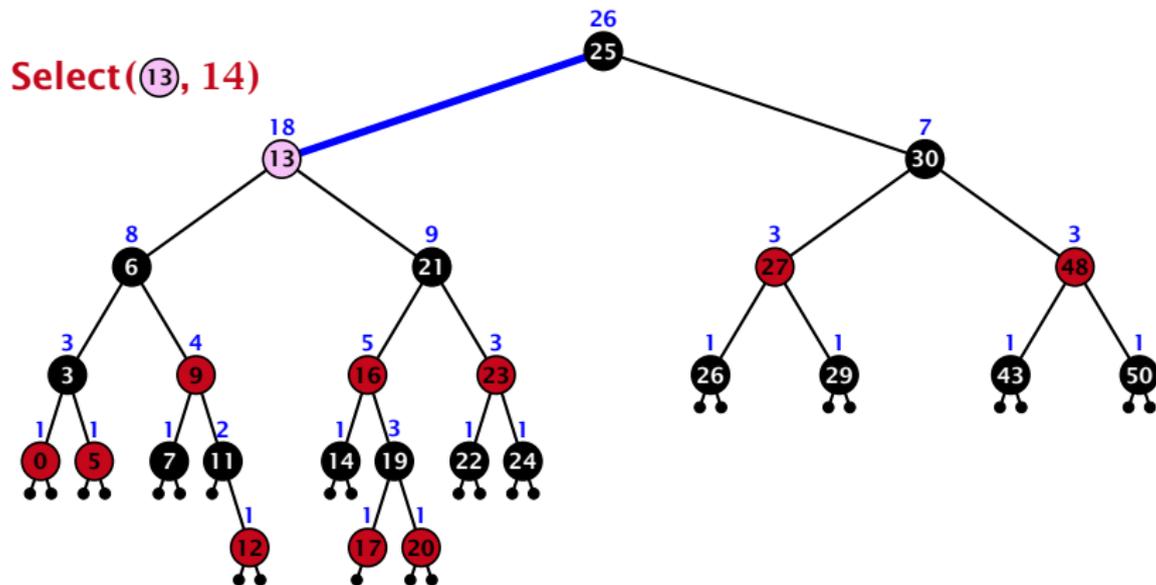
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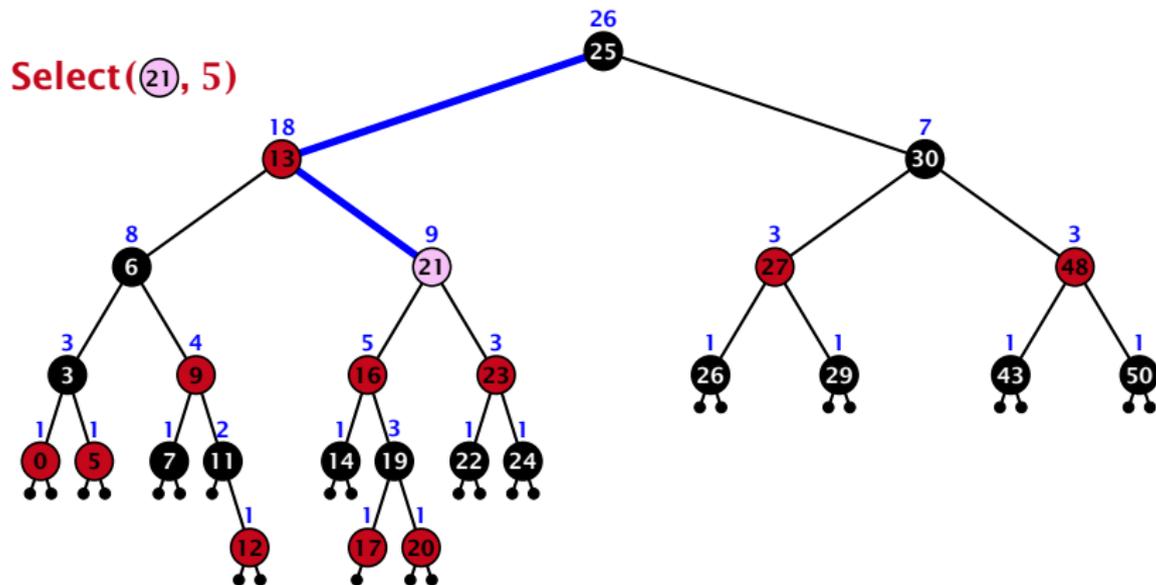
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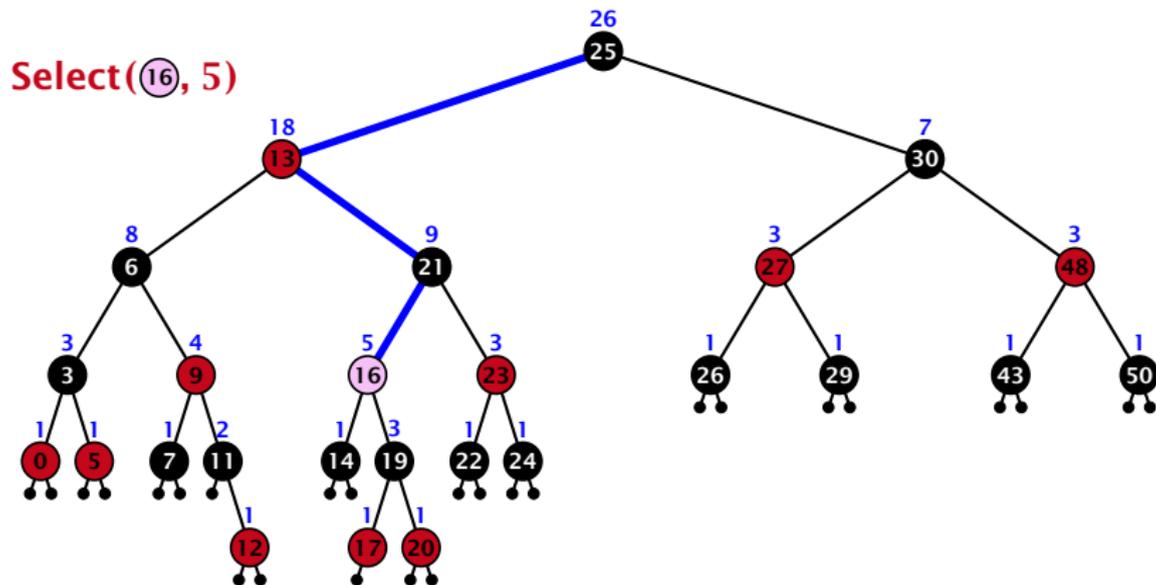
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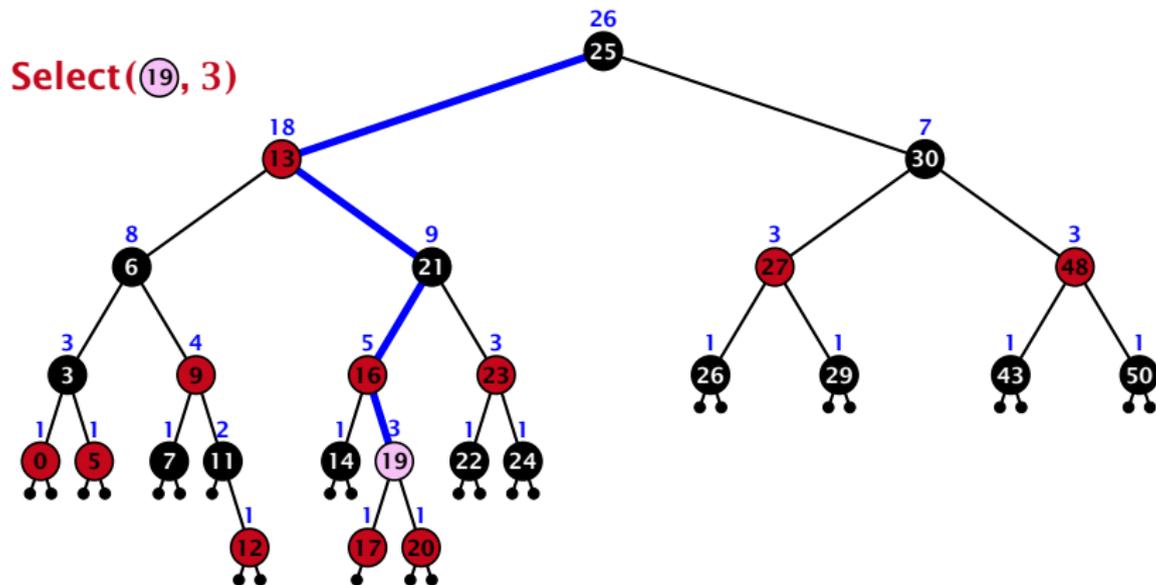
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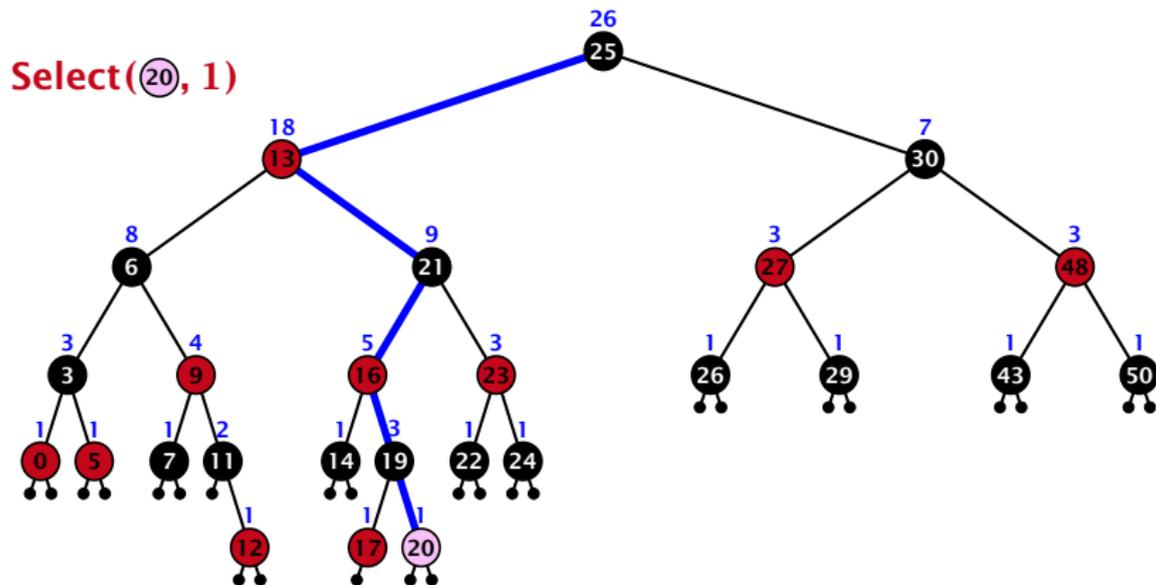
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7.4 Augmenting Data Structures

Goal: Design a data-structure that supports insert, delete, search, and find-by-rank in time $\mathcal{O}(\log n)$.

3. How do we maintain information?

Search(k): Nothing to do.

Insert(x): When going down the search path increase the size field for each visited node. Maintain the size field during rotations.

Delete(x): Directly after splicing out a node traverse the path from the spliced out node upwards, and decrease the size counter on every node on this path. Maintain the size field during rotations.

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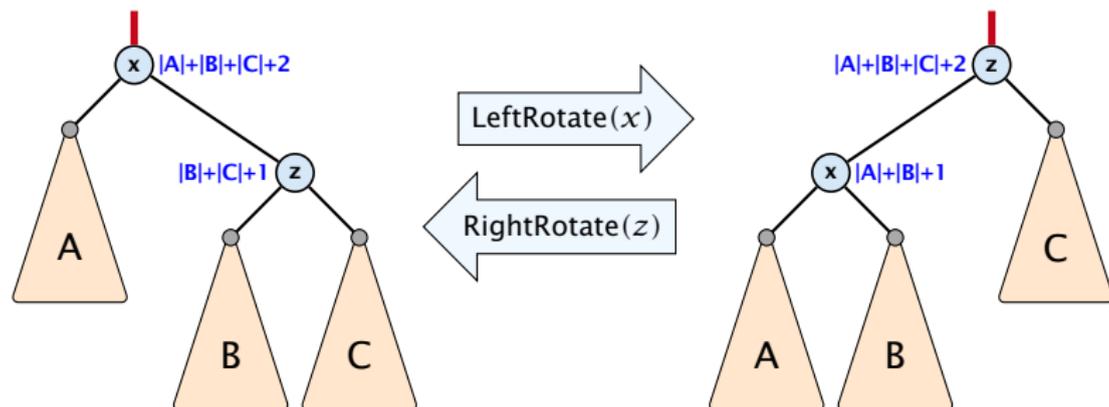
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Rotations

The only operation during the fix-up procedure that alters the tree and requires an update of the size-field:



The nodes x and z are the only nodes changing their size-fields.

The new size-fields can be computed **locally** from the size-fields of the children.

7.5 (a, b) -trees

Definition 7

For $b \geq 2a - 1$ an (a, b) -tree is a search tree with the following properties

1. all leaves have the same distance to the root
2. every internal non-root vertex v has at least a and at most b children
3. the root has degree at least 2 if the tree is non-empty
4. the internal vertices do not contain data, but only keys (external search tree)
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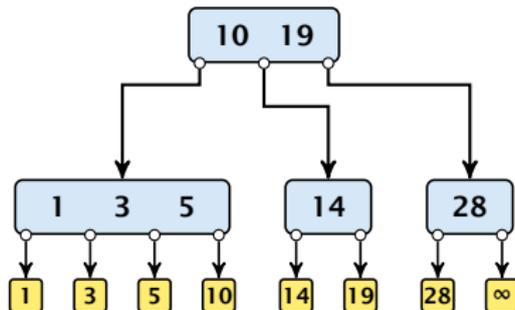
Each internal node v with $d(v)$ children stores $d - 1$ keys k_1, \dots, k_{d-1} . The i -th subtree of v fulfills

$$k_{i-1} < \text{key in } i\text{-th sub-tree} \leq k_i ,$$

where we use $k_0 = -\infty$ and $k_d = \infty$.

7.5 (a, b)-trees

Example 8



7.5 (a, b)-trees

Variants

- ▶ The dummy leaf element may not exist; it only makes implementation more convenient.
- ▶ Variants in which $b = 2a$ are commonly referred to as B -trees.
- ▶ A B -tree usually refers to the variant in which keys and data are stored at internal nodes.
- ▶ A B^+ tree stores the data only at leaf nodes as in our definition. Sometimes the leaf nodes are also connected in a linear list data structure to speed up the computation of successors and predecessors.
- ▶ A B^* tree requires that a node is at least $2/3$ -full as opposed to $1/2$ -full (the requirement of a B -tree).

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Lemma 9

Let T be an (a, b) -tree for $n > 0$ elements (i.e., $n + 1$ leaf nodes) and height h (number of edges from root to a leaf vertex). Then

1. $2a^{h-1} \leq n + 1 \leq b^h$
2. $\log_b(n + 1) \leq h \leq 1 + \log_a\left(\frac{n+1}{2}\right)$

Proof.

Since the root has degree a , each level has at least $2a^{h-1}$ leaf nodes.

Since the root has degree b , each level has at most b^h leaf nodes.

Therefore, $2a^{h-1} \leq n + 1 \leq b^h$.

From $2a^{h-1} \leq n + 1$, we get $h \leq 1 + \log_a\left(\frac{n+1}{2}\right)$.

From $n + 1 \leq b^h$, we get $\log_b(n + 1) \leq h$.



Lemma 9

Let T be an (a, b) -tree for $n > 0$ elements (i.e., $n + 1$ leaf nodes) and height h (number of edges from root to a leaf vertex). Then

1. $2a^{h-1} \leq n + 1 \leq b^h$
2. $\log_b(n + 1) \leq h \leq 1 + \log_a\left(\frac{n+1}{2}\right)$

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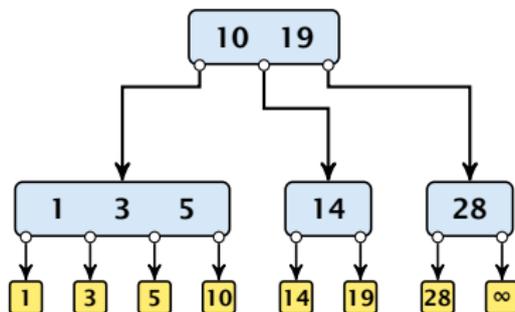
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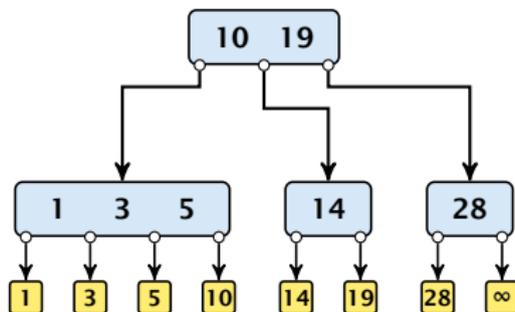


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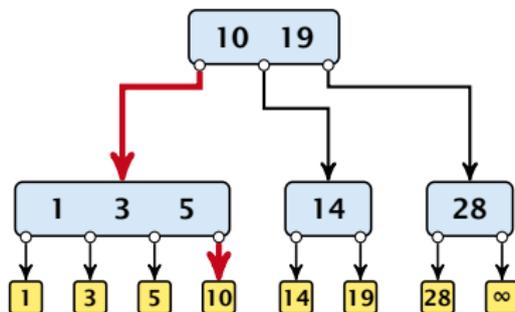
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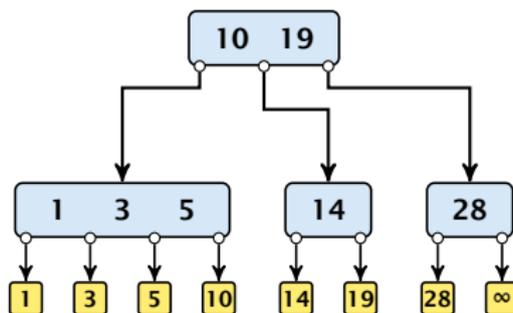
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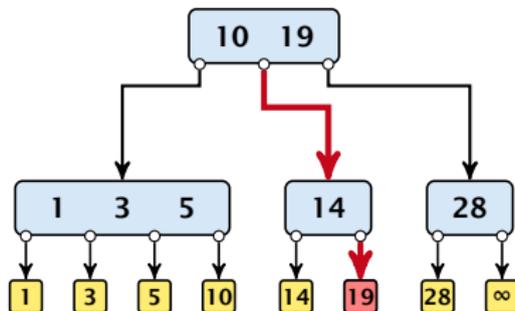
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Search(19)

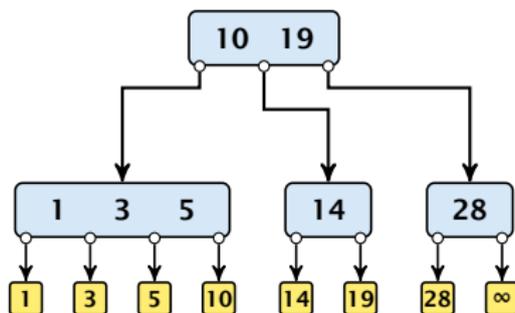


Search

Search(19)

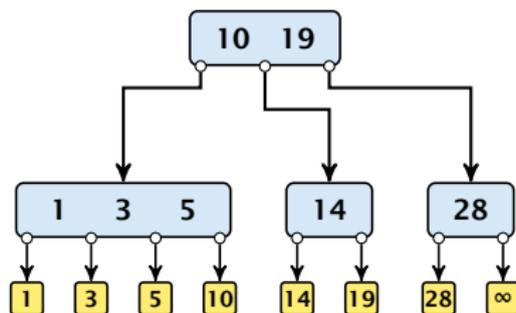


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Time: $\mathcal{O}(b \cdot h) = \mathcal{O}(b \cdot \log n)$, if the individual nodes are organized as linear lists.

Insert element x :

- ▶ Follow the path as if searching for $\text{key}[x]$.
- ▶ If this search ends in leaf ℓ , insert x before this leaf.
- ▶ For this add $\text{key}[x]$ to the key-list of the last internal node v on the path.
- ▶ If after the insert v contains b nodes, do $\text{Rebalance}(v)$.

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Rebalance(v):

- ▶ Let k_i , $i = 1, \dots, b$ denote the keys stored in v .
- ▶ Let $j := \lfloor \frac{b+1}{2} \rfloor$ be the middle element.
- ▶ Create two nodes v_1 , and v_2 . v_1 gets all keys k_1, \dots, k_{j-1} and v_2 gets keys k_{j+1}, \dots, k_b .
- ▶ Both nodes get at least $\lfloor \frac{b-1}{2} \rfloor$ keys, and have therefore degree at least $\lfloor \frac{b-1}{2} \rfloor + 1 \geq a$ since $b \geq 2a - 1$.
- ▶ They get at most $\lceil \frac{b-1}{2} \rceil$ keys, and have therefore degree at most $\lceil \frac{b-1}{2} \rceil + 1 \leq b$ (since $b \geq 2$).
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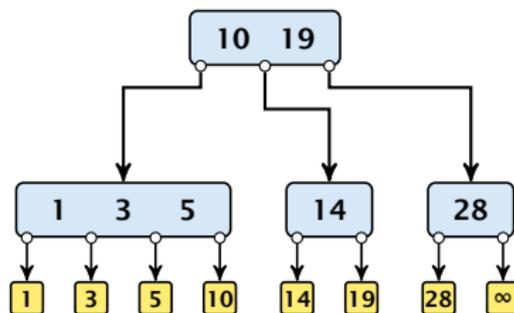
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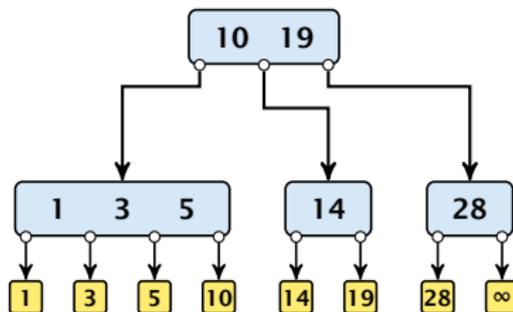
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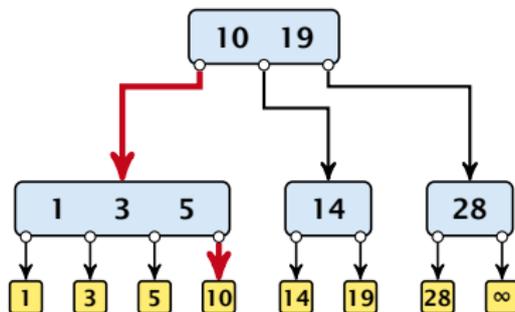
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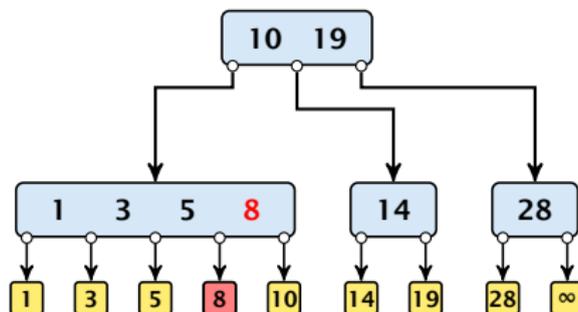
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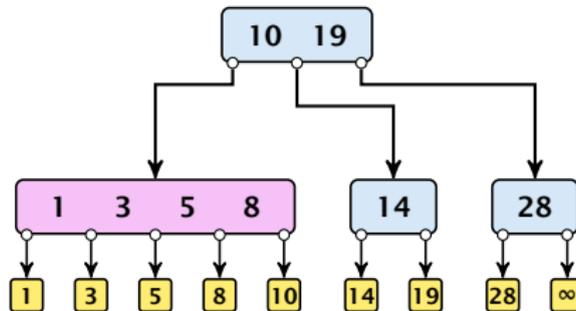
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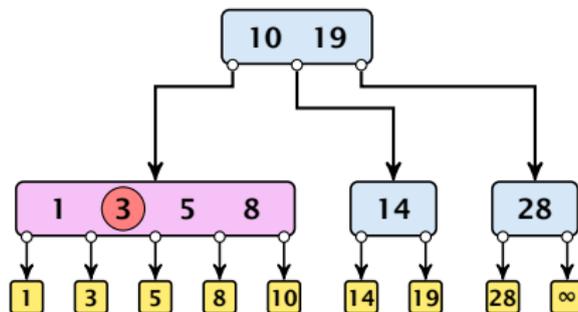
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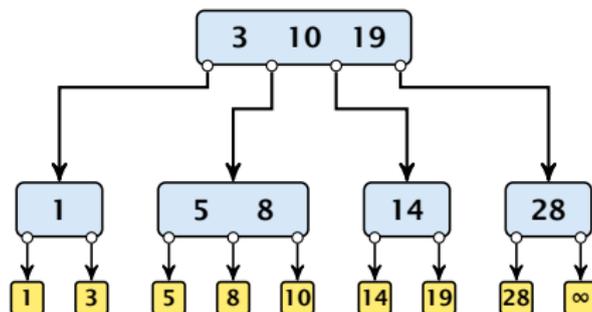


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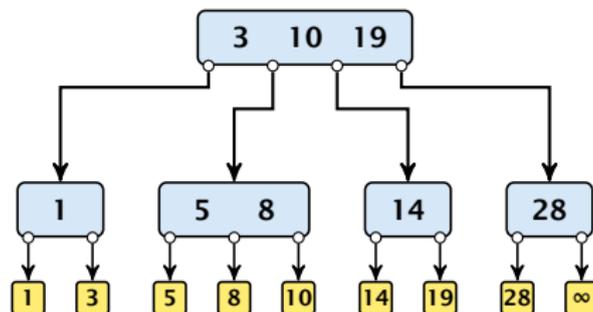


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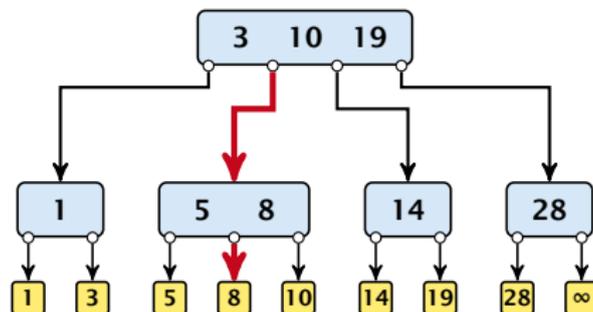
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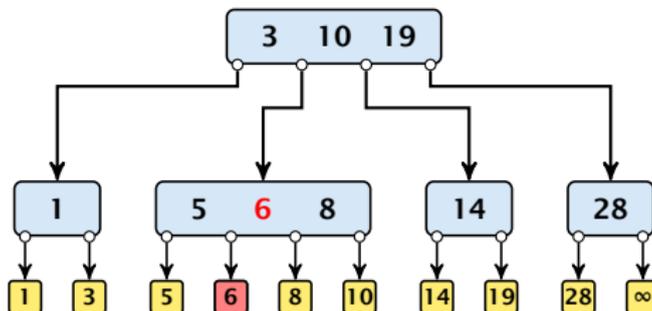
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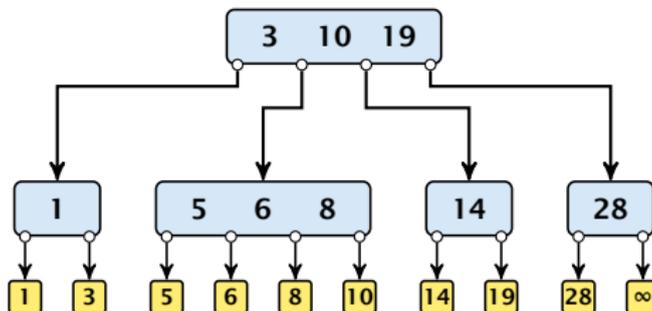
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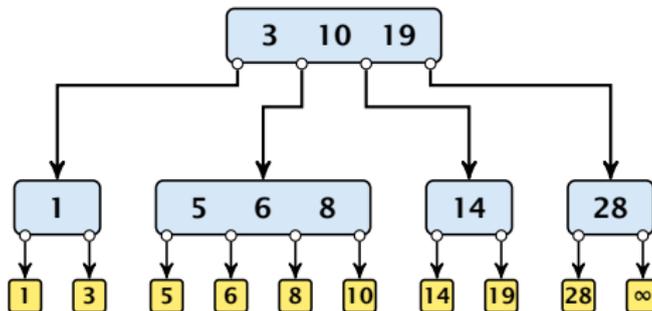
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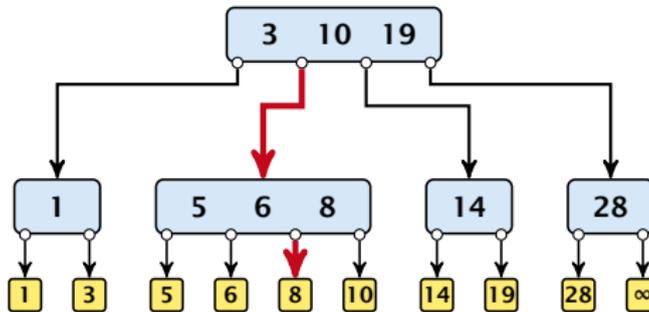
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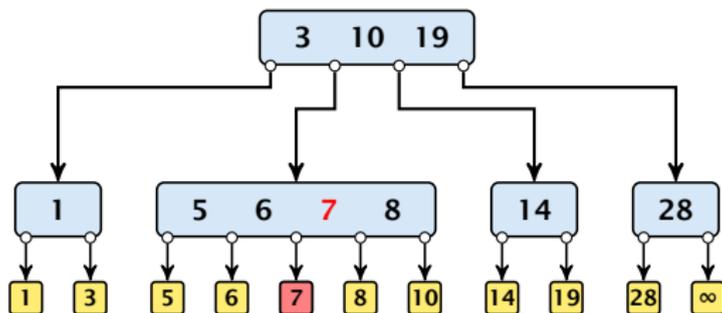
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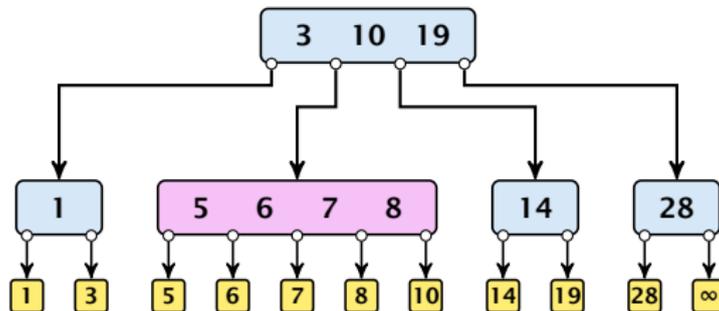
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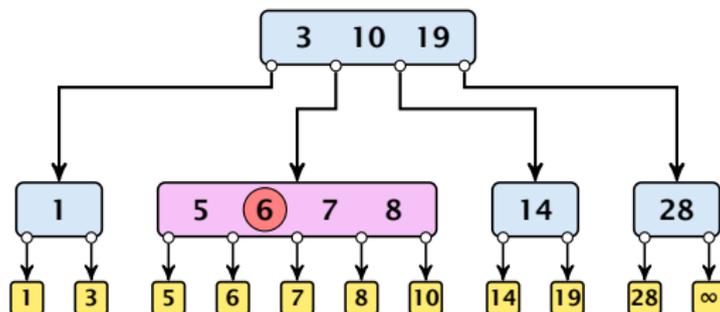
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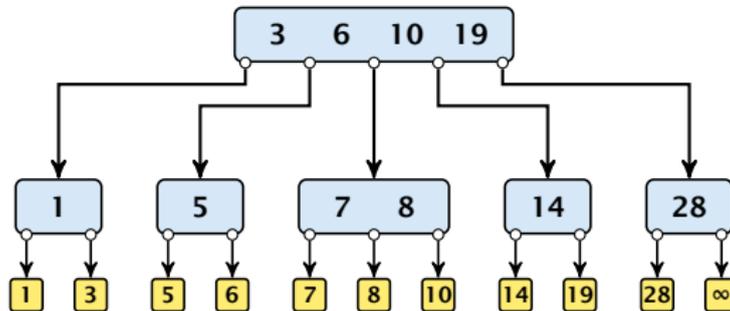
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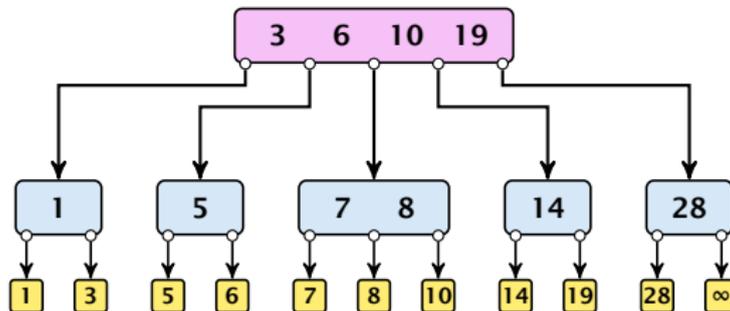
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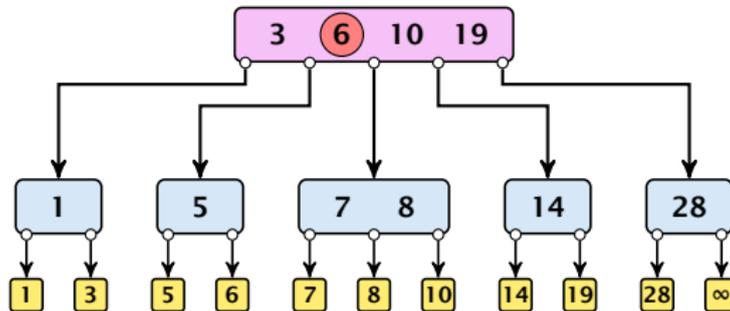
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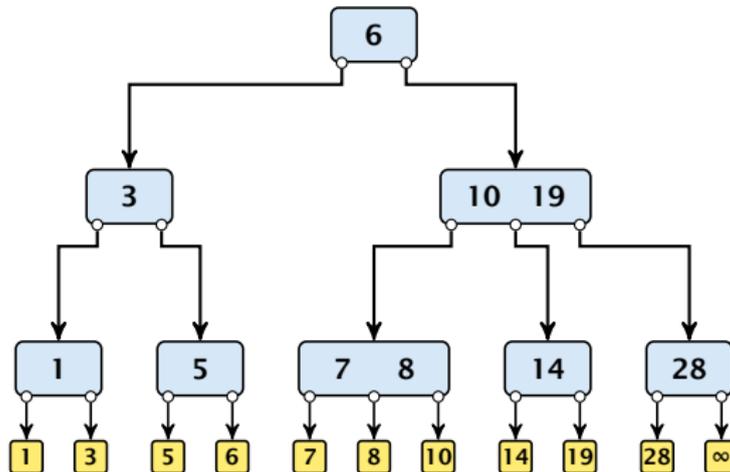
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Delete element x (pointer to leaf vertex):

- ▶ Let v denote the parent of x . If $\text{key}[x]$ is contained in v , remove the key from v , and delete the leaf vertex.
- ▶ Otherwise delete the key of the predecessor of x from v ; delete the leaf vertex; and replace the occurrence of $\text{key}[x]$ in internal nodes by the predecessor key. (Note that it appears in exactly one internal vertex).
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- ▶ If not: merge v with one of its neighbours.
- ▶ The merged node contains at most $(a - 2) + (a - 1) + 1$ keys, and has therefore at most $2a - 1 \leq b$ successors.
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- ▶ During this process the root may become empty. In this case the root is deleted and the height of the tree decreases.

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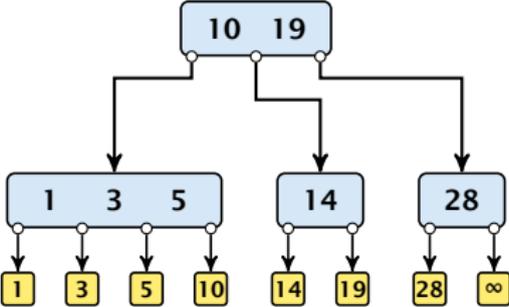
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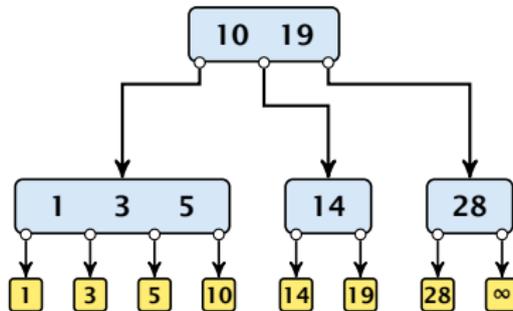
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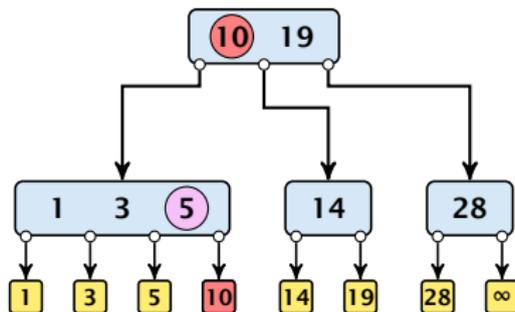
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Delete(10)



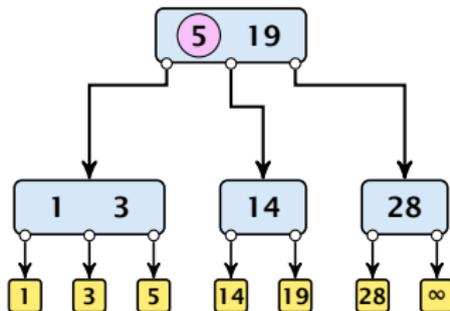
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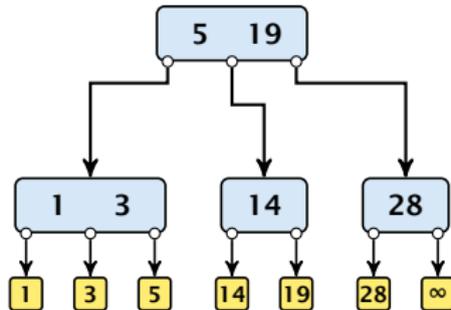


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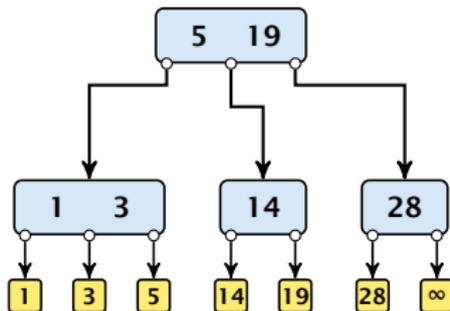


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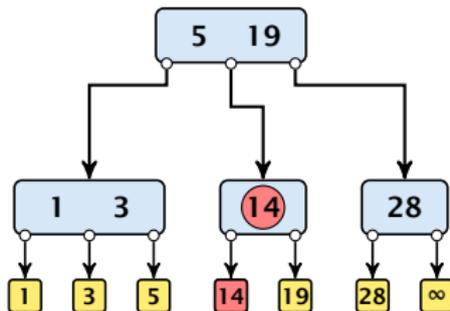
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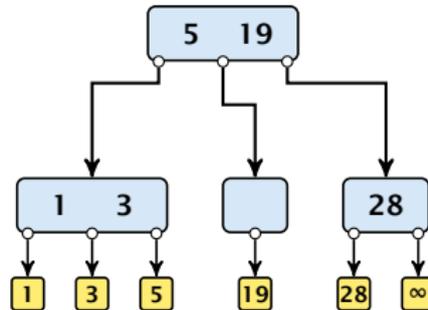
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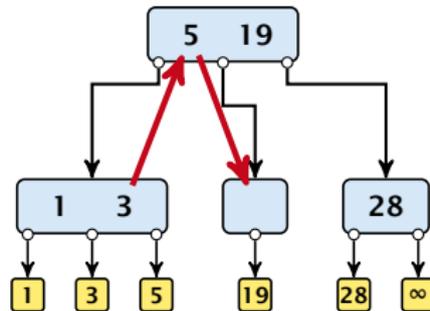
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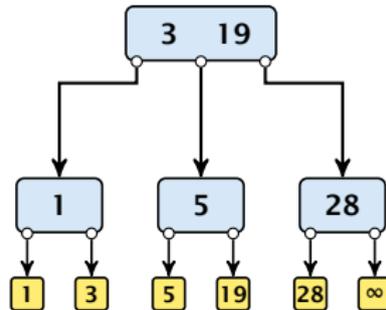
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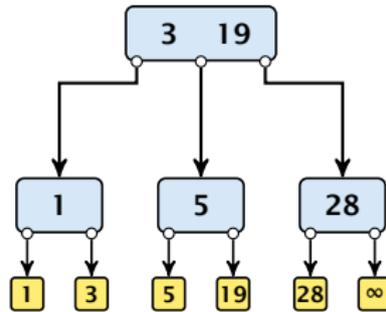


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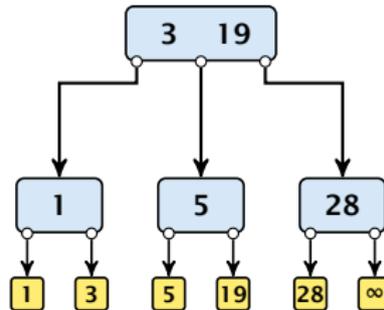


Delete



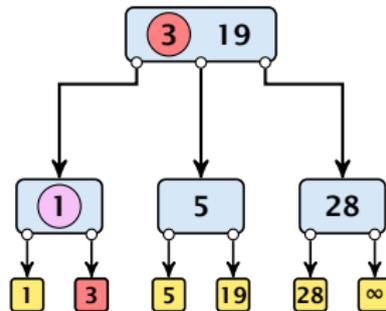
Delete

Delete(3)



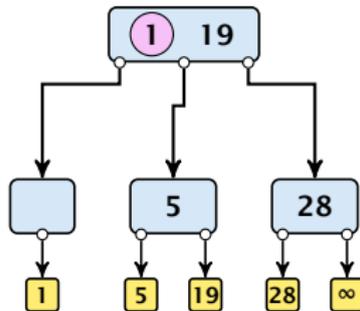
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Delete(3)



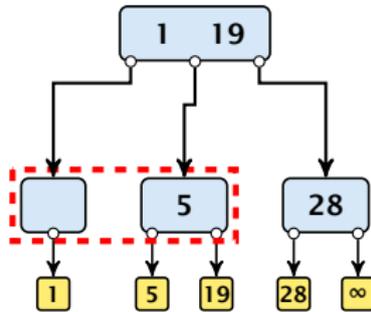
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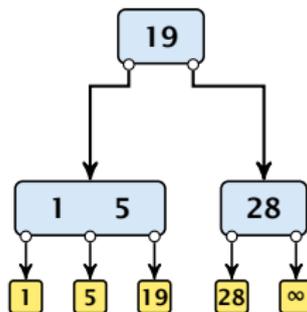
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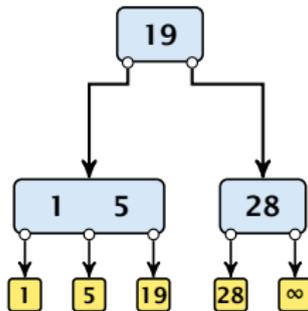


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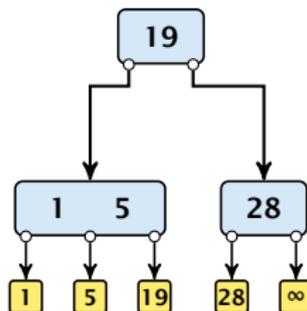


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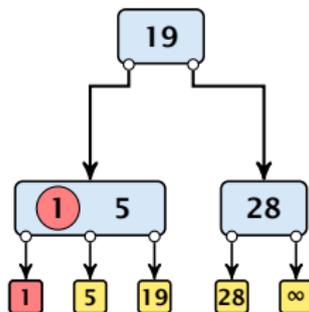
Delete

Delete(1)



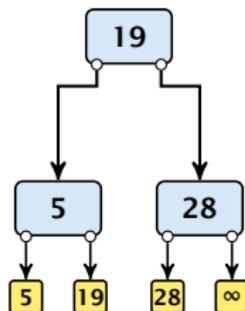
Delete

Delete(1)

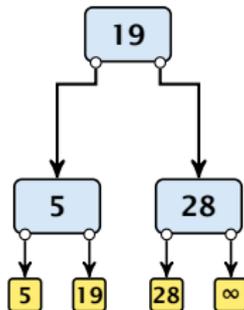


Delete

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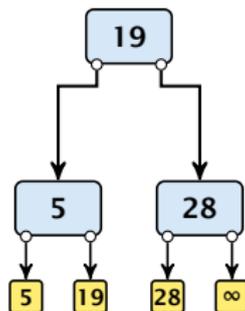


Delete



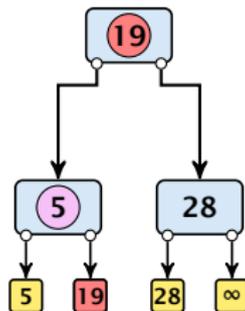
Delete

Delete(19)



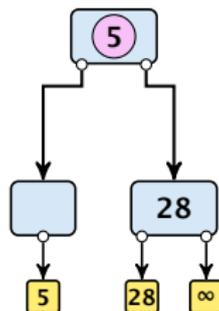
Delete

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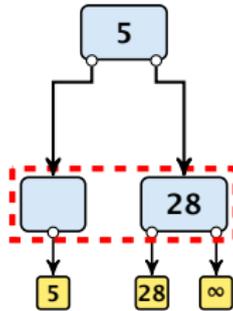
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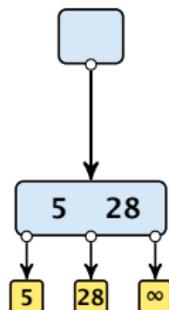
Delete

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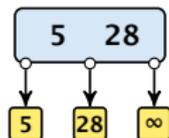
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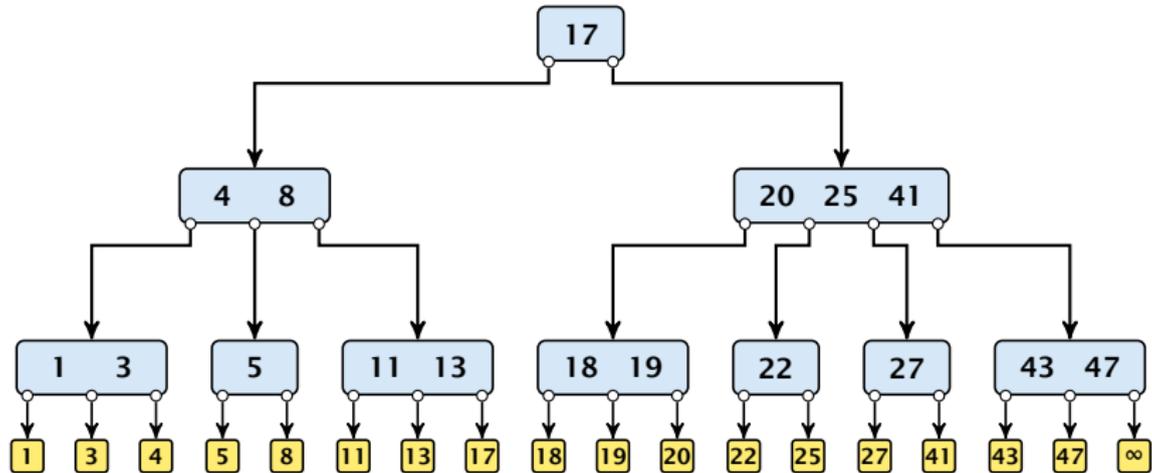
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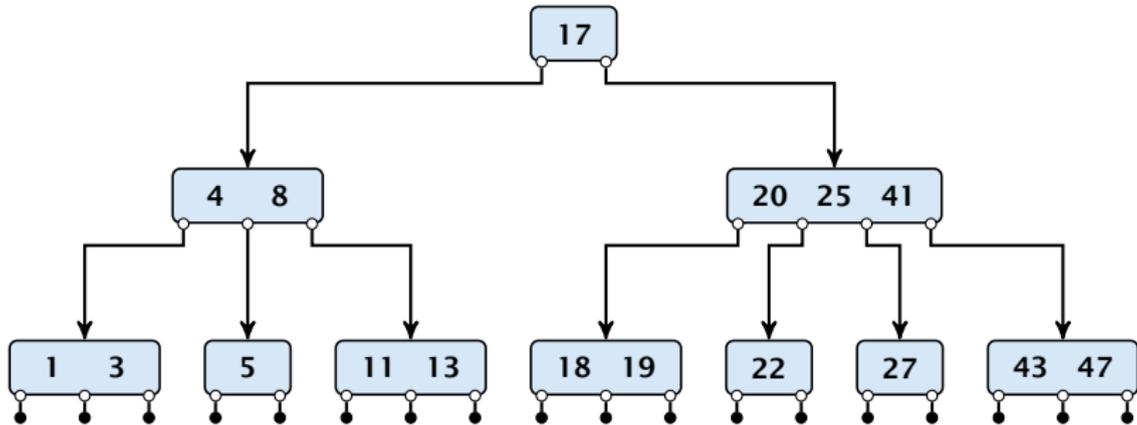
(2, 4)-trees and red black trees

There is a close relation between red-black trees and (2, 4)-trees:



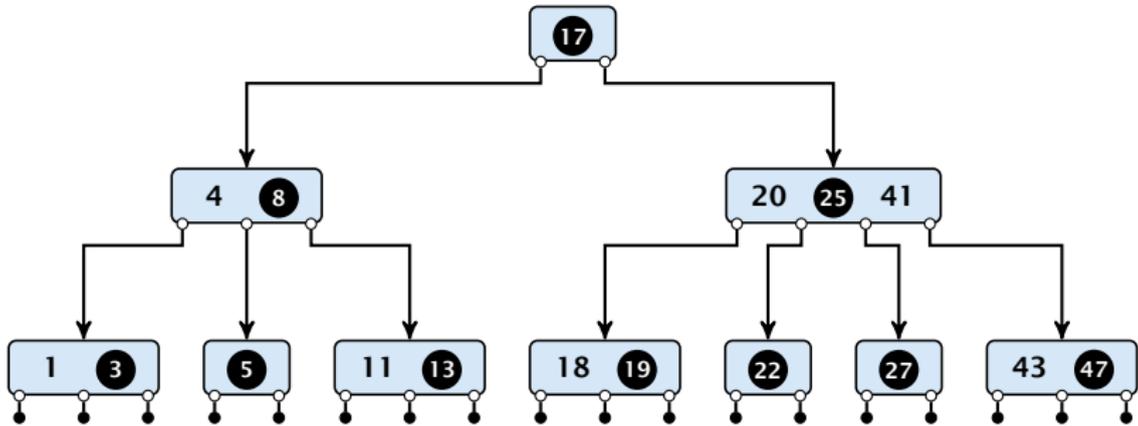
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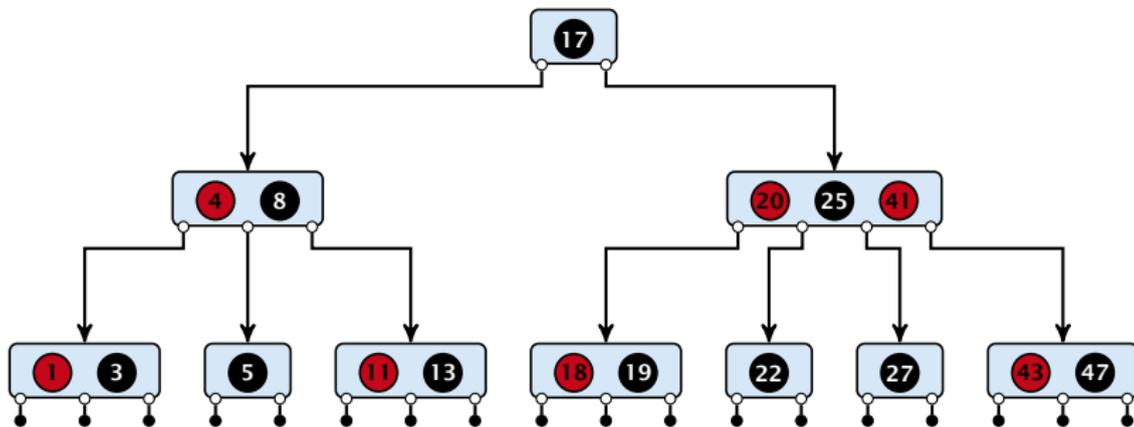
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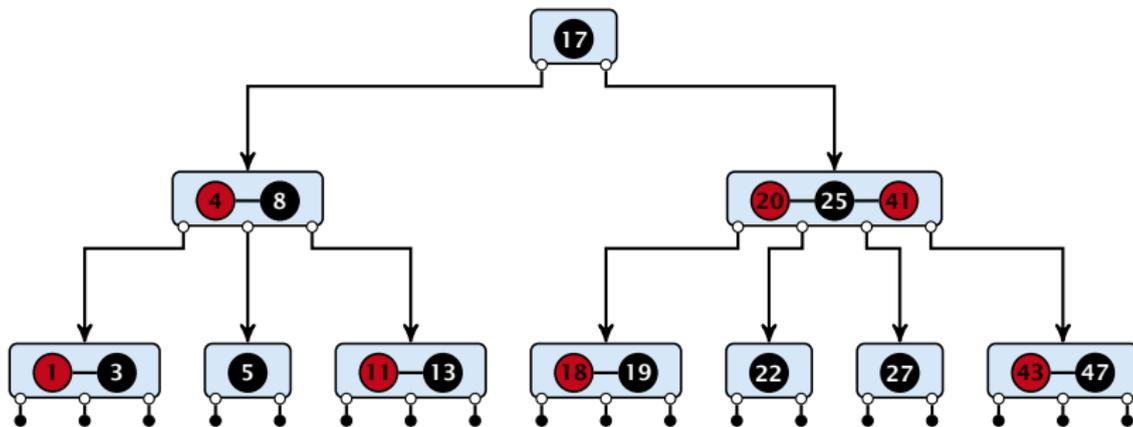
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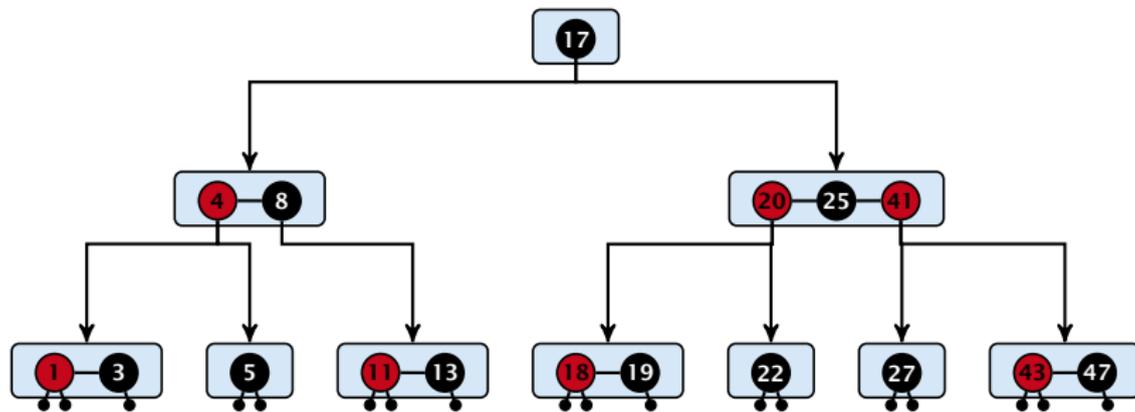
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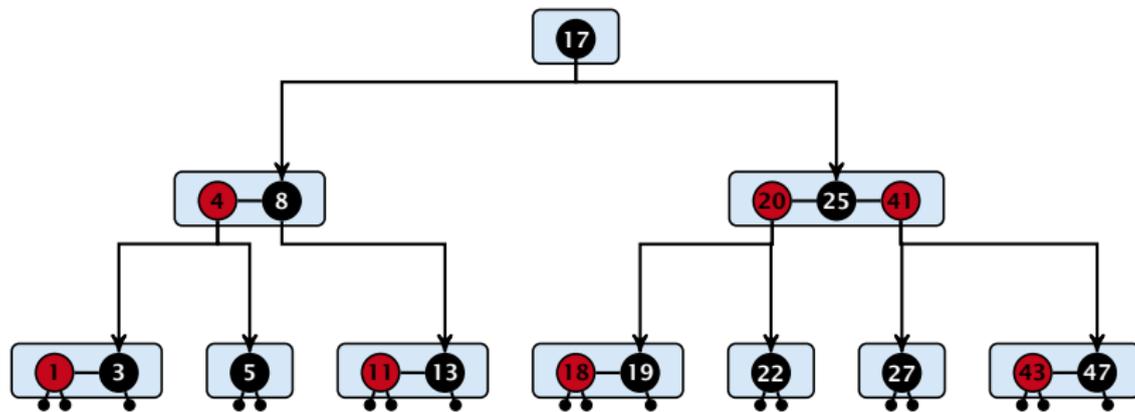
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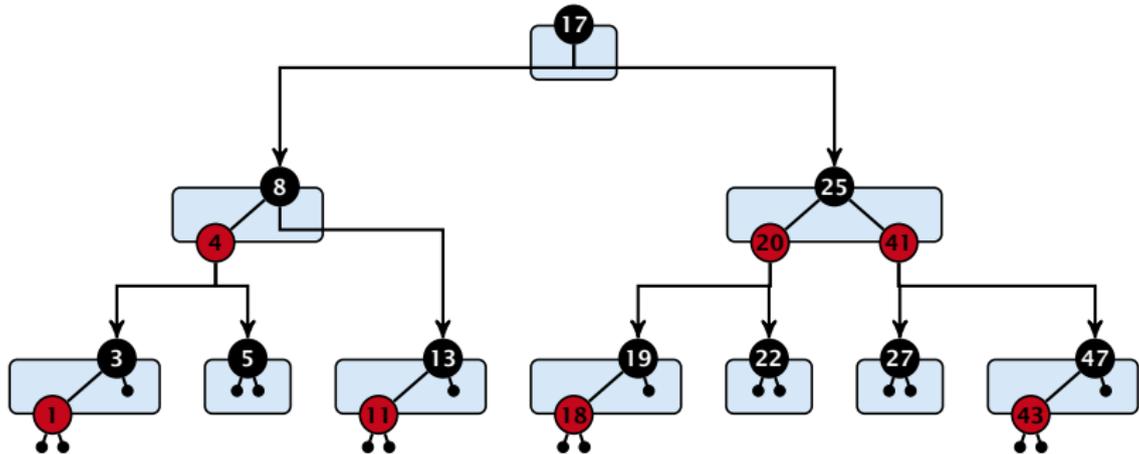
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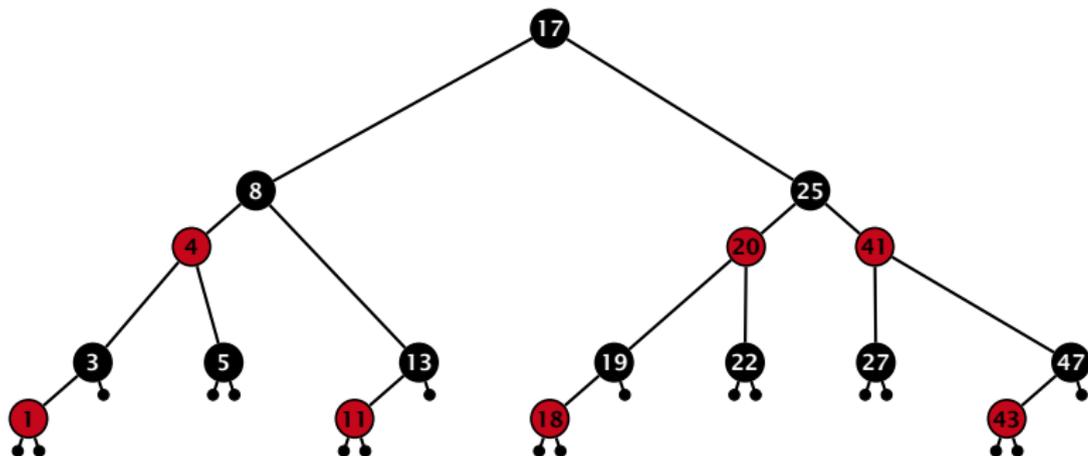
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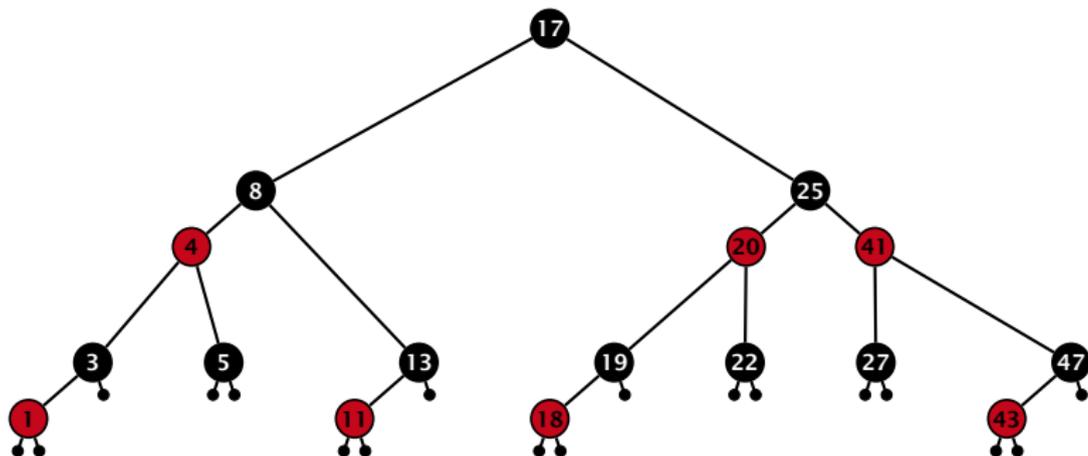
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Note that this correspondence is not unique. In particular, there are different red-black trees that correspond to the same (2, 4)-tree.

7.6 Skip Lists

Why do we not use a list for implementing the ADT Dynamic Set?

- ▶ time for search $\Theta(n)$
- ▶ time for insert $\Theta(n)$ (dominated by searching the item)
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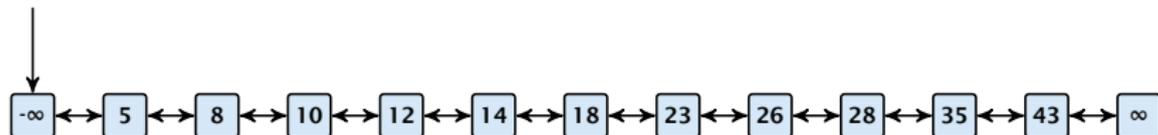
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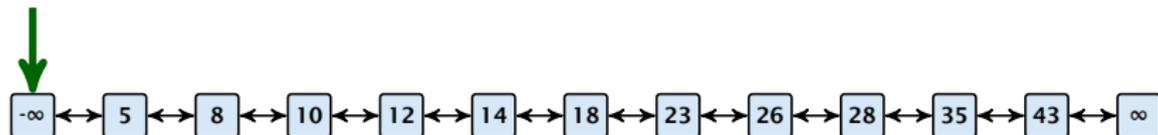
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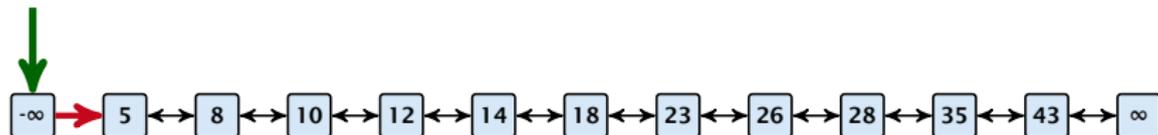
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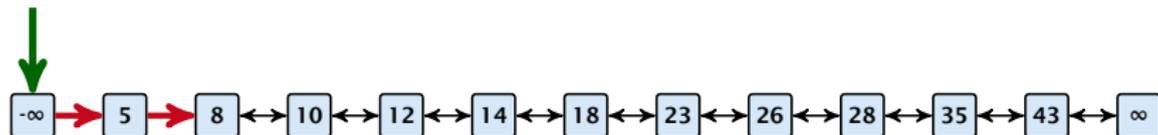
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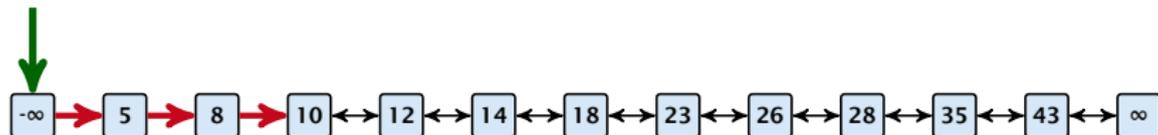
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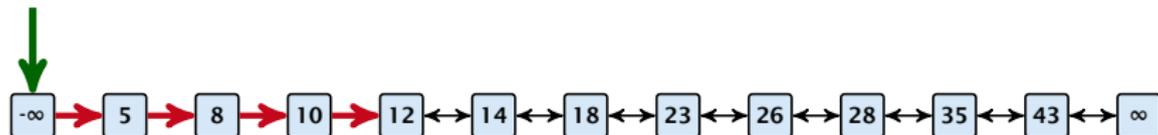
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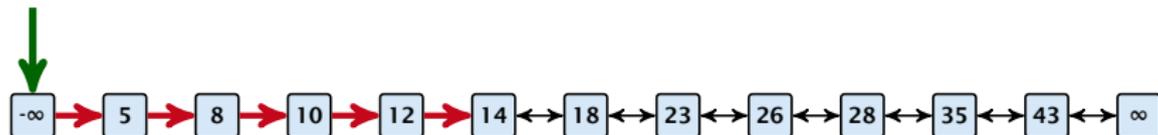
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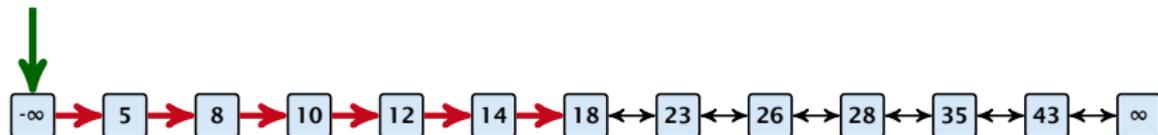
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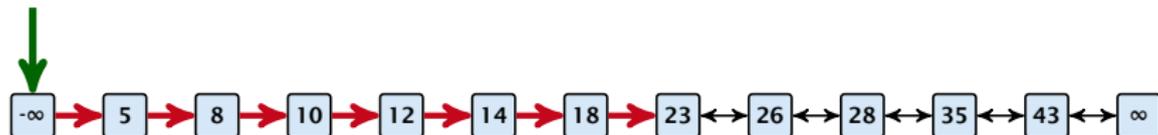
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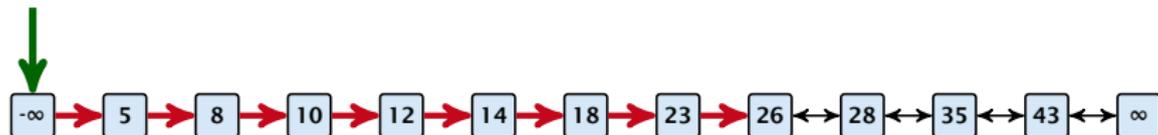
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How can we improve the search-operation?

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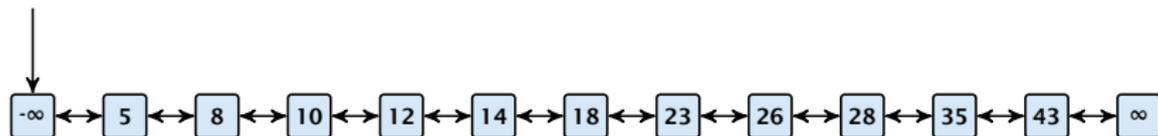
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Add an express lane:

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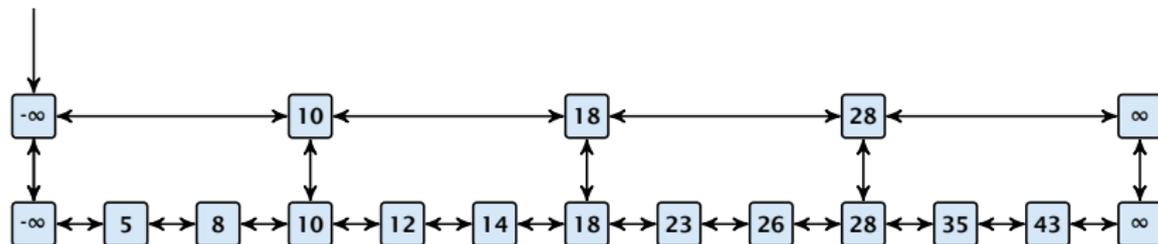
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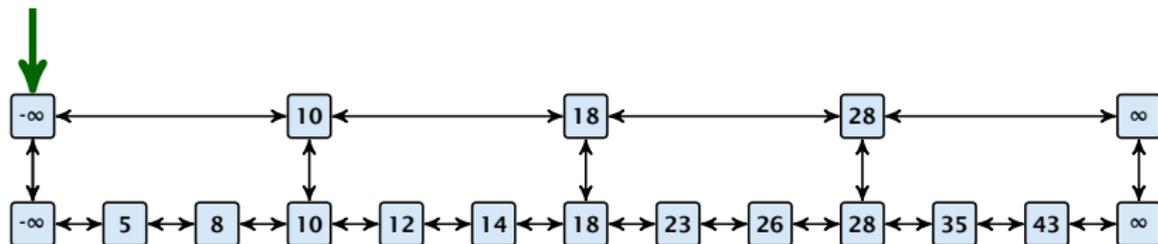
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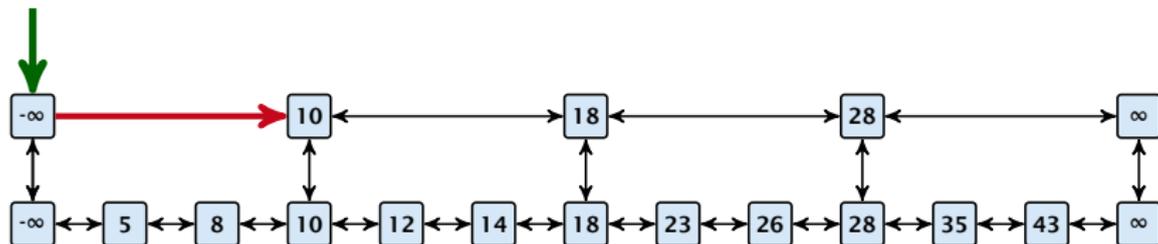
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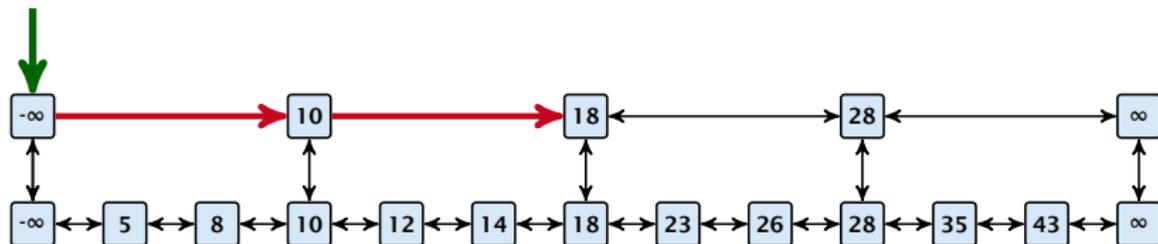
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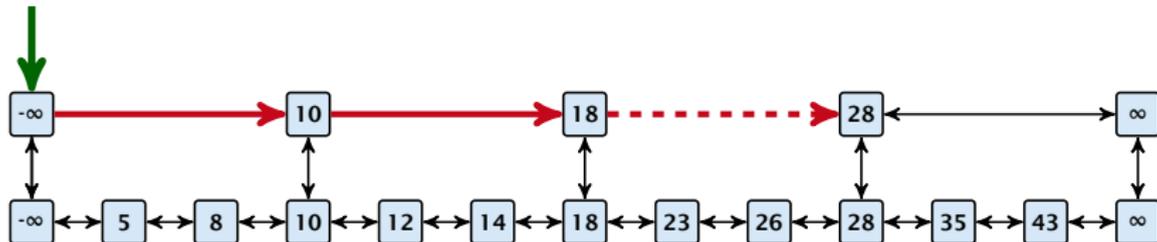
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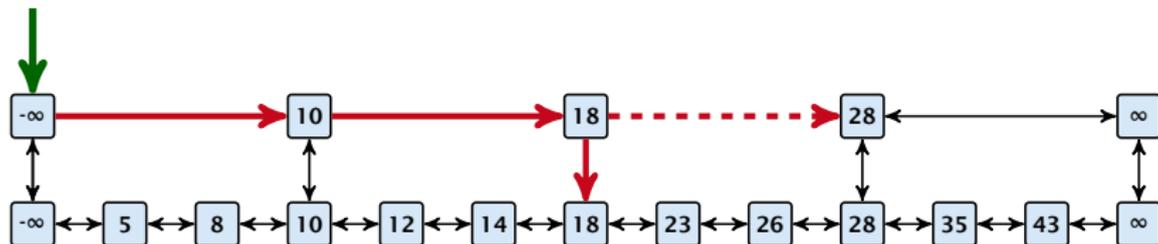
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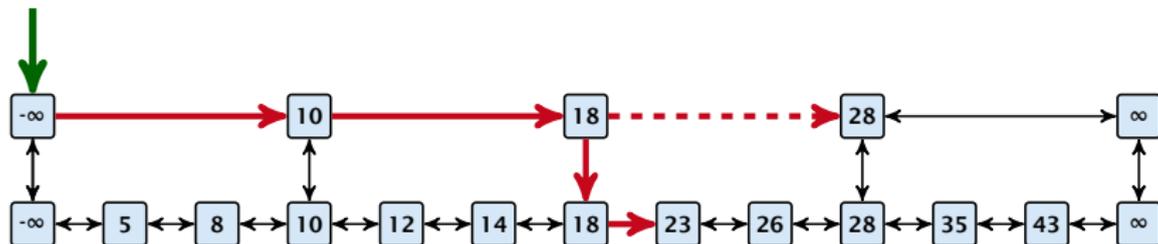
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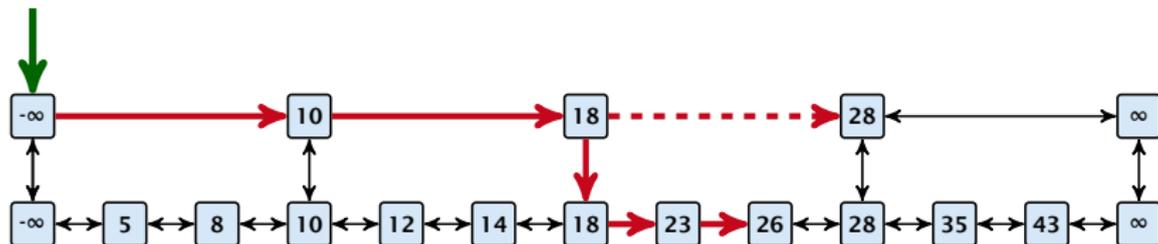
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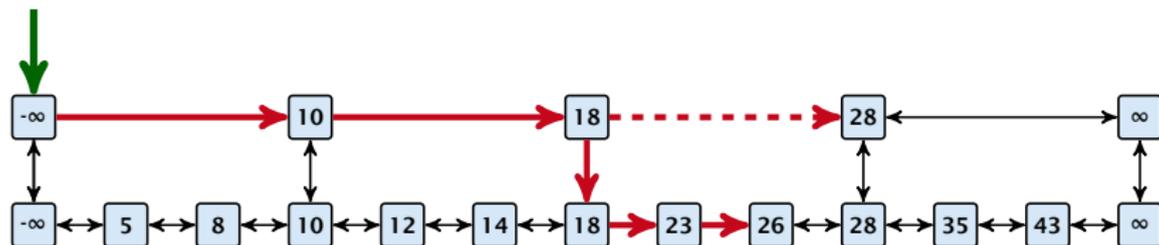
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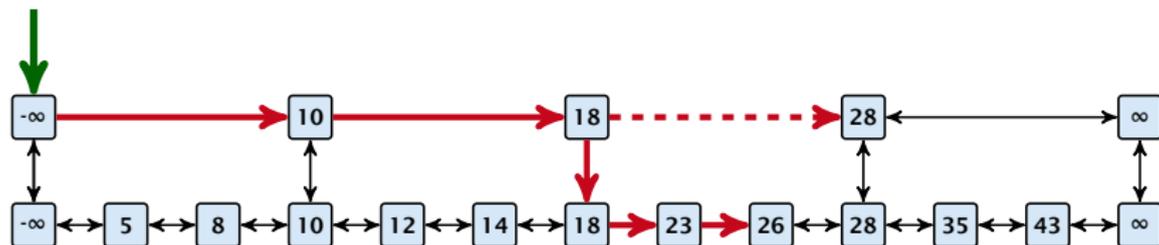


Let $|L_1|$ denote the number of elements in the “express lane”, and $|L_0| = n$ the number of all elements (ignoring dummy elements).

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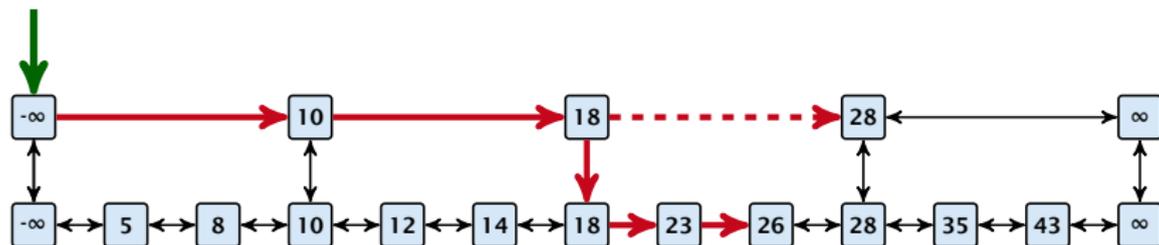
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Choose $|L_1| = \sqrt{n}$. Then search time $\Theta(\sqrt{n})$.

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- ▶ ...
- ▶ At most $|L_k| + \sum_{i=1}^k \frac{L_{i-1}}{L_i} + 3(k + 1)$ steps.

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Choosing $k = \Theta(\log n)$ gives a logarithmic running time.

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How to do insert and delete?

The cost of insert and delete is proportional to the number of elements in the list. Insert or delete may require a lot of reorganization.

Use randomization instead!

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How to do insert and delete?

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Insert:

- ▶ A search operation gives you the insert position for element x in every list.
- ▶ Flip a coin until it shows head, and record the number $t \in \{1, 2, \dots\}$ of trials needed.
- ▶ Insert x into lists L_0, \dots, L_{t-1} .

Delete:

You get all predecessors via backward pointers.

Delete x in all lists it actually appears in.

The time for both operations is dominated by the search time.

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Find all predecessor and successor pointers.

Remove all nodes which appear in it.

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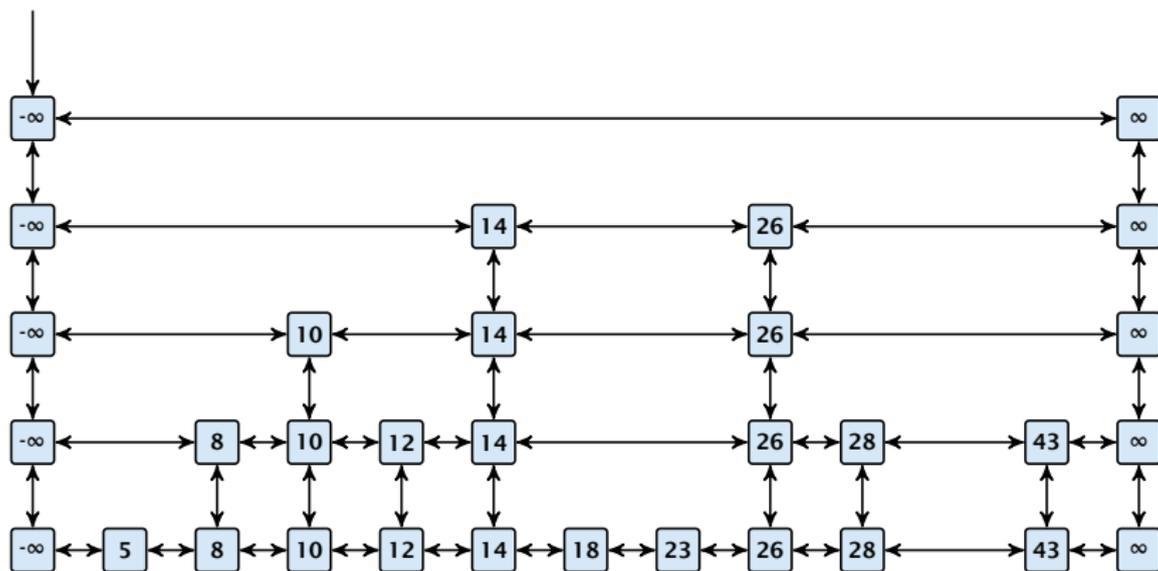
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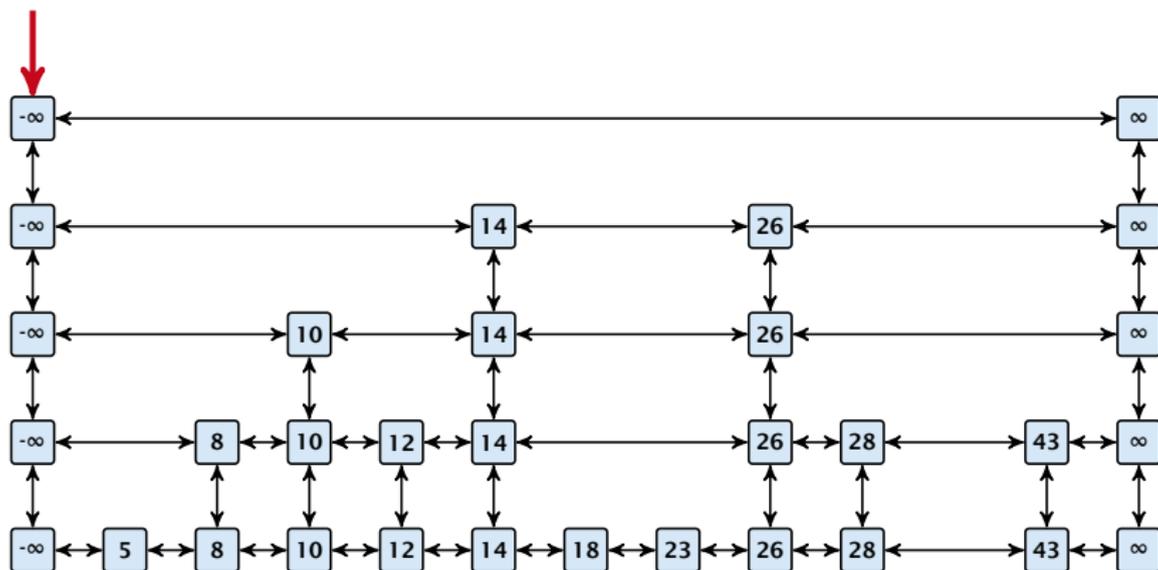
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Insert (35):



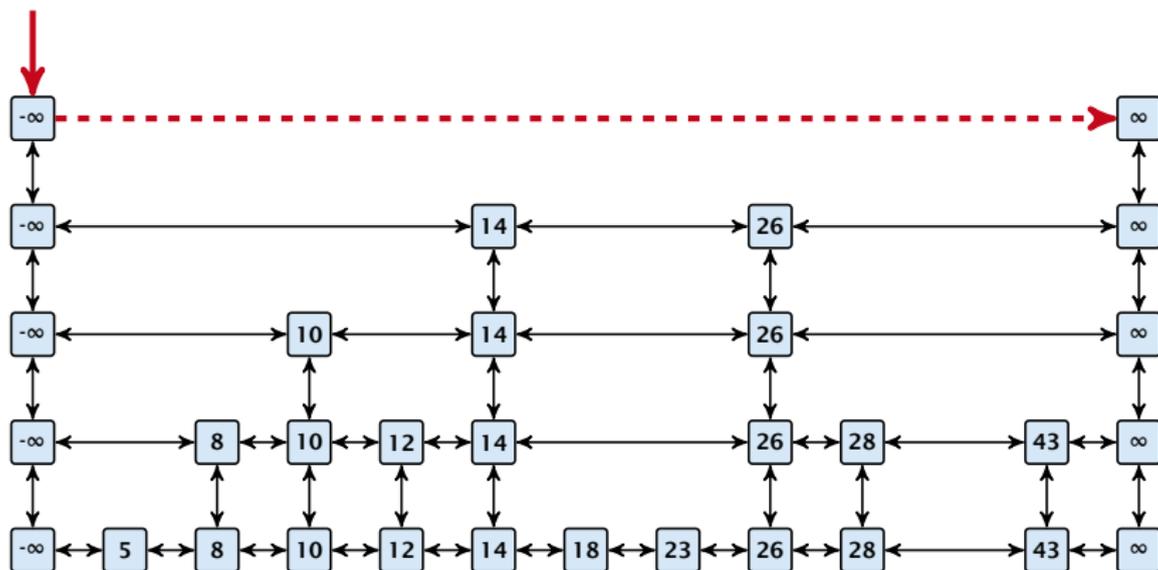
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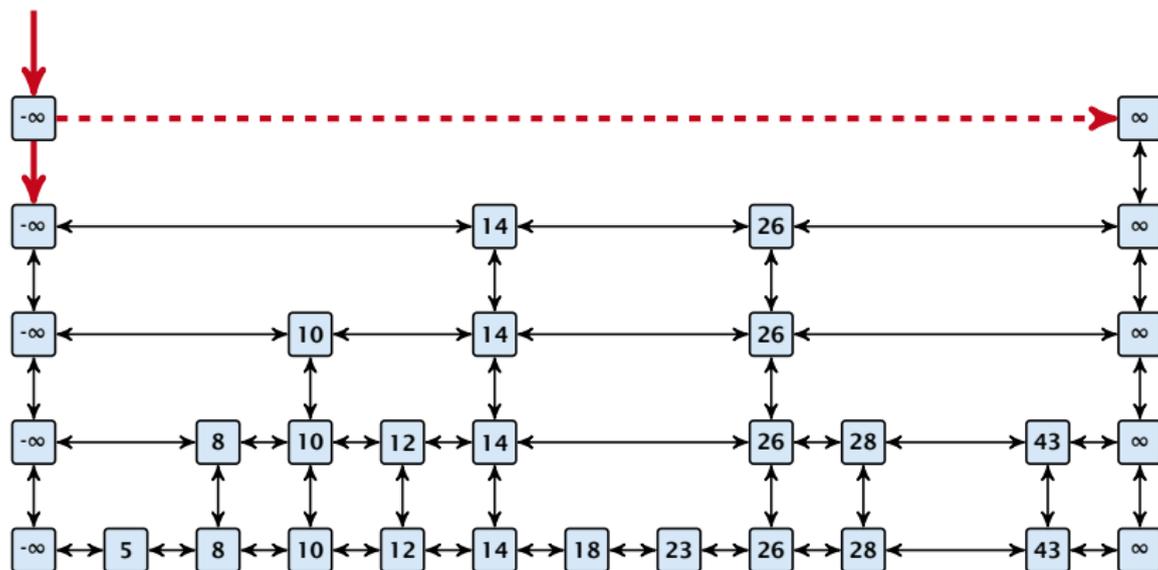
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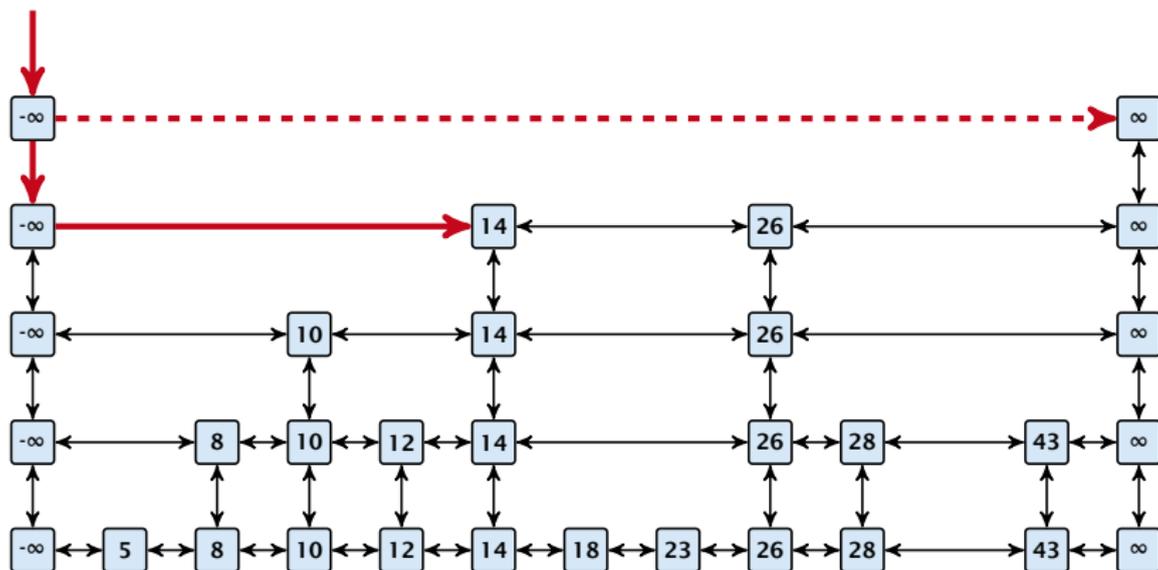
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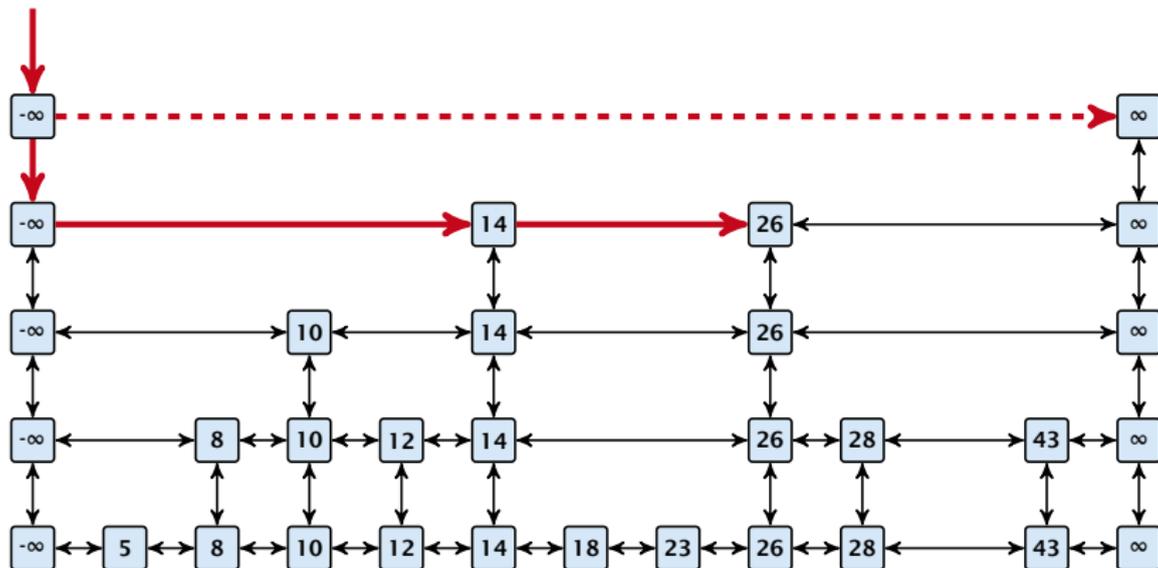
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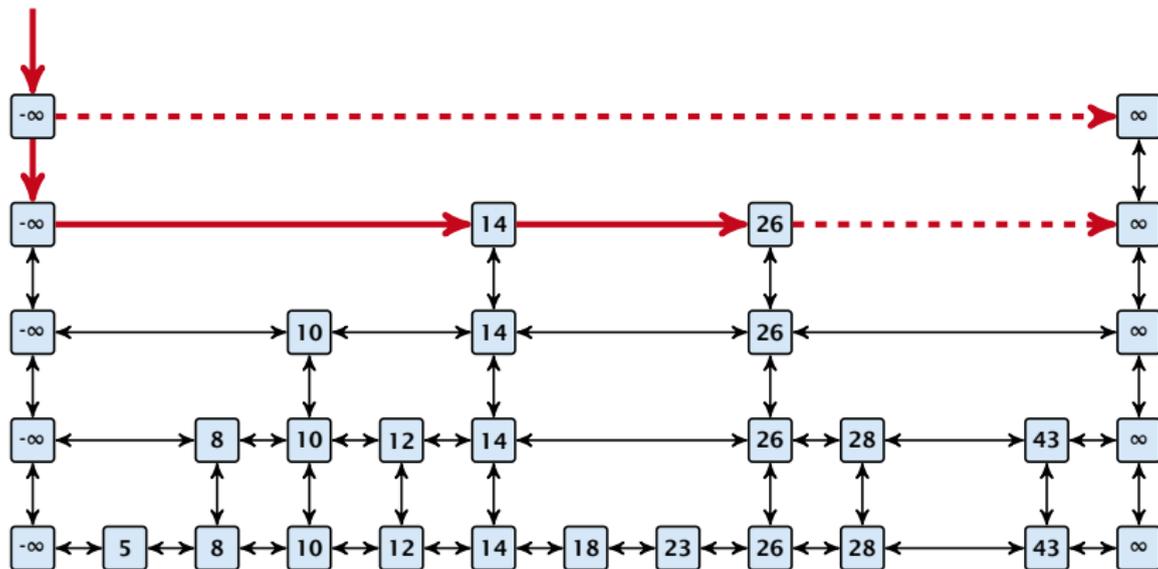
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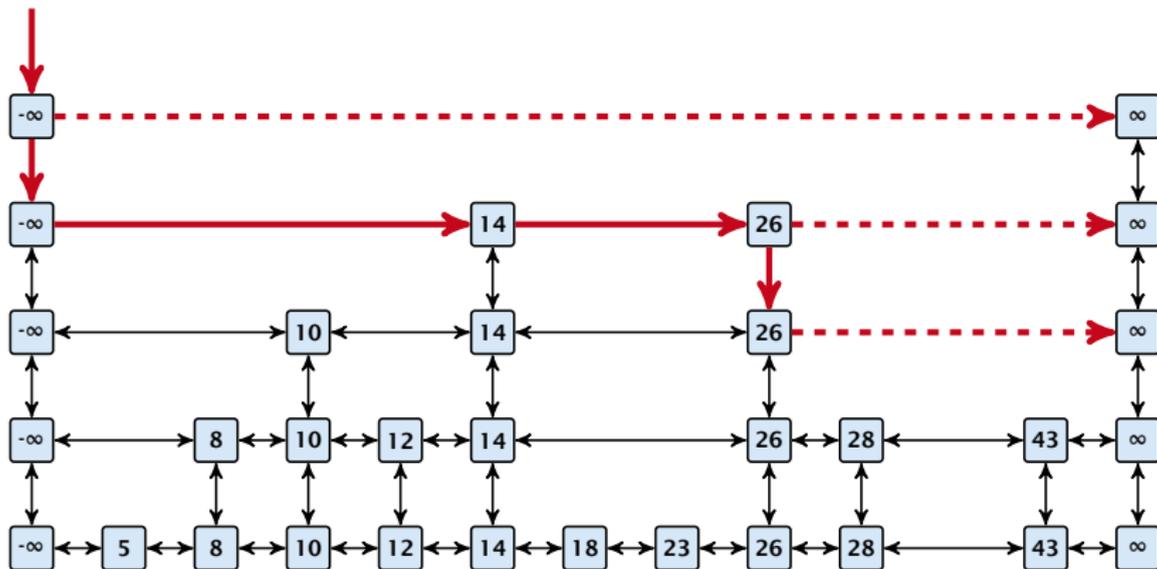
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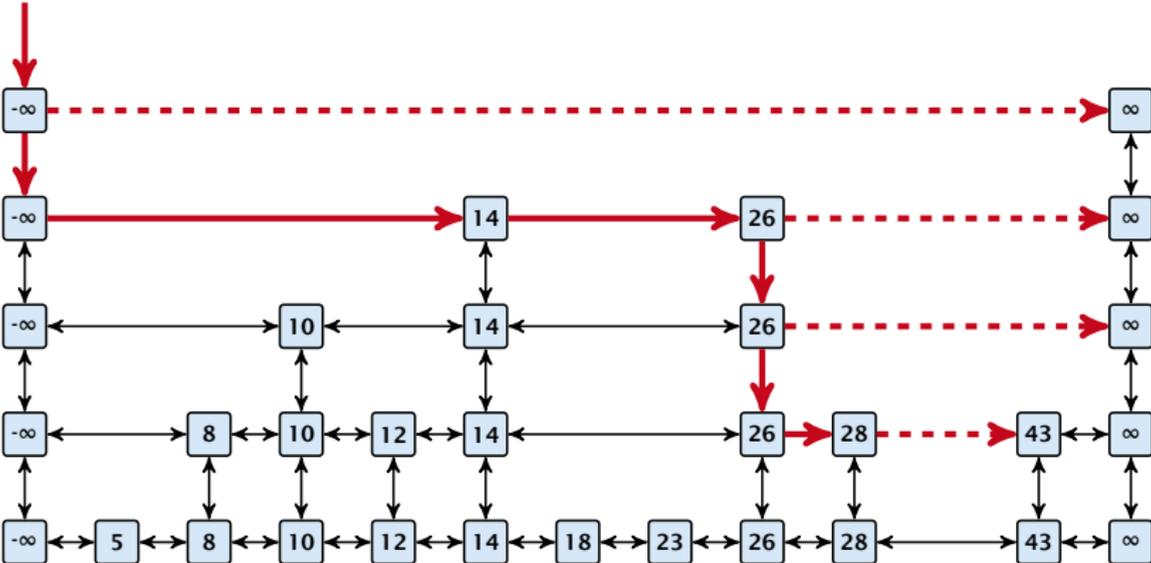
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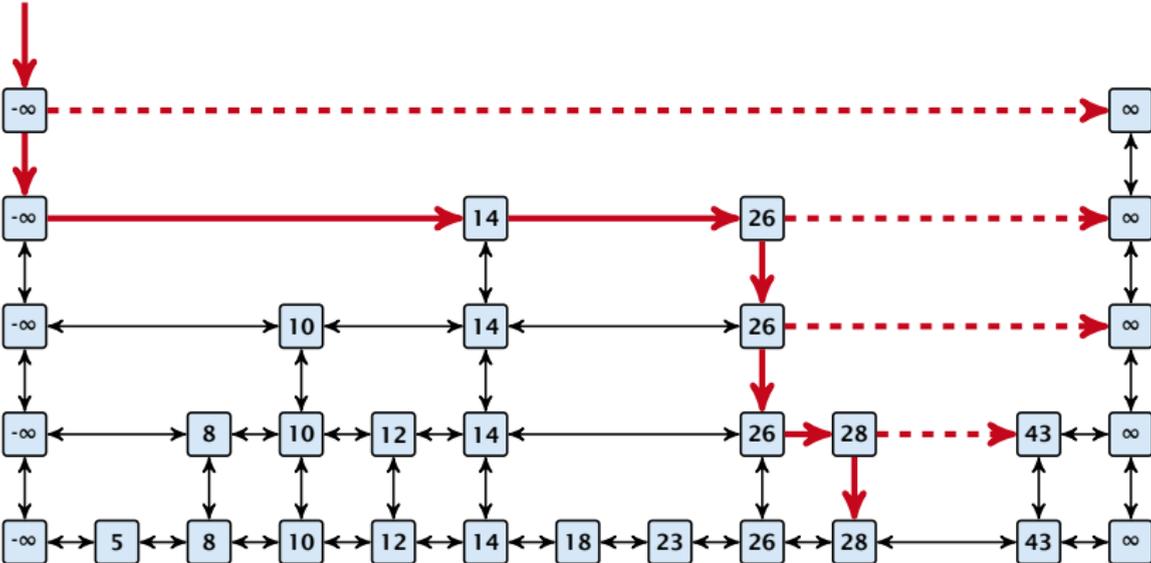
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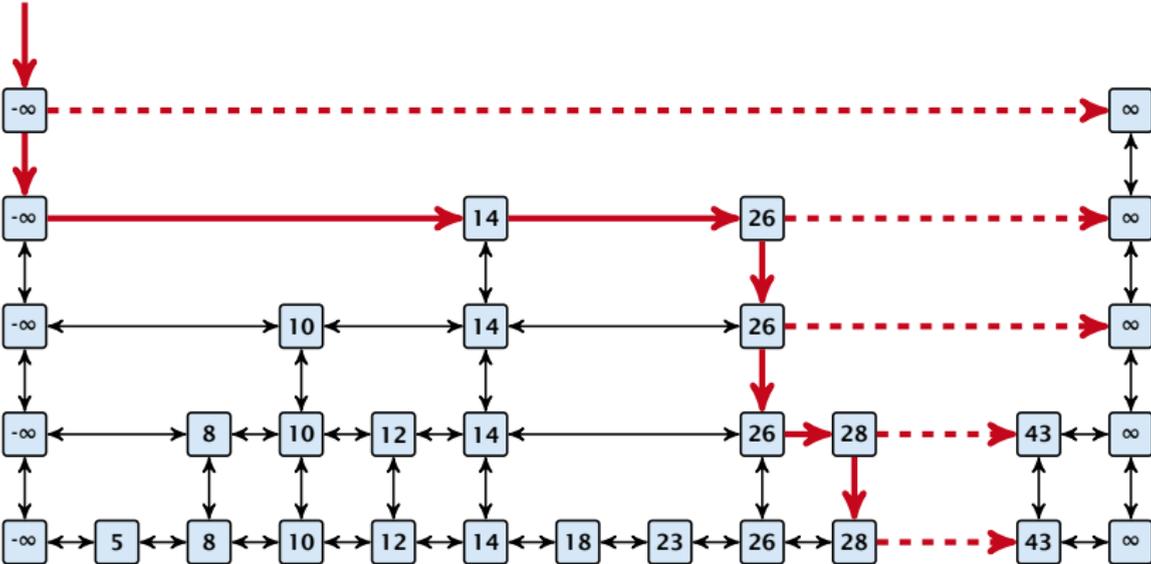
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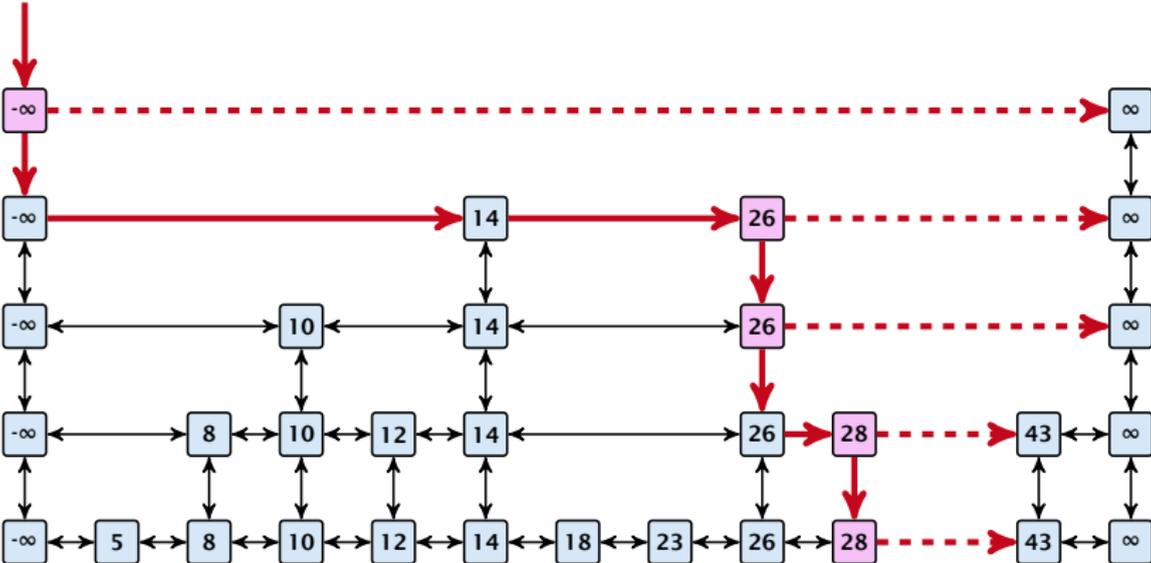
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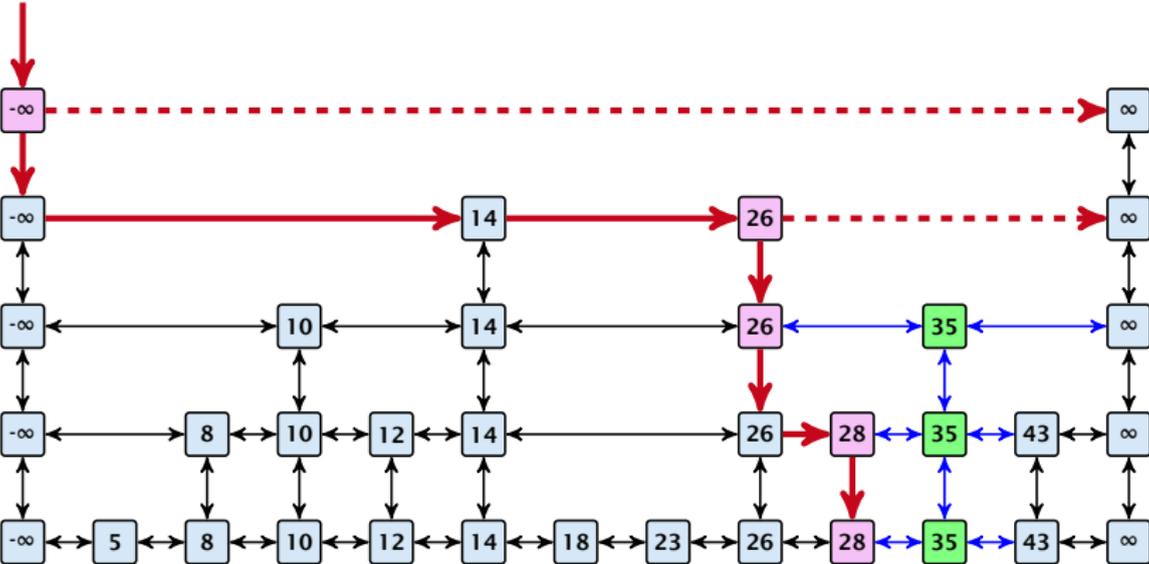
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High Probability

Definition 10 (High Probability)

We say a **randomized** algorithm has running time $\mathcal{O}(\log n)$ with **high probability** if for any constant α the running time is at most $\mathcal{O}(\log n)$ with probability at least $1 - \frac{1}{n^\alpha}$.

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High Probability

Suppose there are a **polynomially** many events E_1, E_2, \dots, E_ℓ , $\ell = n^c$ each holding with high probability (e.g. E_i may be the event that the i -th search in a skip list takes time at most $\mathcal{O}(\log n)$).

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This means $\Pr[E_1 \wedge \dots \wedge E_\ell]$ holds with high probability.

7.6 Skip Lists

Lemma 11

A search (and, hence, also insert and delete) in a skip list with n elements takes time $\mathcal{O}(\log n)$ with high probability (w. h. p.).

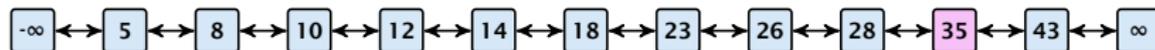
7.6 Skip Lists

Backward analysis:



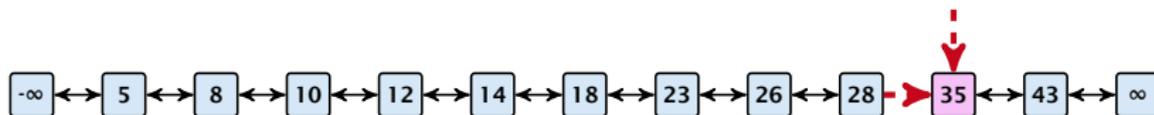
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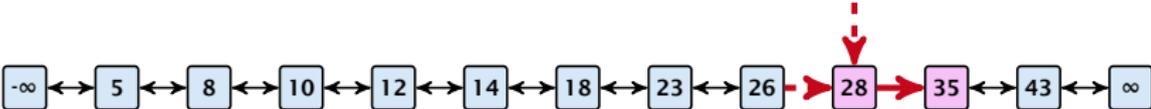
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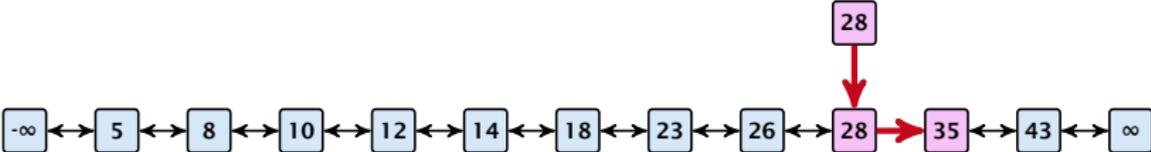
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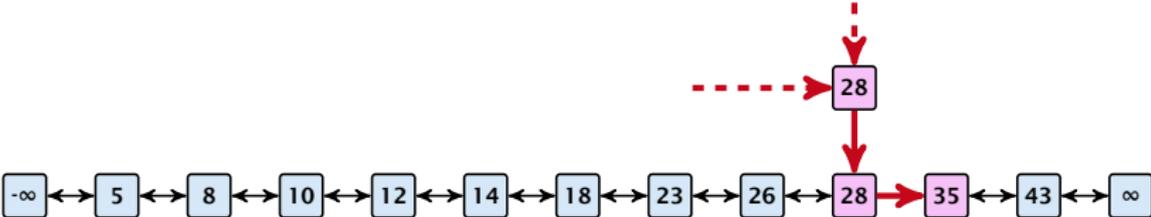
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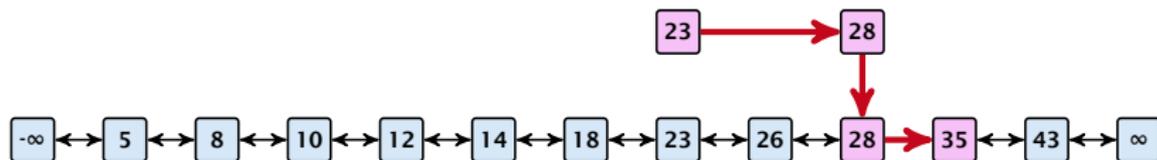
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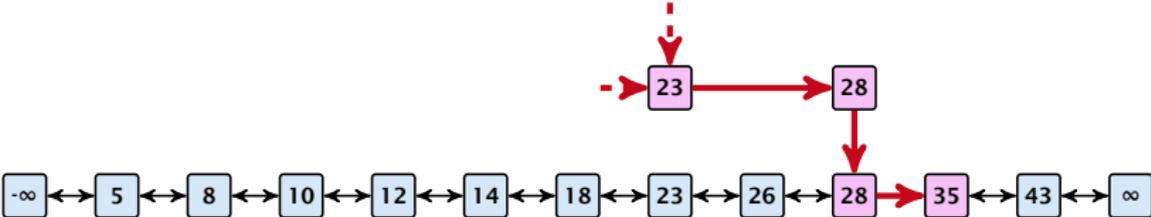
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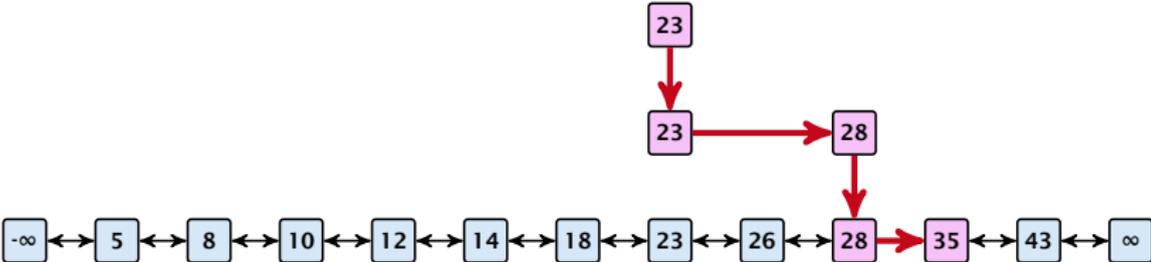
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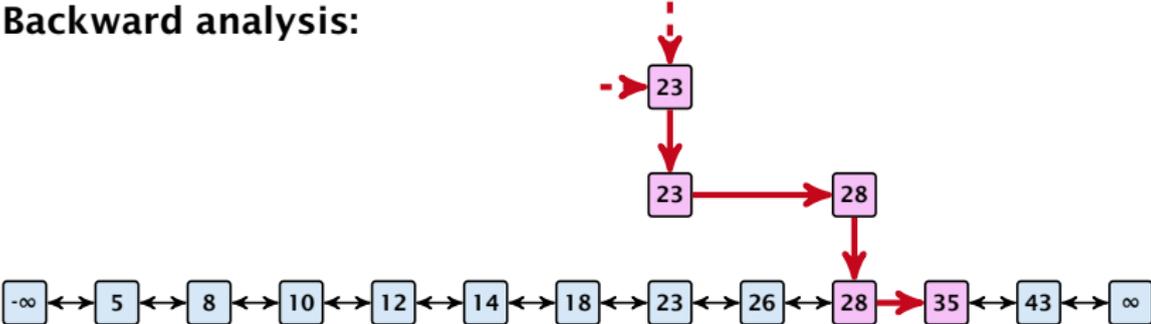
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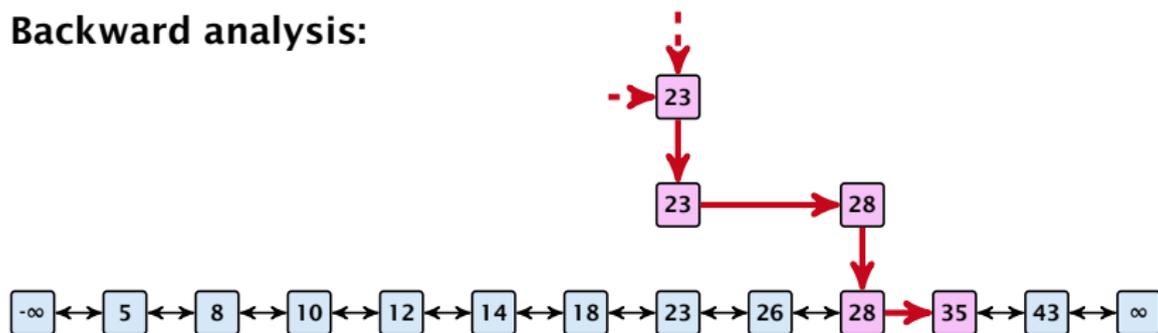
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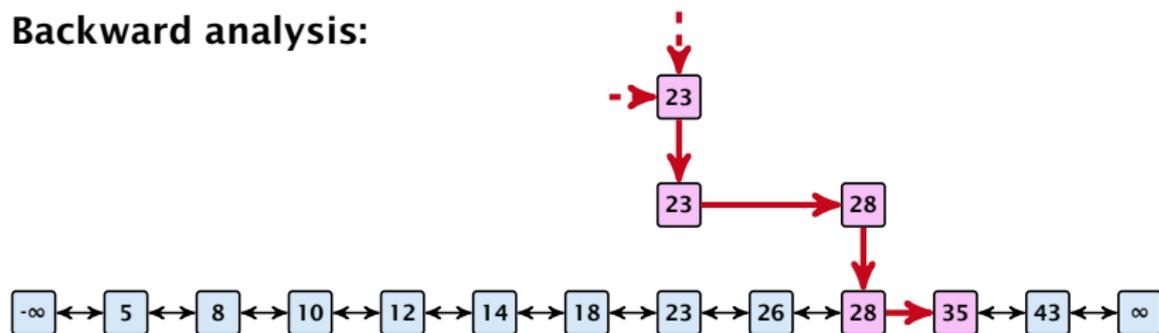
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At each point the path goes up with probability $1/2$ and left with probability $1/2$.

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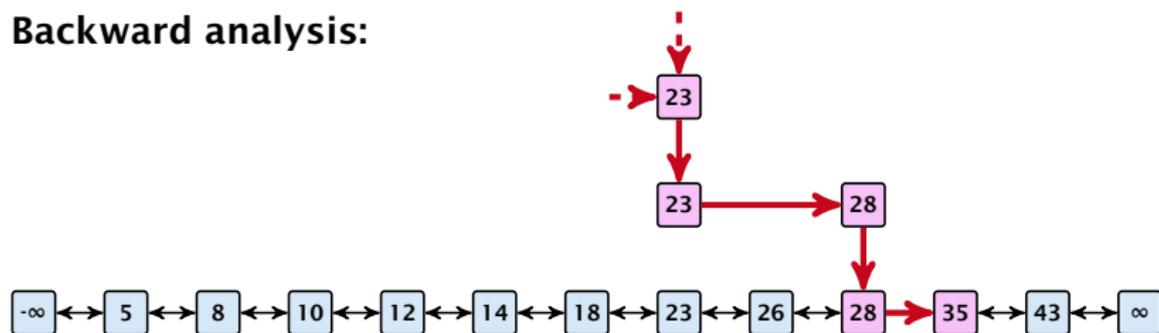
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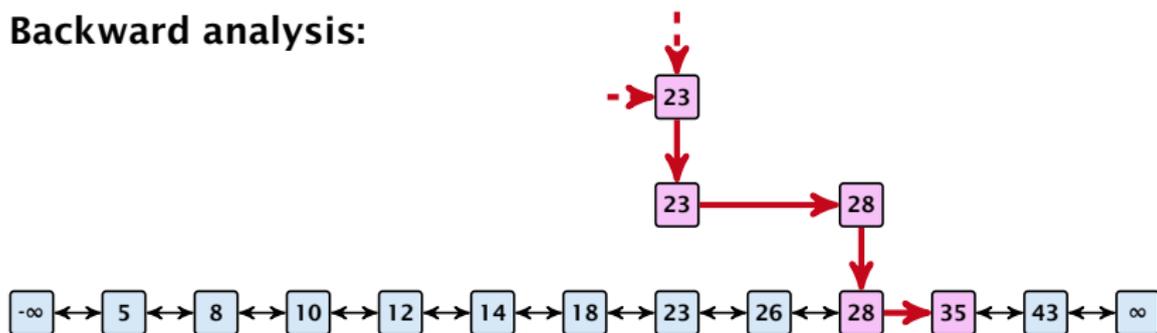
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We show that w.h.p.:

- ▶ A “long” search path must also go very high.
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From this it follows that w.h.p. there are no long paths.

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In particular, this means that during the construction in the backward analysis we see at most k heads (i.e., coin flips that tell you to go up) in z trials.

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7.6 Skip Lists

$$\Pr[E_{z,k}] \leq \Pr[\text{at most } k \text{ heads in } z \text{ trials}]$$

$$\leq \binom{z}{k} 2^{-(z-k)} \leq \left(\frac{ez}{k}\right)^k 2^{-(z-k)} \leq \left(\frac{2ez}{k}\right)^k 2^{-z}$$

choosing $k = \gamma \log n$ with $\gamma \geq 1$ and $z = (\beta + \alpha)\gamma \log n$

$$\leq \left(\frac{2ez}{k}\right)^k 2^{-\beta k} \cdot n^{-\gamma\alpha} \leq \left(\frac{2ez}{2^\beta k}\right)^k \cdot n^{-\alpha}$$

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This means, the search requires at most z steps, w. h. p.

7.7 Hashing

Dictionary:

- ▶ **$S.insert(x)$** : Insert an element x .
- ▶ **$S.delete(x)$** : Delete the element pointed to by x .
- ▶ **$S.search(k)$** : Return a pointer to an element e with $key[e] = k$ in S if it exists; otherwise return null.

So far we have implemented the search for a key by carefully choosing split-elements.

Then the memory location of an object x with key k is determined by successively comparing k to split-elements.

Hashing tries to directly compute the memory location from the given key. The goal is to have constant search time.

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Definitions:

- ▶ Universe U of keys, e.g., $U \subseteq \mathbb{N}_0$. U very large.
- ▶ Set $S \subseteq U$ of keys, $|S| = m \leq |U|$.
- ▶ Array $T[0, \dots, n-1]$ hash-table.
- ▶ Hash function $h : U \rightarrow [0, \dots, n-1]$.

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- ▶ Small storage requirement.
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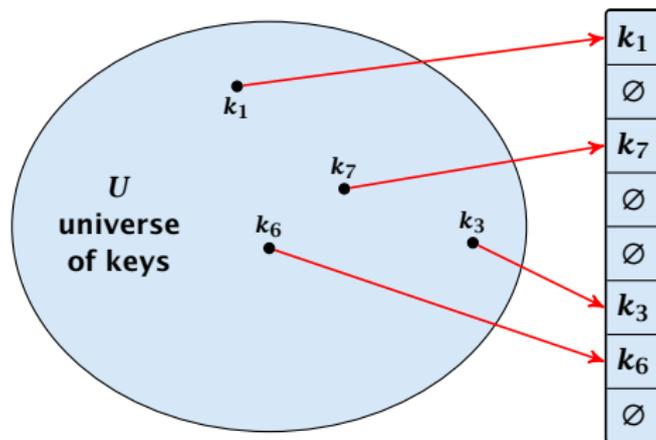
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Direct Addressing

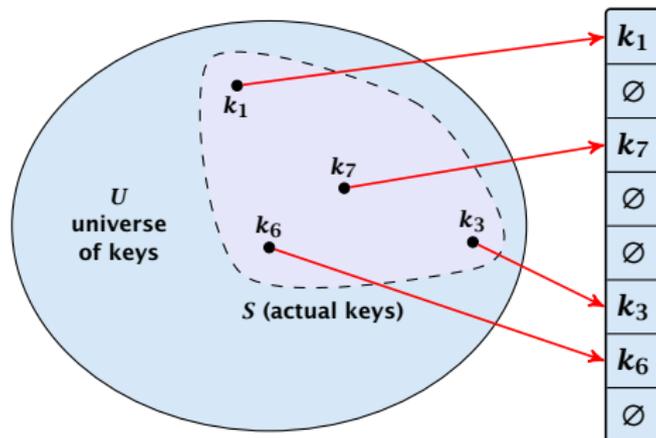
Ideally the hash function maps **all** keys to different memory locations.



This special case is known as **Direct Addressing**. It is usually very unrealistic as the universe of keys typically is quite large, and in particular larger than the available memory.

Perfect Hashing

Suppose that we **know** the set S of actual keys (no insert/no delete). Then we may want to design a **simple** hash-function that maps all these keys to different memory locations.



Such a hash function h is called a **perfect hash function** for set S .

Collisions

If we do not know the keys in advance, the best we can hope for is that the hash function distributes keys evenly across the table.

Problem: Collisions

Usually the universe U is much larger than the table-size n .

Hence, there may be two elements k_1, k_2 from the set S that map to the same memory location (i.e., $h(k_1) = h(k_2)$). This is called a **collision**.

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Typically, collisions do not appear once the size of the set S of actual keys gets close to n , but already when $|S| \geq \omega(\sqrt{n})$.

Lemma 12

*The probability of having a collision when hashing m elements into a table of size n under **uniform hashing** is at least*

$$1 - e^{-\frac{m(m-1)}{2n}} \approx 1 - e^{-\frac{m^2}{2n}}.$$

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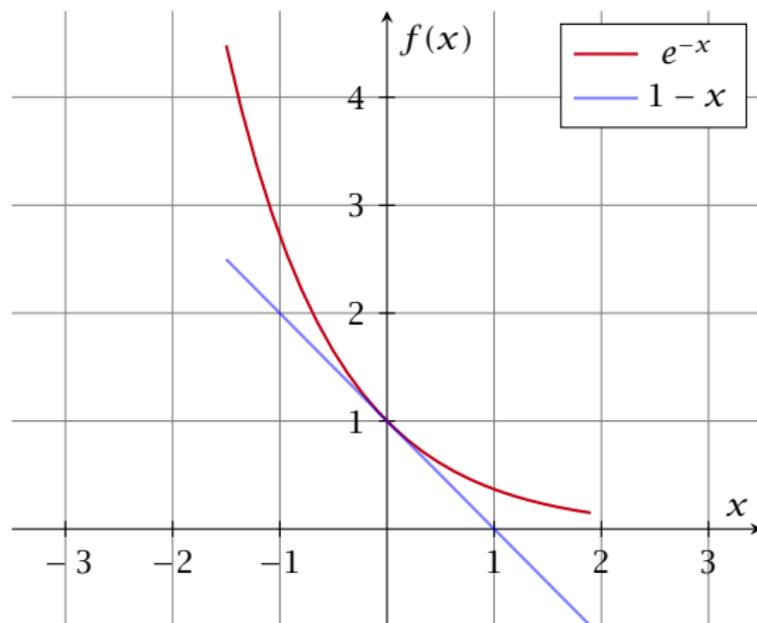
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Here the first equality follows since the ℓ -th element that is hashed has a probability of $\frac{n-\ell+1}{n}$ to not generate a collision under the condition that the previous elements did not induce collisions. □

Collisions



The inequality $1 - x \leq e^{-x}$ is derived by stopping the Taylor-expansion of e^{-x} after the second term.

Resolving Collisions

The methods for dealing with collisions can be classified into the two main types

- ▶ **open addressing**, aka. closed hashing
- ▶ **hashing with chaining**, aka. closed addressing, open hashing.

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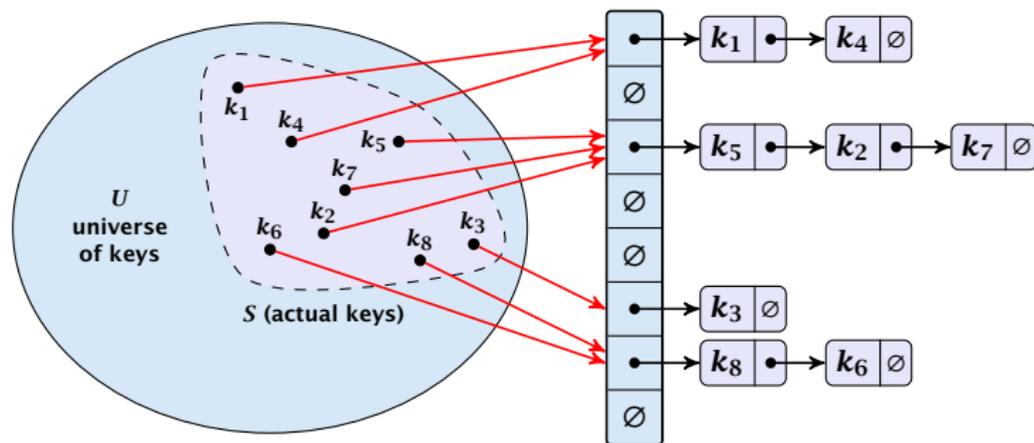
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Hashing with Chaining

Arrange elements that map to the same position in a linear list.

- ▶ Access: compute $h(x)$ and search list for $\text{key}[x]$.
- ▶ Insert: insert at the front of the list.



Hashing with Chaining

Let A denote a strategy for resolving collisions. We use the following notation:

- ▶ A^+ denotes the average time for a **successful** search when using A ;
- ▶ A^- denotes the average time for an **unsuccessful** search when using A ;
- ▶ We parameterize the complexity results in terms of $\alpha := \frac{m}{n}$, the so-called **fill factor** of the hash-table.

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$$A^- = 1 + \alpha .$$

Hashing with Chaining

For a successful search observe that we do **not** choose a list at random, but we consider a random key k in the hash-table and ask for the search-time for k .

This is 1 plus the number of elements that lie before k in k 's list.

Let k_ℓ denote the ℓ -th key inserted into the table.

Let for two keys k_i and k_j , X_{ij} denote the indicator variable for the event that k_i and k_j hash to the same position. Clearly, $\Pr[X_{ij} = 1] = 1/n$ for uniform hashing.

The expected successful search cost is

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$$E \left[\frac{1}{m} \sum_{i=1}^m \left(1 + \sum_{j=i+1}^m X_{ij} \right) \right] = \frac{1}{m} \sum_{i=1}^m \left(1 + \sum_{j=i+1}^m E[X_{ij}] \right)$$

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Hence, the expected cost for a successful search is $A^+ \leq 1 + \frac{\alpha}{2}$.

Hashing with Chaining

Disadvantages:

- ▶ pointers increase memory requirements
- ▶ pointers may lead to bad cache efficiency

Advantages:

- ▶ no à priori limit on the number of elements
- ▶ deletion can be implemented efficiently
- ▶ by using balanced trees instead of linked list one can also obtain worst-case guarantees.

Open Addressing

All objects are stored in the table itself.

Define a function $h(k, j)$ that determines the table-position to be examined in the j -th step. The values $h(k, 0), \dots, h(k, n - 1)$ must form a permutation of $0, \dots, n - 1$.

Search(k): Try position $h(k, 0)$; if it is empty your search fails; otw. continue with $h(k, 1), h(k, 2), \dots$.

Insert(x): Search until you find an empty slot; insert your element there. If your search reaches $h(k, n - 1)$, and this slot is non-empty then your table is full.

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Open Addressing

Choices for $h(k, j)$:

- ▶ **Linear probing:**

$$h(k, i) = h(k) + i \pmod n$$

(sometimes: $h(k, i) = h(k) + ci \pmod n$).

- ▶ Quadratic probing:

$$h(k, i) = h(k) + c_1i + c_2i^2 \pmod n.$$

- ▶ Double hashing:

$$h(k, i) = h_1(k) + ih_2(k) \pmod n.$$

For quadratic probing and double hashing one has to ensure that the search covers all positions in the table (i.e., for double hashing $h_2(k)$ must be relatively prime to n (teilerfremd); for quadratic probing c_1 and c_2 have to be chosen carefully).

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Linear Probing

- ▶ Advantage: **Cache-efficiency**. The new probe position is very likely to be in the cache.
- ▶ Disadvantage: **Primary clustering**. Long sequences of occupied table-positions get longer as they have a larger probability to be hit. Furthermore, they can merge forming larger sequences.

Lemma 13

Let L be the method of linear probing for resolving collisions:

$$L^+ \approx \frac{1}{2} \left(1 + \frac{1}{1 - \alpha} \right)$$

$$L^- \approx \frac{1}{2} \left(1 + \frac{1}{(1 - \alpha)^2} \right)$$

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- ▶ **Secondary clustering**: caused by the fact that all keys mapped to the same position have the same probe sequence.

Lemma 14

Let Q be the method of quadratic probing for resolving collisions:

$$Q^+ \approx 1 + \ln\left(\frac{1}{1-\alpha}\right) - \frac{\alpha}{2}$$

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Double Hashing

- ▶ Any probe into the hash-table usually creates a cache-miss.

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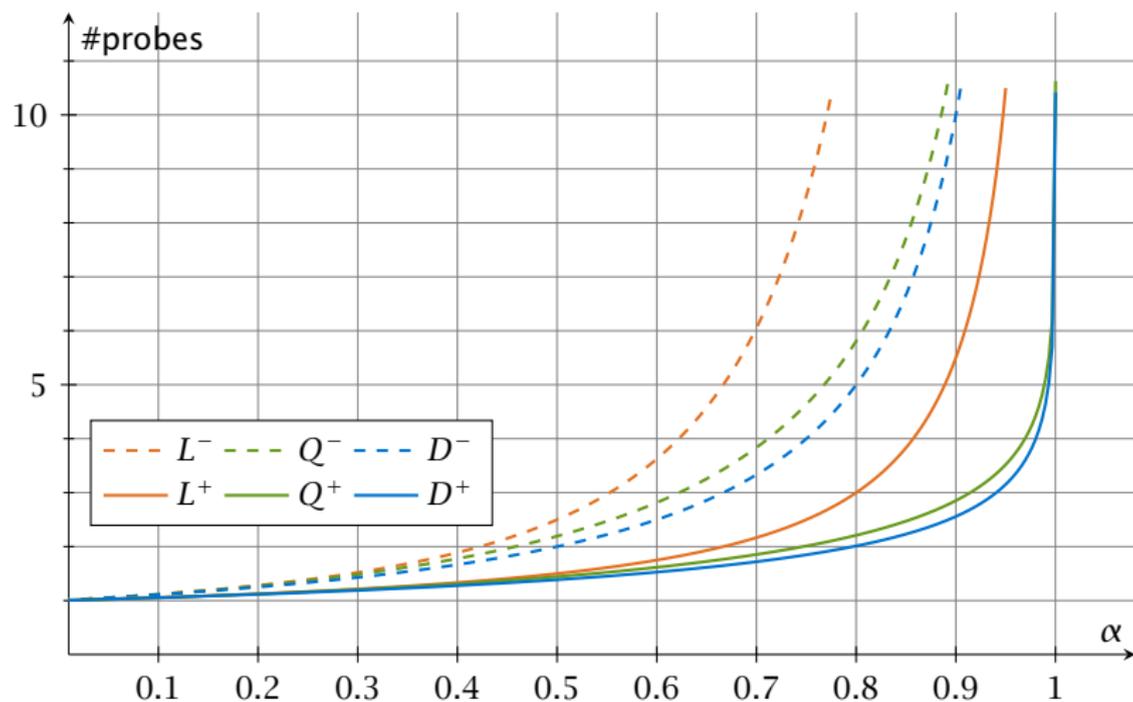
$$D^- \approx \frac{1}{1 - \alpha}$$

Open Addressing

Some values:

α	<i>Linear Probing</i>		<i>Quadratic Probing</i>		<i>Double Hashing</i>	
	L^+	L^-	Q^+	Q^-	D^+	D^-
0.5	1.5	2.5	1.44	2.19	1.39	2
0.9	5.5	50.5	2.85	11.40	2.55	10
0.95	10.5	200.5	3.52	22.05	3.15	20

Open Addressing



Analysis of Idealized Open Address Hashing

We analyze the time for a search in a very idealized Open Addressing scheme.

- ▶ The probe sequence $h(k, 0), h(k, 1), h(k, 2), \dots$ is equally likely to be any permutation of $\langle 0, 1, \dots, n - 1 \rangle$.

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$$\Pr[X \geq i]$$

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$$\Pr[X \geq i] = \frac{m}{n} \cdot \frac{m-1}{n-1} \cdot \frac{m-2}{n-2} \cdot \dots \cdot \frac{m-i+2}{n-i+2}$$

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Analysis of Idealized Open Address Hashing

$E[X]$

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Analysis of Idealized Open Address Hashing

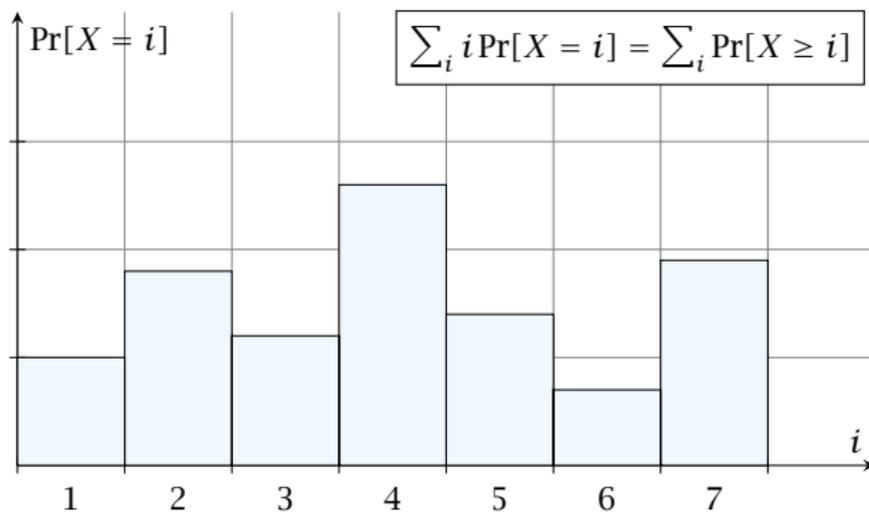
$$E[X] = \sum_{i=1}^{\infty} \Pr[X \geq i] \leq \sum_{i=1}^{\infty} \alpha^{i-1} = \sum_{i=0}^{\infty} \alpha^i = \frac{1}{1-\alpha} .$$

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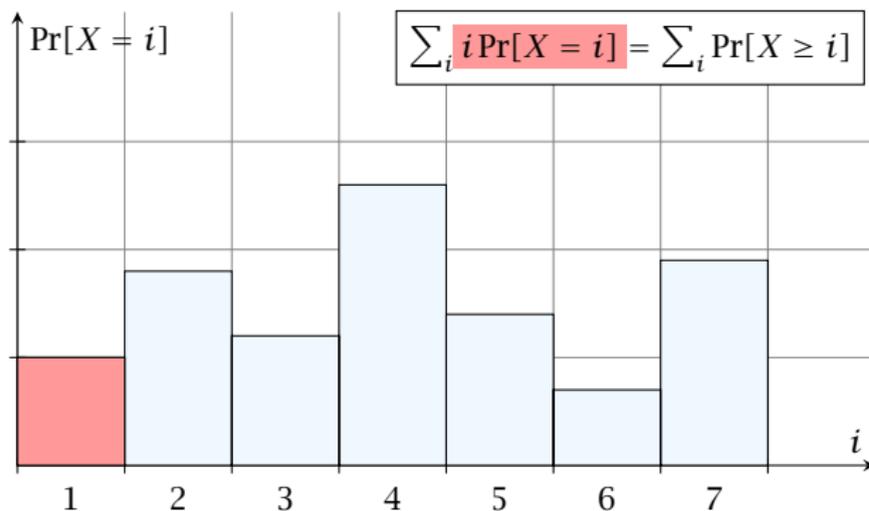
$$\frac{1}{1-\alpha} = 1 + \alpha + \alpha^2 + \alpha^3 + \dots$$

Analysis of Idealized Open Address Hashing



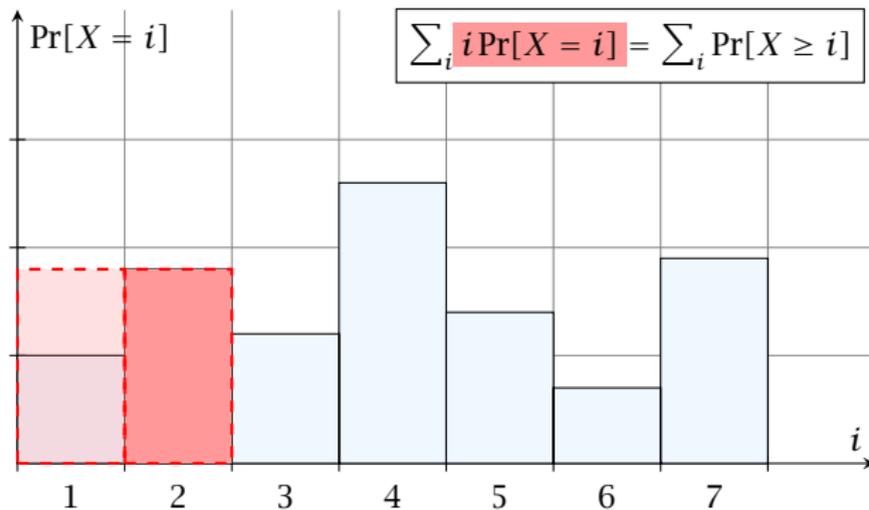
Analysis of Idealized Open Address Hashing

$i = 1$



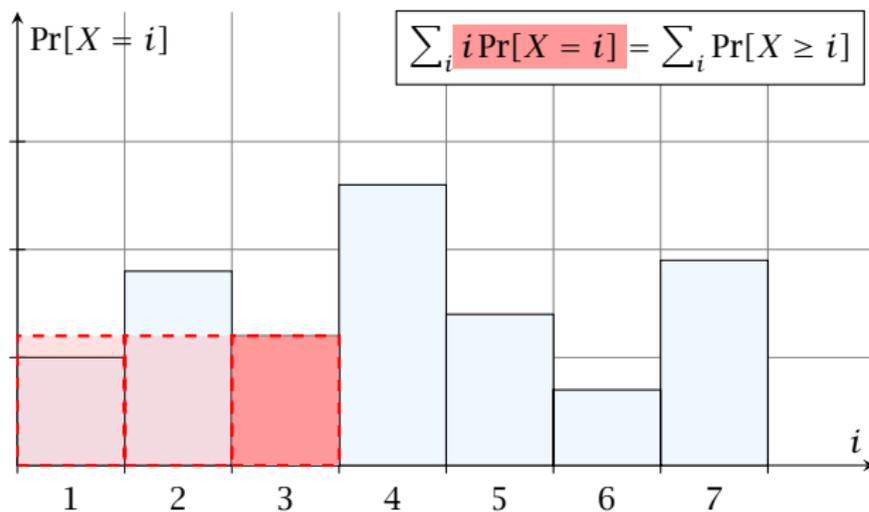
Analysis of Idealized Open Address Hashing

$i = 2$



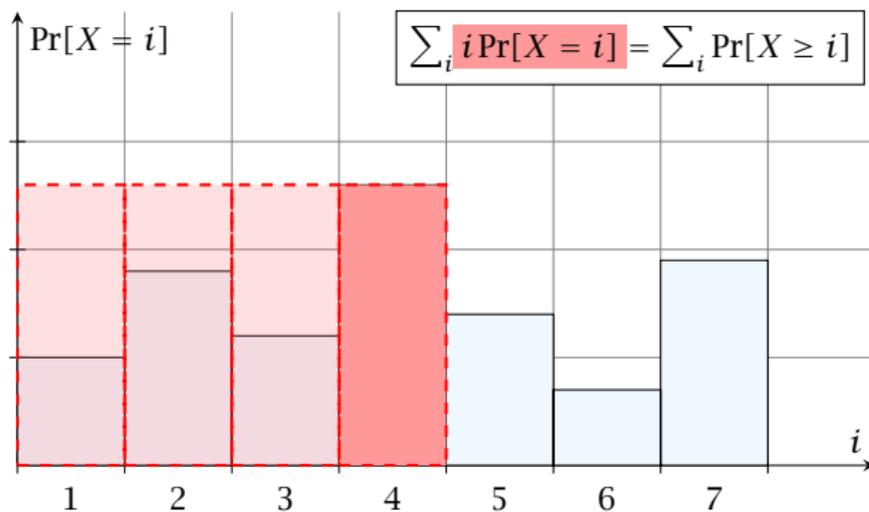
Analysis of Idealized Open Address Hashing

$i = 3$



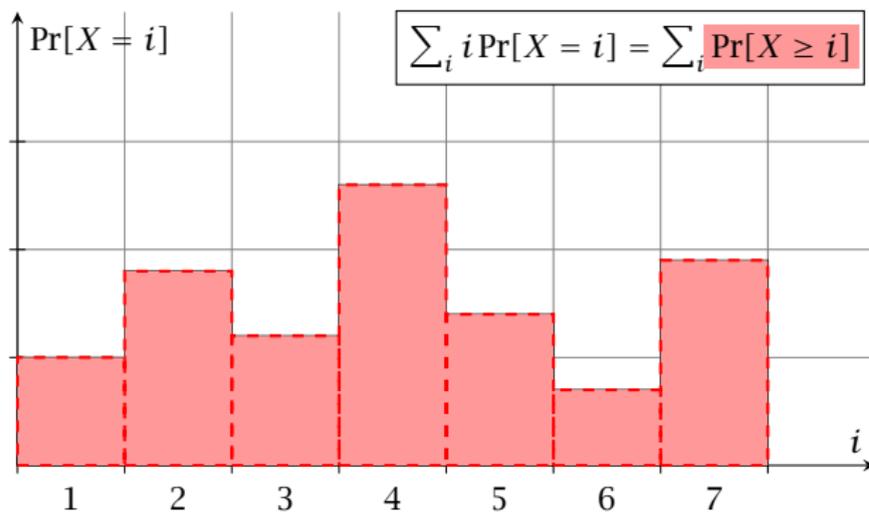
Analysis of Idealized Open Address Hashing

$i = 4$



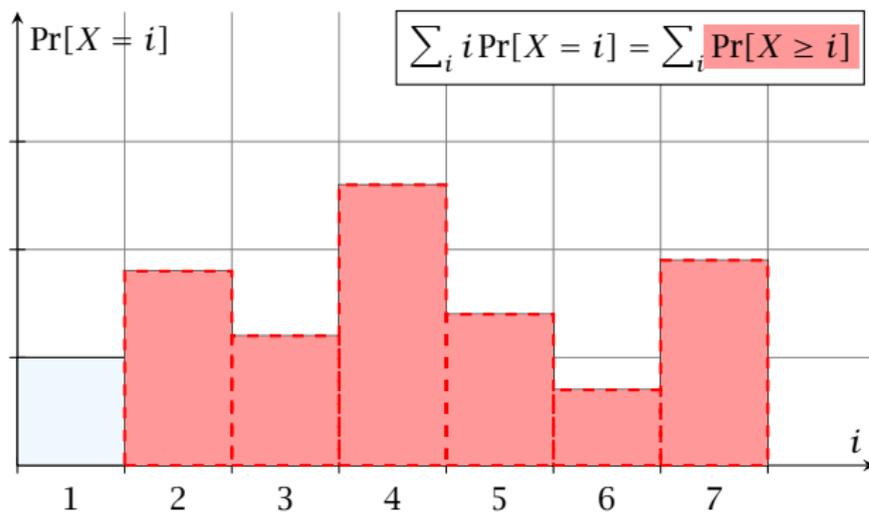
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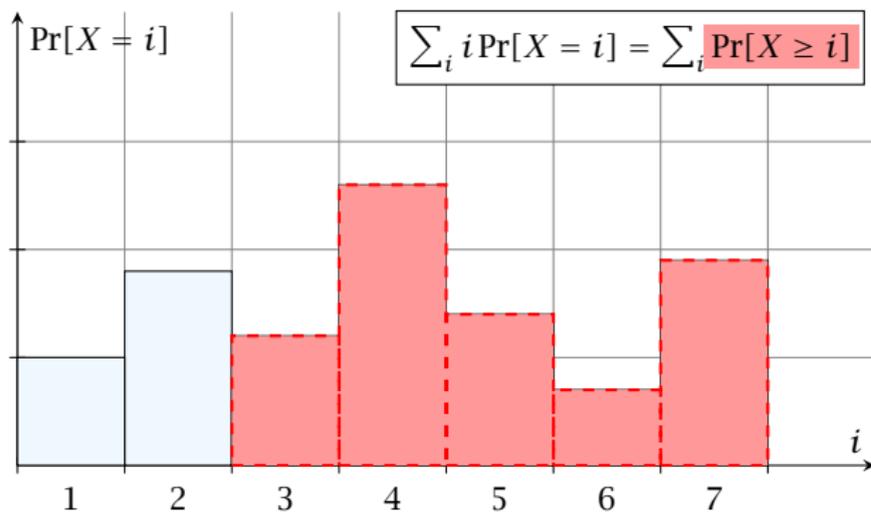
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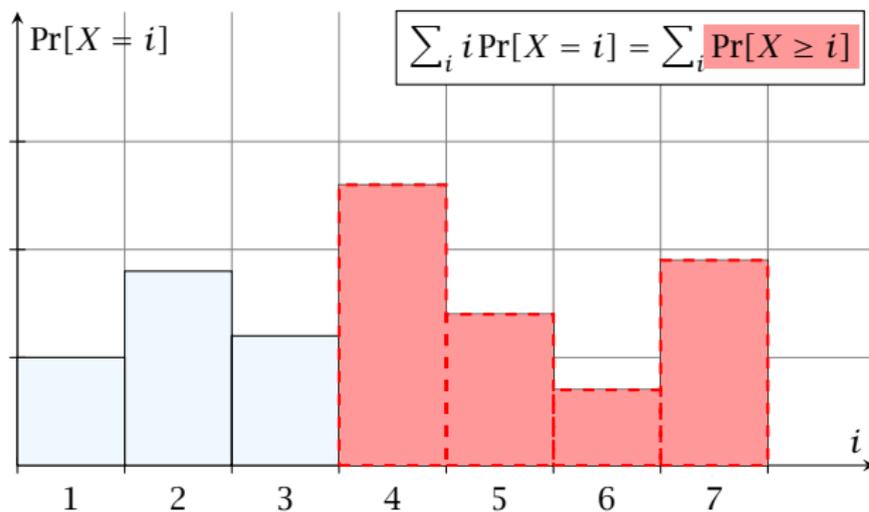
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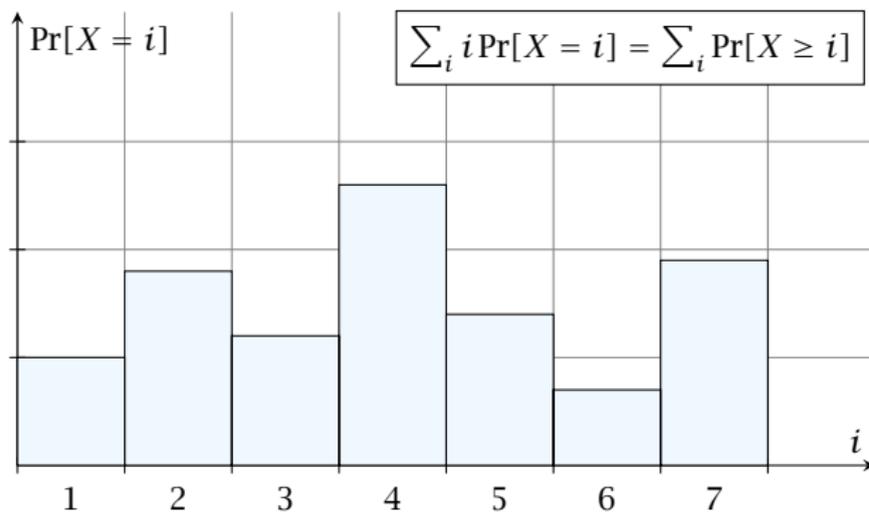


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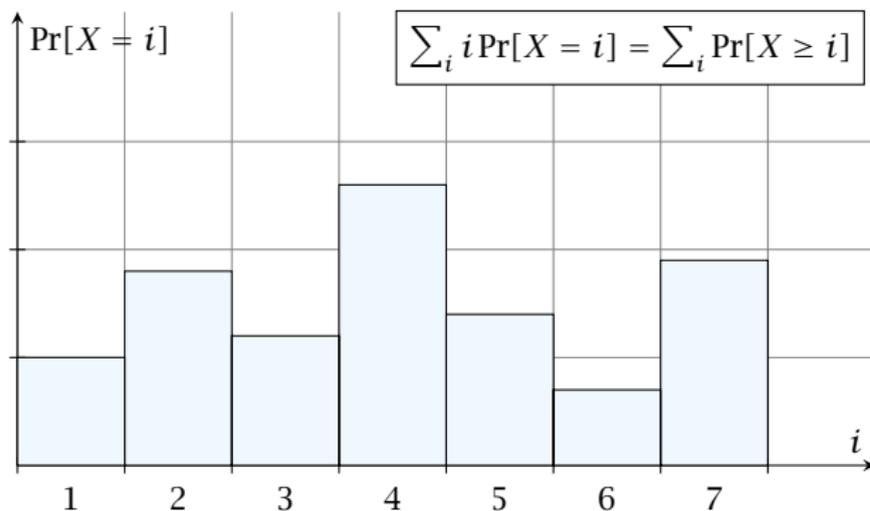
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Analysis of Idealized Open Address Hashing



Analysis of Idealized Open Address Hashing



The j -th rectangle appears in both sums j times. (j times in the first due to multiplication with j ; and j times in the second for summands $i = 1, 2, \dots, j$)

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$$\frac{1}{m} \sum_{i=0}^{m-1} \frac{n}{n-i} = \frac{n}{m} \sum_{i=0}^{m-1} \frac{1}{n-i} = \frac{1}{\alpha} \sum_{k=n-m+1}^n \frac{1}{k}$$

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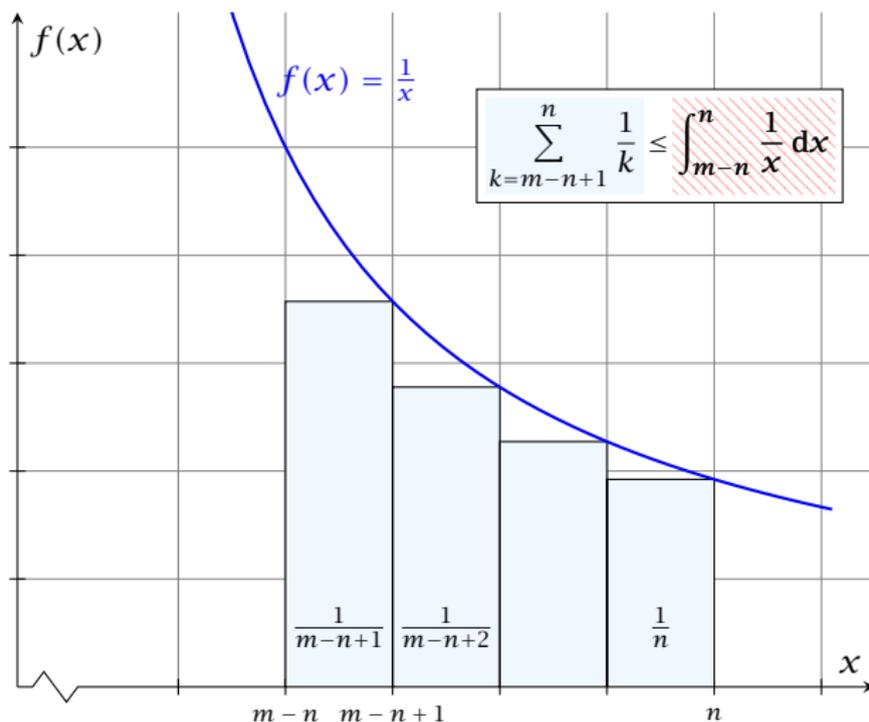
Analysis of Idealized Open Address Hashing

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Analysis of Idealized Open Address Hashing



How do we delete in a hash-table?

- ▶ For hashing with chaining this is not a problem. Simply search for the key, and delete the item in the corresponding list.
- ▶ For open addressing this is difficult.

Deletions in Hashtables

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Deletions in Hashtables

- ▶ Simply removing a key might interrupt the probe sequence of other keys which then cannot be found anymore.
- ▶ One can delete an element by replacing it with a deleted-marker.
 - ▶ Deleted markers interrupt the probe sequence of other keys which then cannot be found anymore.
 - ▶ Deleted markers can be ignored during insertions.
 - ▶ Deleted markers can be deleted during deletions.
 - ▶ Deleted markers can be used to mark deleted elements.
 - ▶ Deleted markers interrupt the probe sequence.
- ▶ The table could fill up with deleted-markers leading to bad performance.
- ▶ If a table contains many deleted-markers (linear fraction of the keys) one can rehash the whole table and amortize the cost for this rehash against the cost for the deletions.

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Algorithm 16 delete(p)

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1:  $T[p] \leftarrow \text{null}$ 
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3: while  $T[p] \neq \text{null}$  do
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p is the index into the table-cell that contains the object to be deleted.

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Universal Hashing

Regardless, of the choice of hash-function there is always an input (a set of keys) that has a very poor worst-case behaviour.

Therefore, so far we assumed that the hash-function is random so that regardless of the input the average case behaviour is good.

However, the assumption of uniform hashing that h is chosen randomly from all functions $f : U \rightarrow [0, \dots, n - 1]$ is clearly unrealistic as there are $n^{|U|}$ such functions. Even writing down such a function would take $|U| \log n$ bits.

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A class \mathcal{H} of hash-functions from the universe U into the set $\{0, \dots, n-1\}$ is called **universal** if for all $u_1, u_2 \in U$ with $u_1 \neq u_2$

$$\Pr[h(u_1) = h(u_2)] \leq \frac{1}{n} ,$$

where the probability is w. r. t. the choice of a random hash-function from set \mathcal{H} .

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- ▶ For any key $u \in U$, and $t \in \{0, \dots, n-1\}$ $\Pr[h(u) = t] = \frac{1}{n}$, i.e., a key is distributed uniformly within the hash-table.
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Let $U := \{0, \dots, p-1\}$ for a prime p . Let $\mathbb{Z}_p := \{0, \dots, p-1\}$, and let $\mathbb{Z}_p^* := \{1, \dots, p-1\}$ denote the set of invertible elements in \mathbb{Z}_p .

Define

$$h_{a,b}(x) := (ax + b \bmod p) \bmod n$$

Lemma 20

The class

$$\mathcal{H} = \{h_{a,b} \mid a \in \mathbb{Z}_p^*, b \in \mathbb{Z}_p\}$$

is a universal class of hash-functions from U to $\{0, \dots, n-1\}$.

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Let $x, y \in U$ be two distinct keys. We have to show that the probability of a collision is only $1/n$.

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- ▶ The hash-function does not generate collisions before the $(\text{mod } n)$ -operation. Furthermore, every choice (a, b) is mapped to a different pair (t_x, t_y) with $t_x := ax + b$ and $t_y := ay + b$.

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There is a one-to-one correspondence between hash-functions (pairs (a, b) , $a \neq 0$) and pairs (t_x, t_y) , $t_x \neq t_y$.

Therefore, we can view the first step (before the mod n -operation) as choosing a pair (t_x, t_y) , $t_x \neq t_y$ uniformly at random.

What happens when we do the mod n operation?

Fix a value t_x . There are $p - 1$ possible values for choosing t_y .

From the range $0, \dots, p - 1$ the values $t_x, t_x + n, t_x + 2n, \dots$ map to t_x after the modulo-operation. These are at most $\lceil p/n \rceil$ values.

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As $t_y \neq t_x$ there are

$$\left| \frac{t_y}{n} - \frac{t_x}{n} \right| = \frac{|t_y - t_x|}{n} < \frac{n-1}{n}$$

possibilities for choosing t_y such that the final hash-value creates a collision.

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This happens with probability at most $\frac{1}{n}$.

Universal Hashing

Universal Hashing

It is also possible to show that \mathcal{H} is an (almost) pairwise independent class of hash-functions.

$$\Pr_{t_x \neq t_y \in \mathbb{Z}_p^2} \left[\begin{array}{l} t_x \bmod n = h_1 \\ t_y \bmod n = h_2 \end{array} \right]$$

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Note that the middle is the probability that $h(x) = h_1$ and $h(y) = h_2$. The total number of choices for (t_x, t_y) is $p(p-1)$. The number of choices for t_x (t_y) such that $t_x \bmod n = h_1$ ($t_y \bmod n = h_2$) lies between $\lfloor \frac{p}{n} \rfloor$ and $\lceil \frac{p}{n} \rceil$.

Universal Hashing

Definition 21

Let $d \in \mathbb{N}$; $q \geq (d + 1)n$ be a prime; and let $\bar{a} \in \{0, \dots, q - 1\}^{d+1}$. Define for $x \in \{0, \dots, q - 1\}$

$$h_{\bar{a}}(x) := \left(\sum_{i=0}^d a_i x^i \bmod q \right) \bmod n .$$

Let $\mathcal{H}_n^d := \{h_{\bar{a}} \mid \bar{a} \in \{0, \dots, q - 1\}^{d+1}\}$. The class \mathcal{H}_n^d is $(e, d + 1)$ -independent.

Note that in the previous case we had $d = 1$ and chose $a_d \neq 0$.

Universal Hashing

For the coefficients $\vec{a} \in \{0, \dots, q-1\}^{d+1}$ let $f_{\vec{a}}$ denote the polynomial

$$f_{\vec{a}}(x) = \left(\sum_{i=0}^d a_i x^i \right) \bmod q$$

The polynomial is defined by $d + 1$ distinct points.

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Fix $\ell \leq d + 1$; let $x_1, \dots, x_\ell \in \{0, \dots, q - 1\}$ be keys, and let t_1, \dots, t_ℓ denote the corresponding hash-function values.

Let $A^\ell = \{h_{\bar{a}} \in \mathcal{H} \mid h_{\bar{a}}(x_i) = t_i \text{ for all } i \in \{1, \dots, \ell\}\}$

Then

$$h_{\bar{a}} \in A^\ell \Leftrightarrow h_{\bar{a}} = f_{\bar{a}} \bmod n \text{ and}$$

$$f_{\bar{a}}(x_i) \in \underbrace{\{t_i + \alpha \cdot n \mid \alpha \in \{0, \dots, \lceil \frac{q}{n} \rceil - 1\}\}}_{=: B_i}$$

In order to obtain the cardinality of A^ℓ we choose our polynomial by fixing $d + 1$ points.

We first fix the values for inputs x_1, \dots, x_ℓ .

We have

$$|B_1| \cdot \dots \cdot |B_\ell|$$

possibilities to do this (so that $h_{\bar{a}}(x_i) = t_i$).

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Now, we choose $d - \ell + 1$ other inputs and choose their value arbitrarily. We have $q^{d-\ell+1}$ possibilities to do this.

Therefore we have

$$|B_1| \cdot \dots \cdot |B_\ell| \cdot q^{d-\ell+1} \leq \left\lceil \frac{q}{n} \right\rceil^\ell \cdot q^{d-\ell+1}$$

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Therefore the probability of choosing $h_{\bar{a}}$ from A_ℓ is only

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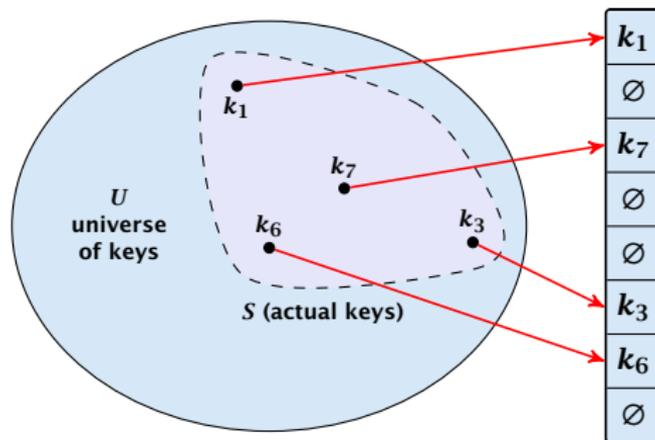
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This shows that the \mathcal{H} is $(e, d + 1)$ -universal.

The last step followed from $q \geq (d + 1)n$, and $\ell \leq d + 1$.

Perfect Hashing

Suppose that we **know** the set S of actual keys (no insert/no delete). Then we may want to design a **simple** hash-function that maps all these keys to different memory locations.



Perfect Hashing

Let $m = |S|$. We could simply choose the hash-table size very large so that we don't get any collisions.

Using a universal hash-function the expected number of collisions is

$$E[\#\text{Collisions}] = \binom{m}{2} \cdot \frac{1}{n} .$$

If we choose $n = m^2$ the expected number of collisions is strictly less than $\frac{1}{2}$.

Can we get an upper bound on the probability of having collisions?

The probability of having 1 or more collisions can be at most $\frac{1}{2}$ as otherwise the expectation would be larger than $\frac{1}{2}$.

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We can find such a hash-function by a few trials.

However, a hash-table size of $n = m^2$ is very very high.

We construct a two-level scheme. We first use a hash-function that maps elements from S to m buckets.

Let m_j denote the number of items that are hashed to the j -th bucket. For each bucket we choose a second hash-function that maps the elements of the bucket into a table of size m_j^2 . The second function can be chosen such that all elements are mapped to different locations.

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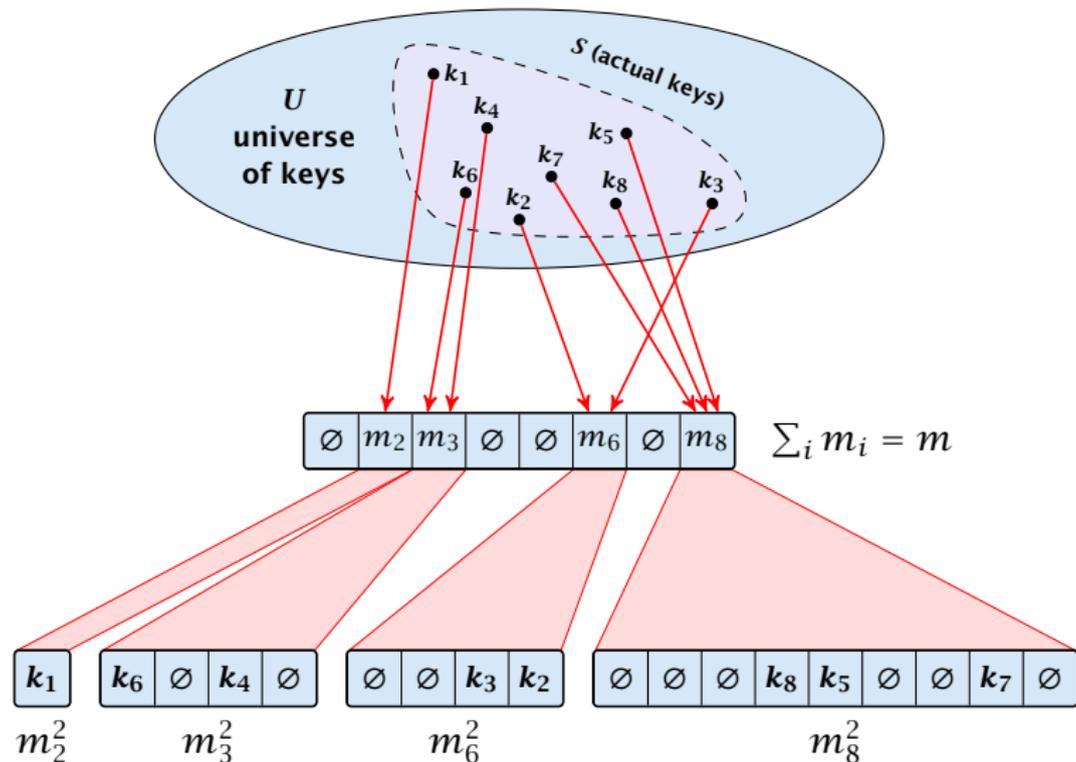
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$$= 2 \binom{m}{2} \frac{1}{m} + m = 2m - 1 .$$

Perfect Hashing

We need only $\mathcal{O}(m)$ time to construct a hash-function h with $\sum_j m_j^2 = \mathcal{O}(4m)$, because with probability at least $1/2$ a random function from a universal family will have this property.

Then we construct a hash-table h_j for every bucket. This takes expected time $\mathcal{O}(m_j)$ for every bucket. A random function h_j is collision-free with probability at least $1/2$. We need $\mathcal{O}(m_j)$ to test this.

We only need that the hash-functions are chosen from a universal family!!!

Cuckoo Hashing

Goal:

Try to generate a hash-table with constant worst-case search time in a dynamic scenario.

Two hash-tables $T_1[0, \dots, m-1]$ and $T_2[0, \dots, m-1]$, with hash functions h_1 and h_2 .

An object x is either stored at location $T_1[h_1(x)]$ or $T_2[h_2(x)]$.

Insertion/deletion clearly takes constant time if the above constraints are met.

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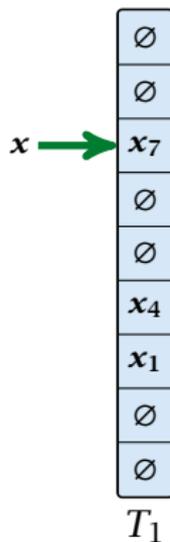
T_1



T_2

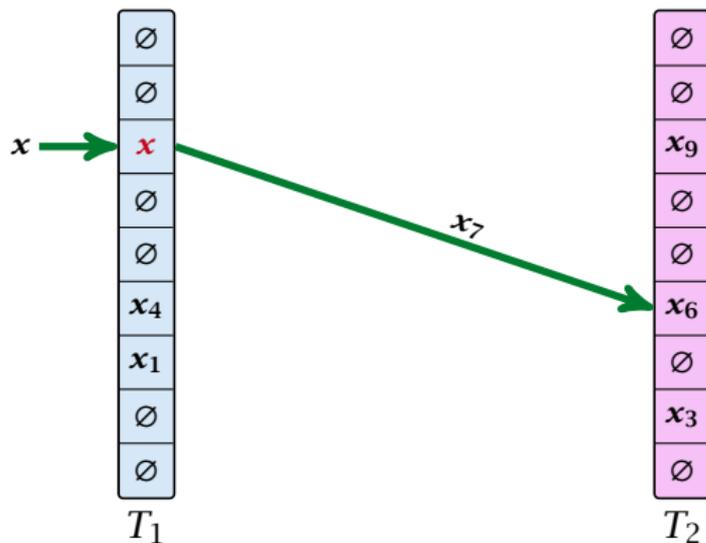
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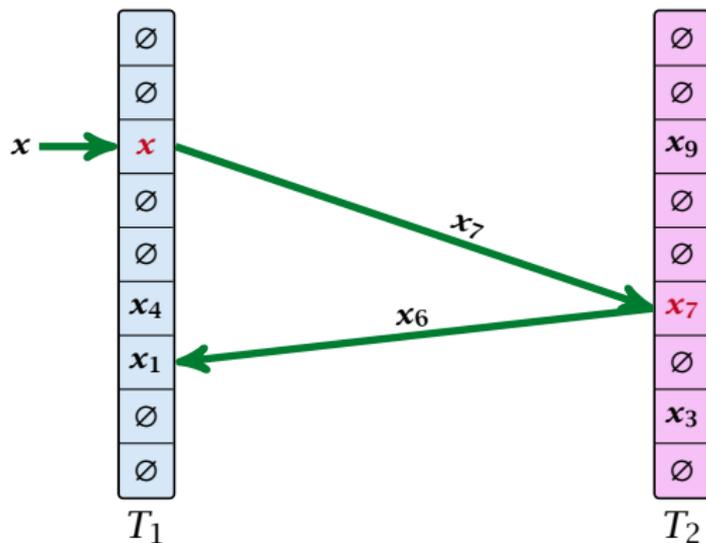
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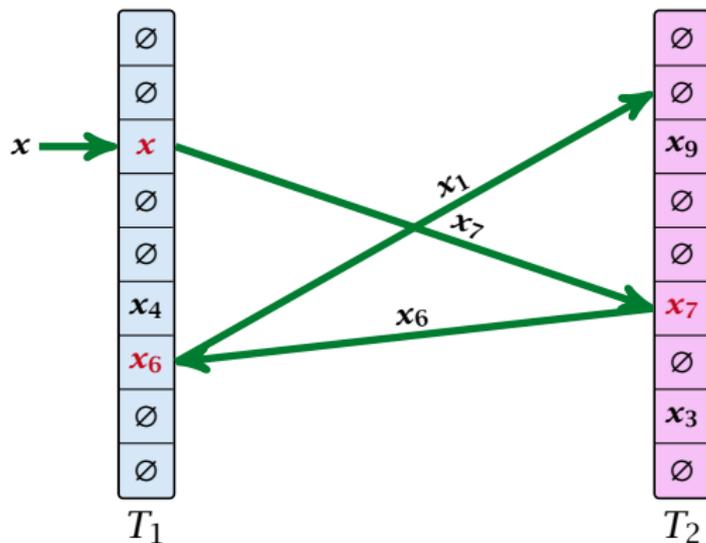
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Algorithm 17 Cuckoo-Insert(x)

```
1: if  $T_1[h_1(x)] = x \vee T_2[h_2(x)] = x$  then return  
2: steps  $\leftarrow$  1  
3: while steps  $\leq$  maxsteps do  
4:     exchange  $x$  and  $T_1[h_1(x)]$   
5:     if  $x = \text{null}$  then return  
6:     exchange  $x$  and  $T_2[h_2(x)]$   
7:     if  $x = \text{null}$  then return  
8:     steps  $\leftarrow$  steps + 1  
9: rehash() // change hash-functions; rehash everything  
10: Cuckoo-Insert( $x$ )
```

Cuckoo Hashing

- ▶ We call one iteration through the while-loop a **step** of the algorithm.
- ▶ We call a sequence of iterations through the while-loop without the termination condition becoming true a **phase** of the algorithm.
- ▶ We say a phase is **successful** if it is not terminated by the maxstep-condition, but the while loop is left because $x = \text{null}$.

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Cuckoo Hashing

What is the expected time for an insert-operation?

We first analyze the probability that we end-up in an infinite loop (that is then terminated after maxsteps steps).

Formally what is the probability to enter an infinite loop that touches s different keys?

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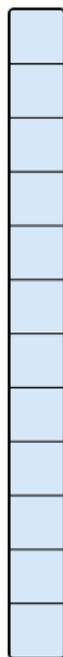
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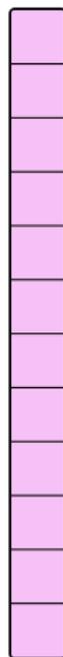
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Cuckoo Hashing: Insert

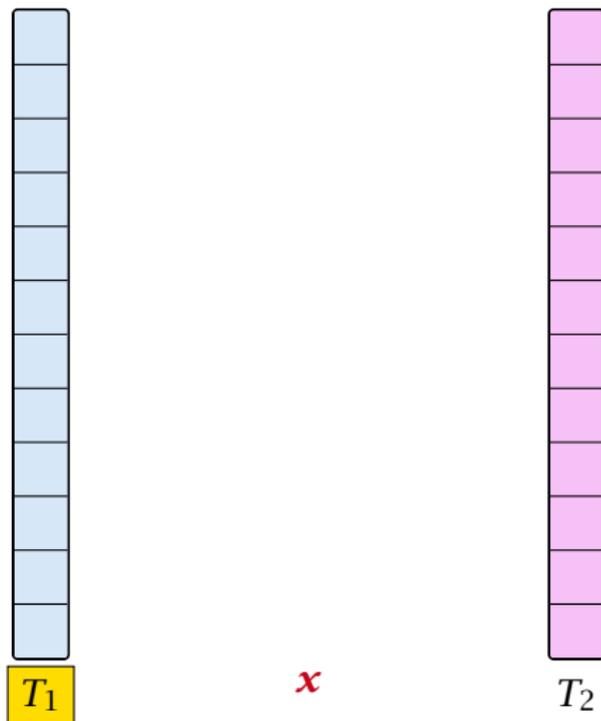


T_1

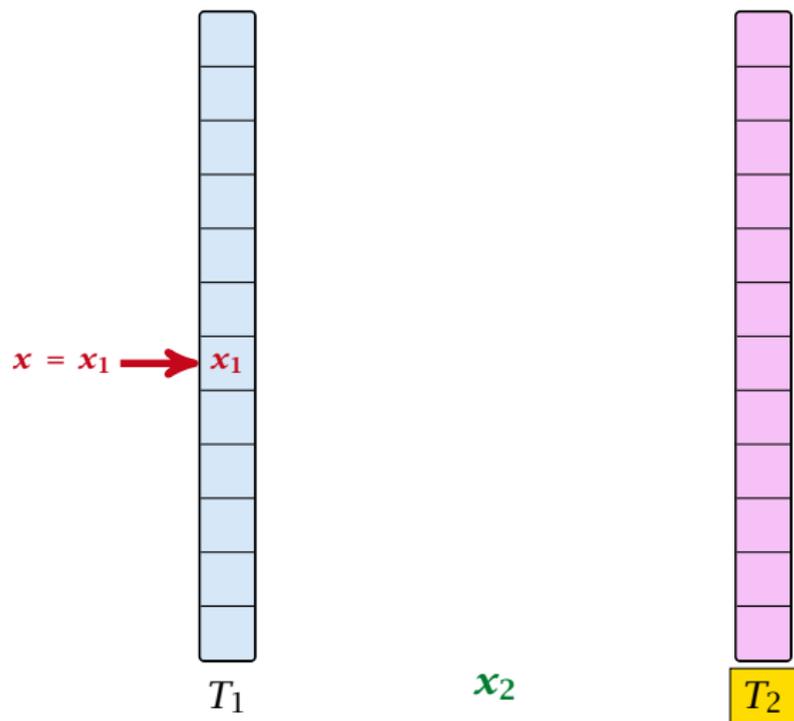


T_2

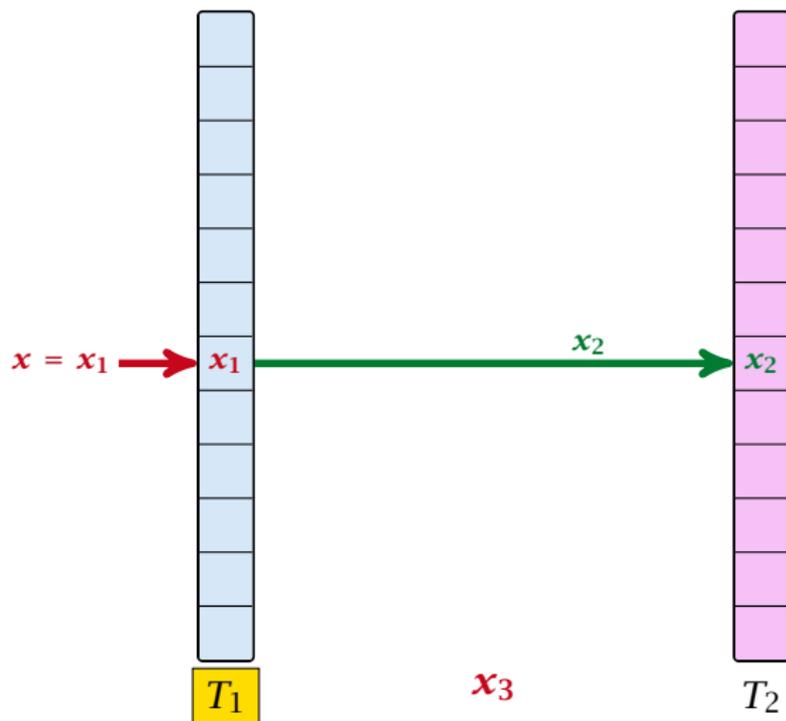
Cuckoo Hashing: Insert



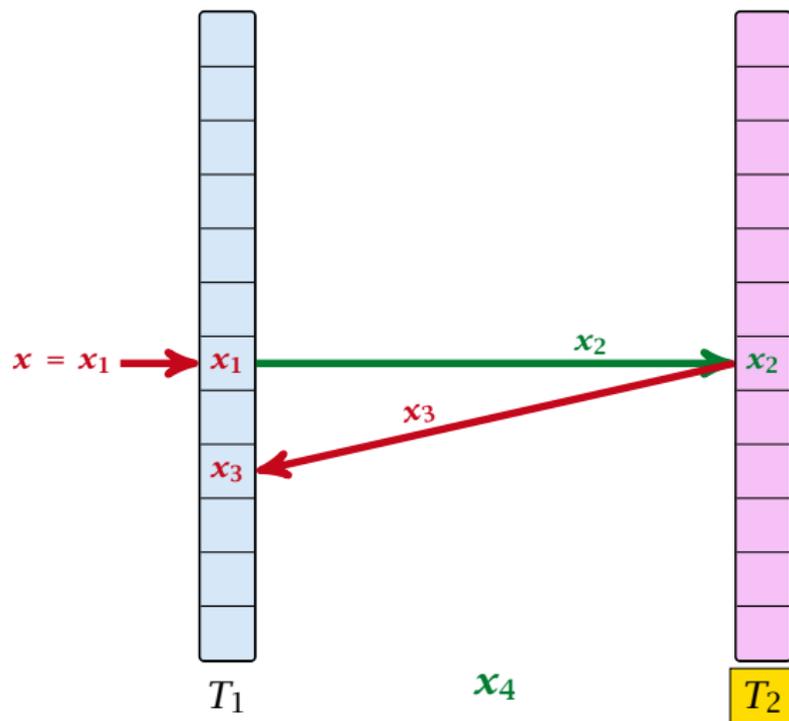
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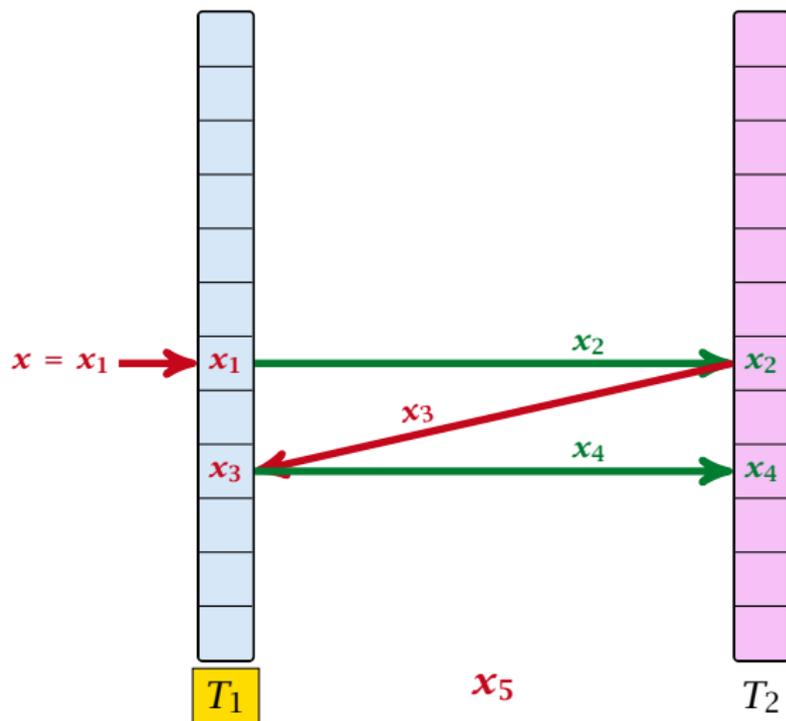
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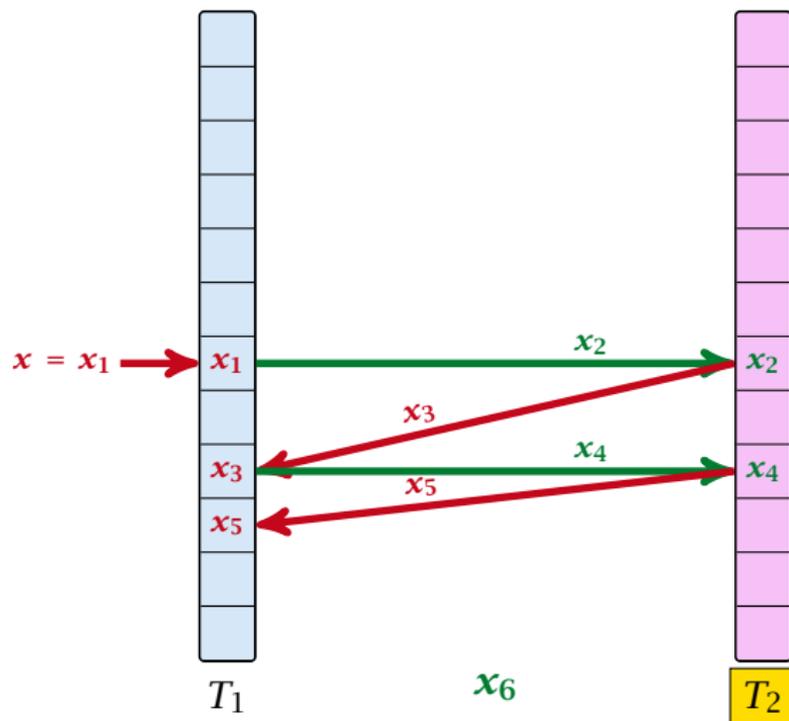
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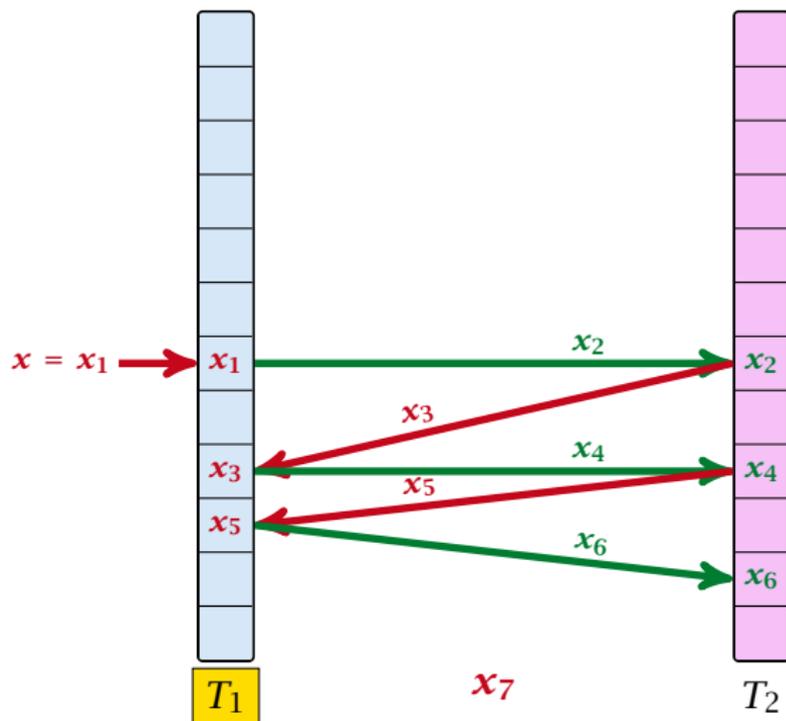
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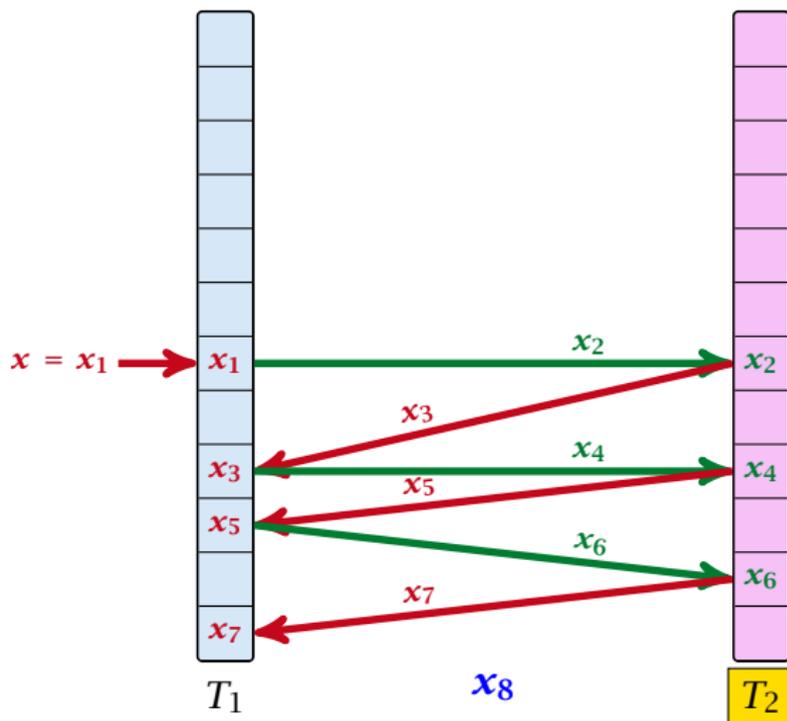
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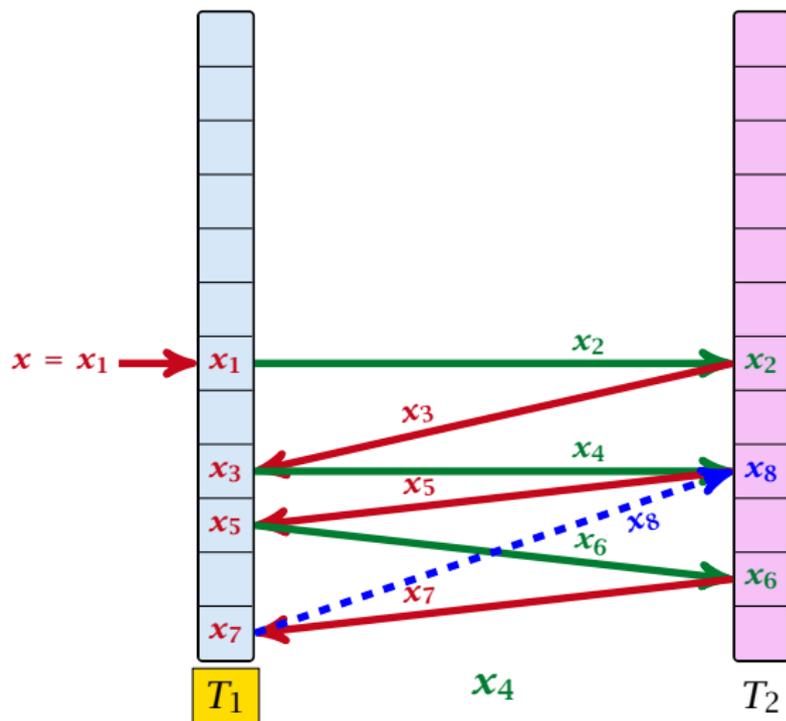
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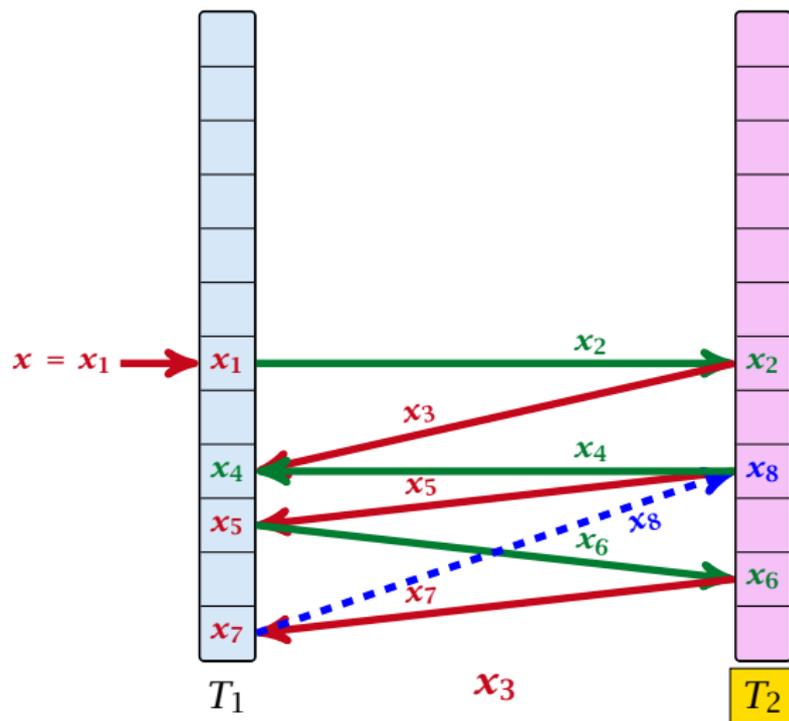
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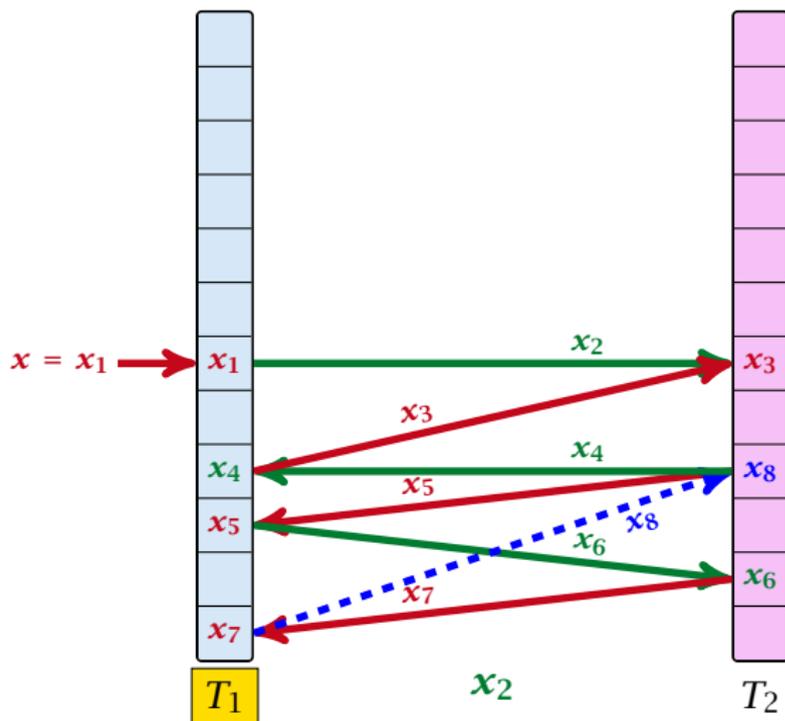
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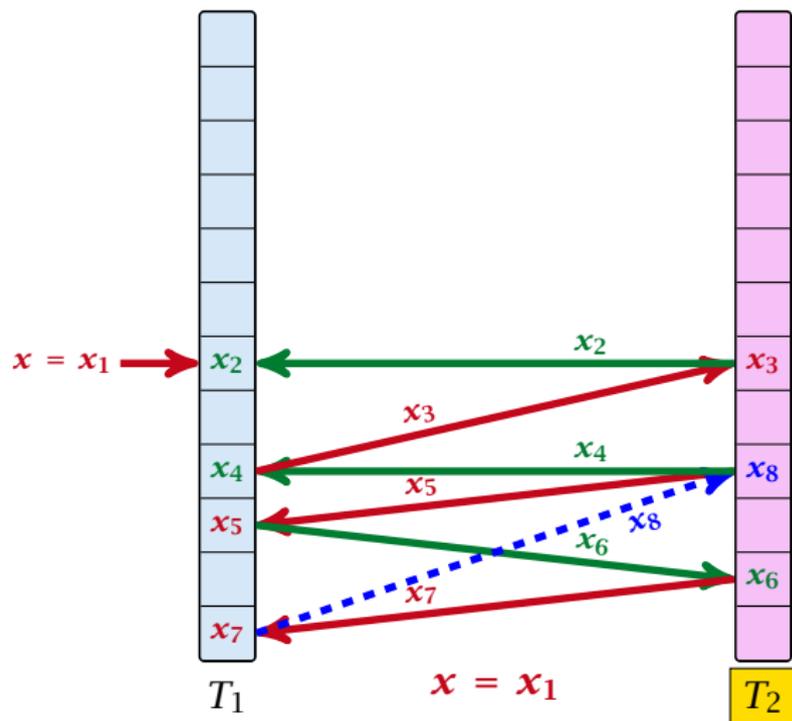
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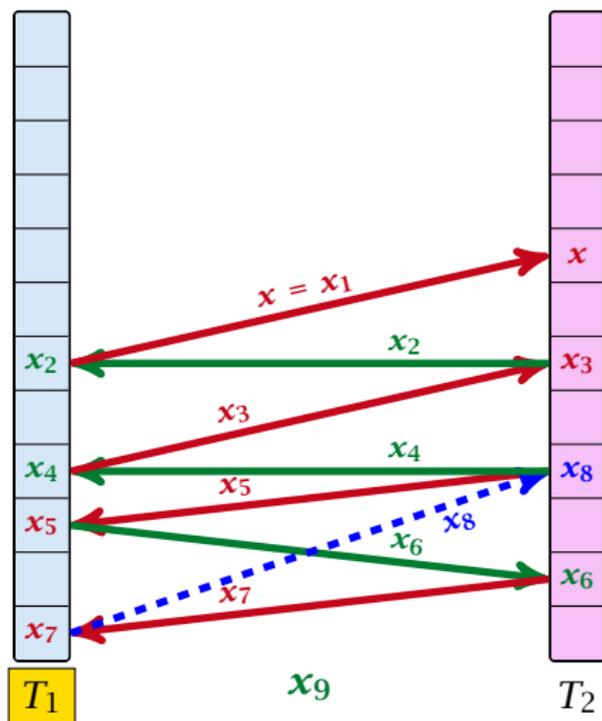
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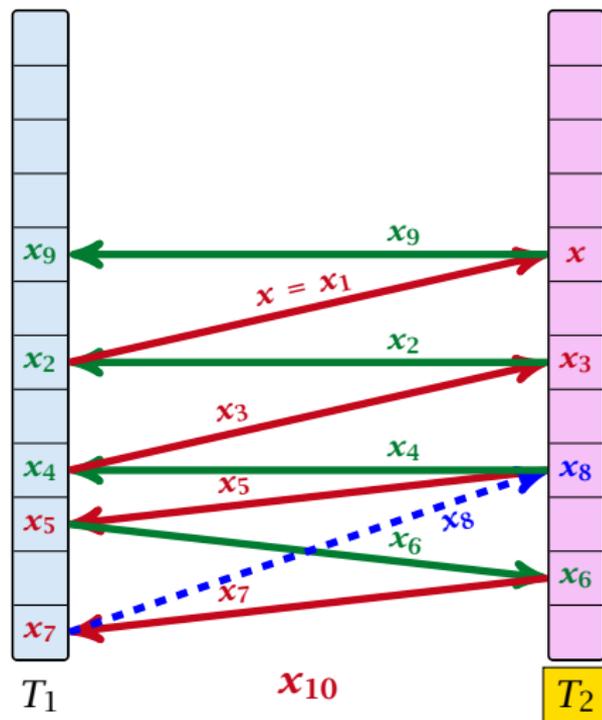
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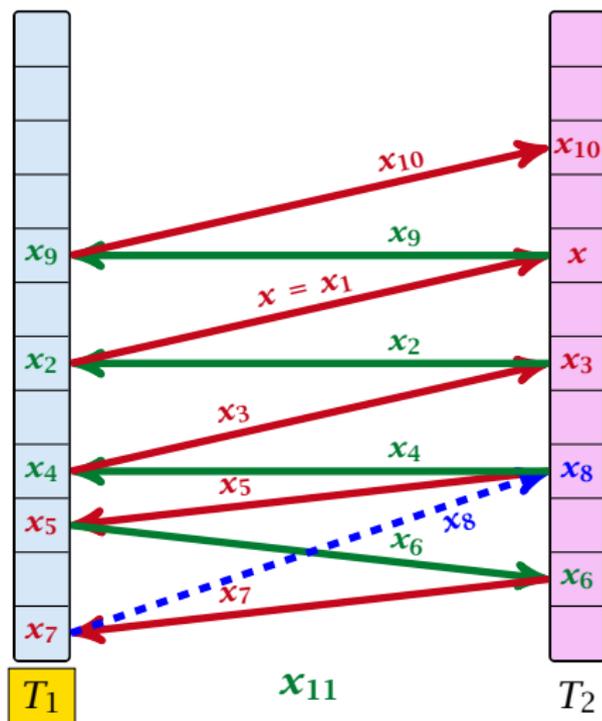
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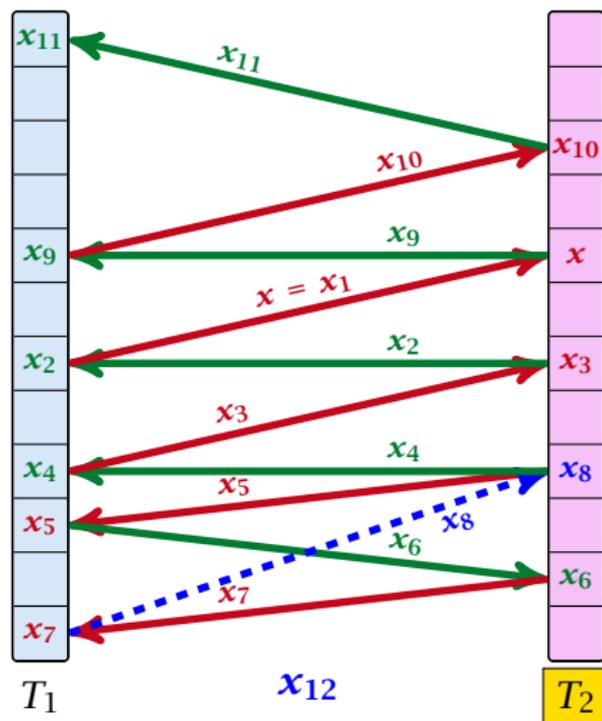
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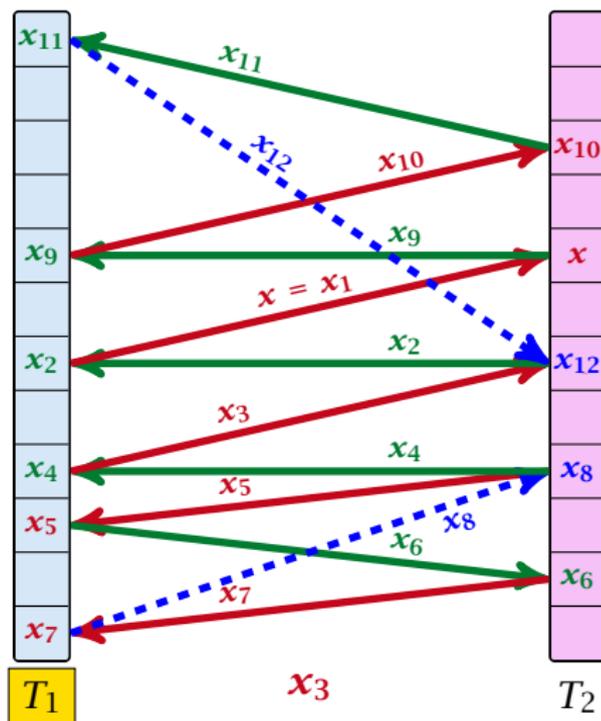
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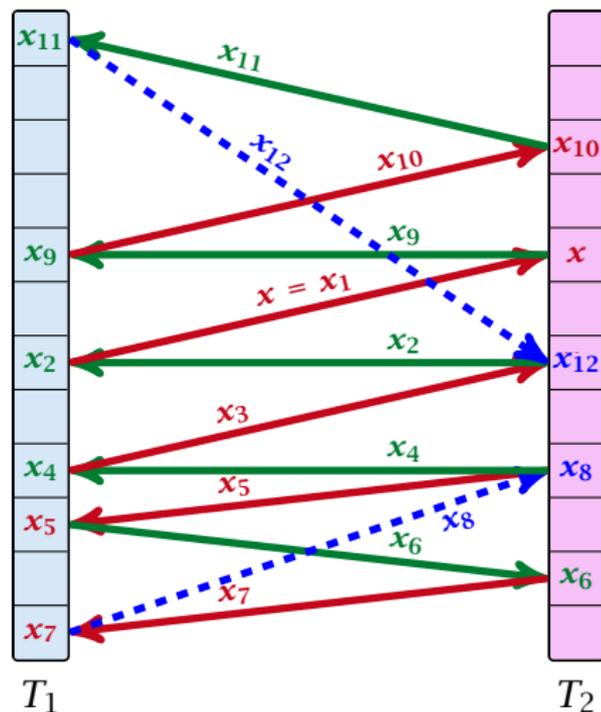
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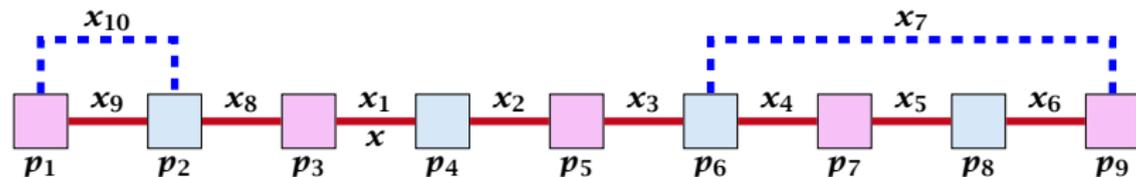
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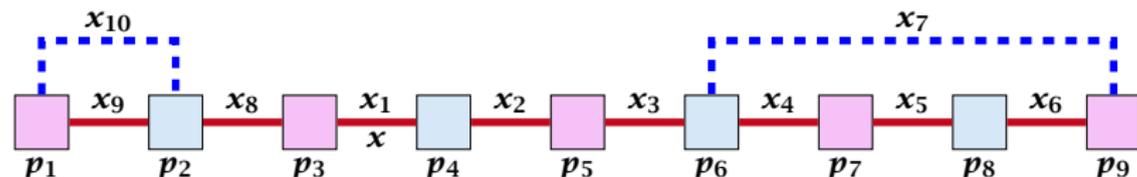


Cuckoo Hashing



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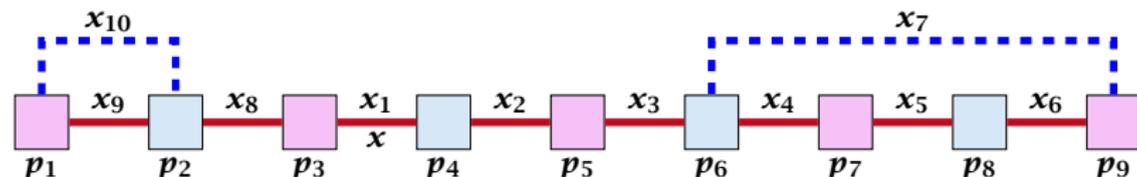
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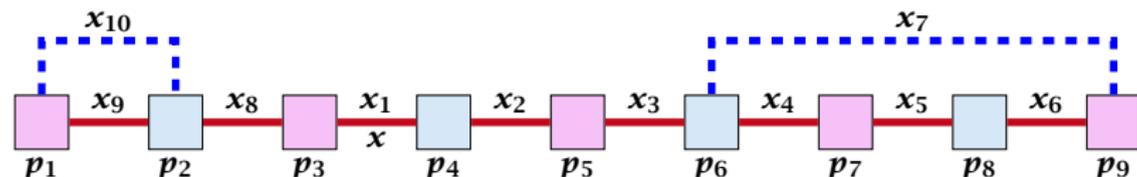
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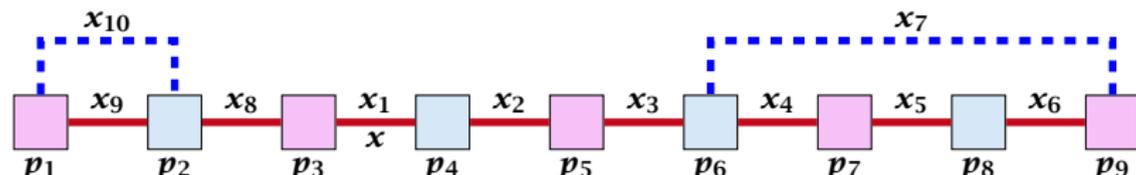
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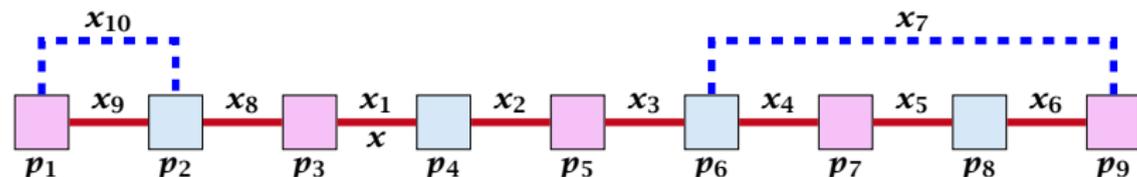
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$$h_1(x_\ell) = p_i \quad \text{and} \quad h_2(x_\ell) = p_j$$

Observation:

If during a phase the insert-procedure runs into a cycle there must exist an active cycle structure of size $s \geq 3$.

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Cuckoo Hashing

What is the probability that all keys in a cycle-structure of size s correctly map into their T_1 -cell?

This probability is at most $\frac{\mu}{n^s}$ since h_1 is a (μ, s) -independent hash-function.

What is the probability that all keys in the cycle-structure of size s correctly map into their T_2 -cell?

This probability is at most $\frac{\mu}{n^s}$ since h_2 is a (μ, s) -independent hash-function.

These events are independent.

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The number of cycle-structures of size s is at most

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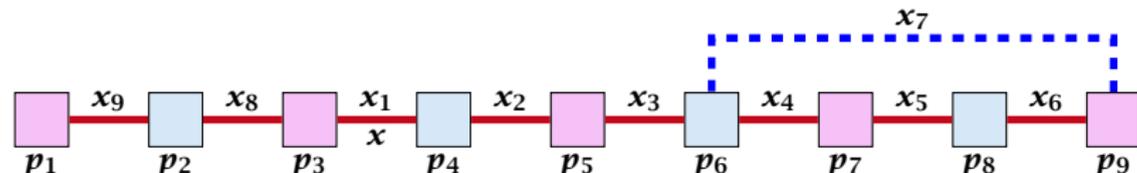
Hence,

$$\Pr[\text{cycle}] = \mathcal{O}\left(\frac{1}{m^2}\right).$$

Cuckoo Hashing

Now, we analyze the probability that a phase is not successful without running into a closed cycle.

Cuckoo Hashing



Sequence of visited keys:

$x = x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_3, x_2, x_1 = x, x_8, x_9, \dots$

Cuckoo Hashing

Consider the sequence of not necessarily distinct keys starting with x in the order that they are visited during the phase.

Lemma 22

If the sequence is of length p then there exists a sub-sequence of at least $\frac{p+2}{3}$ keys starting with x of distinct keys.

Cuckoo Hashing

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Lemma 22

*If the sequence is of length p then there exists a sub-sequence of at least $\frac{p+2}{3}$ keys starting with x of **distinct** keys.*

Cuckoo Hashing

Proof.

Let i be the number of keys (including x) that we see before the first repeated key. Let j denote the total number of distinct keys.

The sequence is of the form:

$$x = x_1 \rightarrow x_2 \rightarrow \cdots \rightarrow x_i \rightarrow x_r \rightarrow x_{r-1} \rightarrow \cdots \rightarrow x_1 \rightarrow x_{i+1} \rightarrow \cdots \rightarrow x_j$$

As $r \leq i - 1$ the length p of the sequence is

$$p = i + r + (j - i) \leq i + j - 1 .$$

Either sub-sequence $x_1 \rightarrow x_2 \rightarrow \cdots \rightarrow x_i$ or sub-sequence $x_1 \rightarrow x_{i+1} \rightarrow \cdots \rightarrow x_j$ has at least $\frac{p+2}{3}$ elements. □

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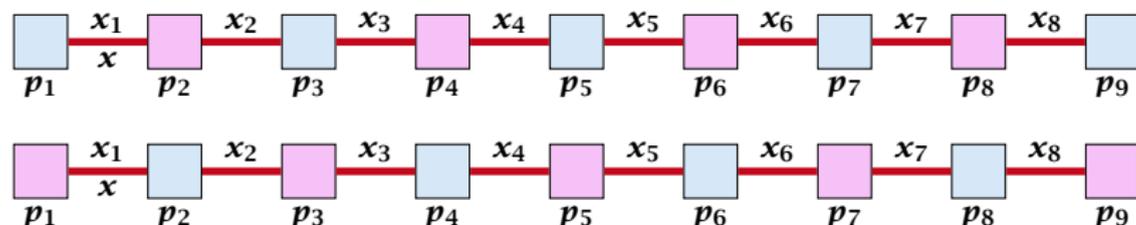
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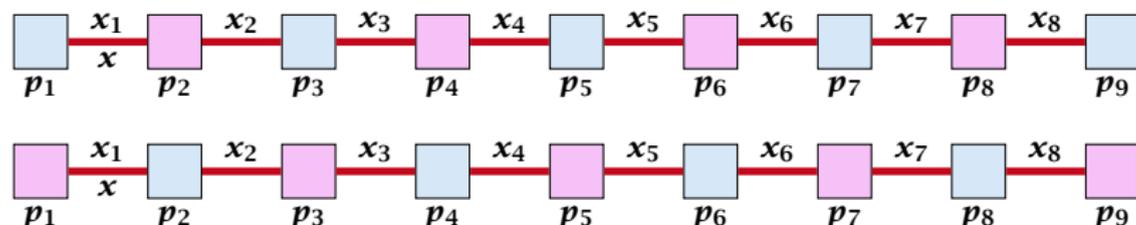
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Cuckoo Hashing



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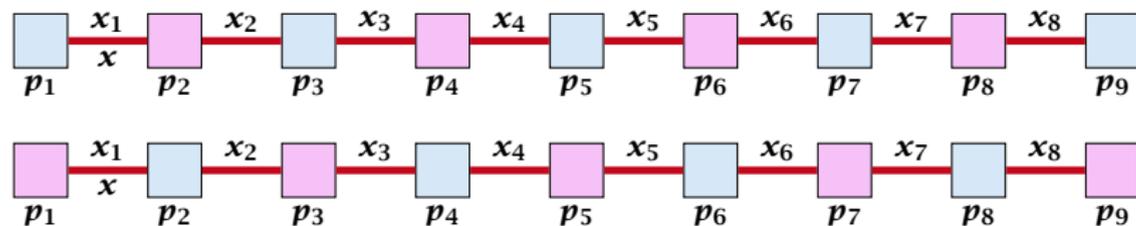
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A **path-structure of size s** is defined by

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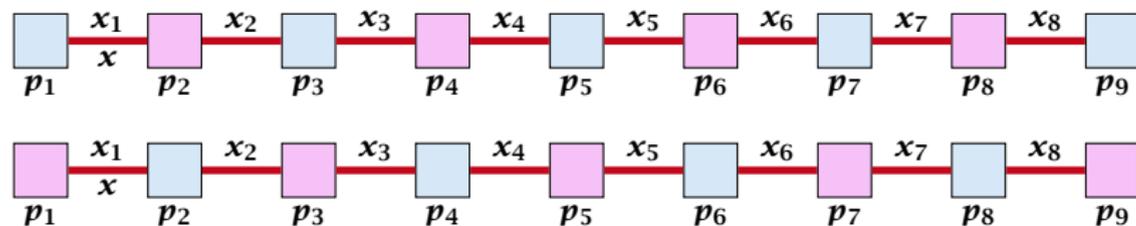
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Observation:

If a phase takes at least t steps without running into a cycle there must exist an active path-structure of size $(2t + 2)/3$.

Cuckoo Hashing

The probability that a given path-structure of size s is active is at most $\frac{\mu^2}{n^{2s}}$.

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The probability that there exists an active path-structure of size s is at most

$$2 \cdot n^{s+1} \cdot m^{s-1} \cdot \frac{\mu^2}{n^{2s}}$$

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$$\begin{aligned} 2 \cdot n^{s+1} \cdot m^{s-1} \cdot \frac{\mu^2}{n^{2s}} \\ \leq 2\mu^2 \left(\frac{m}{n}\right)^{s-1} \end{aligned}$$

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This gives $\text{maxsteps} = \Theta(\log m)$.

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So far we estimated

$$\Pr[\text{cycle}] \leq \mathcal{O}\left(\frac{1}{m^2}\right)$$

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A phase that is not successful induces cost $\mathcal{O}(m)$ for doing a complete rehash (this dominates the cost for the steps in the phase).

The probability that a phase is not successful is $p = \mathcal{O}(1/m^2)$ (probability $\mathcal{O}(1/m^2)$ of running into a cycle and probability $\mathcal{O}(1/m^2)$ of reaching maxsteps without running into a cycle).

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What kind of hash-functions do we need?

Since maxsteps is $\Theta(\log m)$ the largest size of a path-structure or cycle-structure contains just $\Theta(\log m)$ different keys.

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How do we make sure that $n \geq (1 + \epsilon)m$?

- ▶ Let $\alpha := 1/(1 + \epsilon)$.
- ▶ Keep track of the number of elements in the table. When $m \geq \alpha n$ we double n and do a complete re-hash (*table-expand*).
- ▶ Whenever m drops below $\alpha n/4$ we divide n by 2 and do a rehash (*table-shrink*).
- ▶ Note that right after a change in table-size we have $m = \alpha n/2$. In order for a table-expand to occur at least $\alpha n/2$ insertions are required. Similar, for a table-shrink at least $\alpha n/4$ deletions must occur.
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Lemma 23

Cuckoo Hashing has an expected constant insert-time and a worst-case constant search-time.

Note that the above lemma only holds if the fill-factor (number of keys/total number of hash-table slots) is at most $\frac{1}{2(1+c)}$.

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