

## Technique 2: Rounding the Dual Solution.

### Relaxation for Set Cover

**Primal:**

$$\begin{array}{ll} \min & \sum_{i \in I} w_i x_i \\ \text{s.t. } \forall u & \sum_{i: u \in S_i} x_i \geq 1 \\ & x_i \geq 0 \end{array}$$

**Dual:**

$$\begin{array}{ll} \max & \sum_{u \in U} y_u \\ \text{s.t. } \forall i & \sum_{u: u \in S_i} y_u \leq w_i \\ & y_u \geq 0 \end{array}$$

## Technique 2: Rounding the Dual Solution.

### Rounding Algorithm:

Let  $I$  denote the index set of sets for which the dual constraint is tight. This means for all  $i \in I$

$$\sum_{u:u \in S_i} y_u = w_i$$

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### Lemma 3

*The resulting index set is an  $f$ -approximation.*

#### **Proof:**

Every  $u \in U$  is covered.

- ▶ Suppose there is a  $u$  that is not covered.
- ▶ This means  $\sum_{u:u \in S_i} y_u < w_i$  for all sets  $S_i$  that contain  $u$ .
- ▶ But then  $y_u$  could be increased in the dual solution without violating any constraint. This is a contradiction to the fact that the dual solution is optimal.

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**Proof:**

$$\begin{aligned}\sum_{i \in I} w_i &= \sum_{i \in I} \sum_{u: u \in S_i} y_u \\ &= \sum_u |\{i \in I : u \in S_i\}| \cdot y_u \\ &\leq \sum_u f_u y_u \\ &\leq f \sum_u y_u \\ &\leq f \text{cost}(x^*) \\ &\leq f \cdot \text{OPT}\end{aligned}$$

Let  $I$  denote the solution obtained by the first rounding algorithm and  $I'$  be the solution returned by the second algorithm. Then

$$I \subseteq I' .$$

This means  $I'$  is never better than  $I$ .

- ▶ Suppose that we take  $S_i$  in the first algorithm. I.e.,  $i \in I$ .
- ▶ This means  $x_i \geq \frac{1}{f}$ .
- ▶ Because of **Complementary Slackness Conditions** the corresponding constraint in the dual must be tight.
- ▶ Hence, the second algorithm will also choose  $S_i$ .