Technique 2: Rounding the Dual Solution.

Relaxation for Set Cover

Primal:

$$\begin{array}{lrl} \min & \sum_{i \in I} w_i x_i & \\ \text{s.t. } \forall u & \sum_{i: u \in S_i} x_i & \geq 1 \\ & x_i & \geq 0 \end{array}$$

Dual:

$$\begin{array}{lrl} \max & \sum_{u \in U} y_u & \\ \text{s.t. } \forall i & \sum_{u: u \in S_i} y_u & \leq w_i \\ & y_u & \geq 0 \end{array}$$

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Rounding Algorithm:

Let I denote the index set of sets for which the dual constraint is tight. This means for all $i \in I$

$$\sum_{u: u \in S_i} y_u = w_i$$

Technique 2: Rounding the Dual Solution.

Lemma 3

The resulting index set is an f-approximation.

Proof:

Every $u \in U$ is covered.

- Suppose there is a u that is not covered.
- This means $\sum_{u: u \in S_i} y_u < w_i$ for all sets S_i that contain u.
- But then y_u could be increased in the dual solution without violating any constraint. This is a contradiction to the fact that the dual solution is optimal.

Technique 2: Rounding the Dual Solution.

Proof:

$$\begin{aligned} \sum_{i \in I} w_i &= \sum_{i \in I} \sum_{u: u \in S_i} y_u \\ &= \sum_u |\{i \in I : u \in S_i\}| \cdot y_u \\ &\leq \sum_u f_u y_u \\ &\leq f \sum_u y_u \\ &\leq f \operatorname{cost}(x^*) \\ &\leq f \cdot \mathrm{OPT} \end{aligned}$$

Let I denote the solution obtained by the first rounding algorithm and I' be the solution returned by the second algorithm. Then

$$I \subseteq I' \ .$$

This means I' is never better than I.

- Suppose that we take S_i in the first algorithm. I.e., $i \in I$.
- This means $x_i \geq \frac{1}{f}$.
- Because of Complementary Slackness Conditions the corresponding constraint in the dual must be tight.
- Hence, the second algorithm will also choose S_i.

Technique 3: The Primal Dual Method

The previous two rounding algorithms have the disadvantage that it is necessary to solve the LP. The following method also gives an f-approximation without solving the LP.

For estimating the cost of the solution we only required two properties.

1. The solution is dual feasible and, hence,

$$\sum_u y_u \leq \text{cost}(x^*) \leq \text{OPT}$$

   where x^* is an optimum solution to the primal LP.
2. The set I contains only sets for which the dual inequality is tight.

Of course, we also need that I is a cover.

Technique 3: The Primal Dual Method

Algorithm 1 PrimalDual

1: $y \leftarrow 0$
2: $I \leftarrow \emptyset$
3: **while** exists $u \notin \bigcup_{i \in I} S_i$ **do**
4:     increase dual variable y_i until constraint for some new set S_ℓ becomes tight
5:     $I \leftarrow I \cup \{\ell\}$

Technique 4: The Greedy Algorithm

Algorithm 1 Greedy

1: $I \leftarrow \emptyset$
2: $\hat{S}_j \leftarrow S_j$ for all j
3: **while** I not a set cover **do**
4:     $\ell \leftarrow \arg\min_{j:\hat{S}_j \neq 0} \frac{w_j}{|\hat{S}_j|}$
5:     $I \leftarrow I \cup \{\ell\}$
6:     $\hat{S}_j \leftarrow \hat{S}_j - S_\ell$ for all j

In every round the Greedy algorithm takes the set that covers remaining elements in the most cost-effective way.

We choose a set such that the ratio between cost and still uncovered elements in the set is minimized.