A crucial ingredient for the design and analysis of approximation algorithms is a technique to obtain an upper bound (for maximization problems) or a lower bound (for minimization problems).

Therefore Linear Programs or Integer Linear Programs play a vital role in the design of many approximation algorithms.

Definition 2

An Integer Linear Program or Integer Program is a Linear Program in which all variables are required to be integral.

Definition 3

A Mixed Integer Program is a Linear Program in which a subset of the variables are required to be integral.

Many important combinatorial optimization problems can be formulated in the form of an Integer Program.

Note that solving Integer Programs in general is NP-complete!

Set Cover

Given a ground set U, a collection of subsets $S_1, \dots, S_k \subseteq U$, where the i-th subset S_i has weight/cost w_i. Find a collection $I \subseteq \{1, \dots, k\}$ such that

$$\forall u \in U \exists i \in I : \; u \in S_i \quad \text{(every element is covered)}$$

and

$$\sum_{i \in I} w_i \quad \text{is minimized.}$$

IP-Formulation of Set Cover

$$\begin{array}{lrrcc}
\min & & \sum_i w_i x_i & & \\
\text{s.t.} & \forall u \in U & \sum_{i:u\in S_i} x_i & \geq & 1 \\
 & \forall i \in \{1,\dots,k\} & x_i & \geq & 0 \\
 & \forall i \in \{1,\dots,k\} & x_i & \text{integral} &
\end{array}$$

IP-Formulation of Set Cover

$$
\begin{array}{llcl}
\min & & \sum_i w_i x_i & \\
\text{s.t.} & \forall u \in U & \sum_{i:u\in S_i} x_i & \geq & 1 \\
 & \forall i \in \{1,\dots,k\} & x_i & \in & \{0,1\}
\end{array}
$$

Vertex Cover

Given a graph $G = (V, E)$ and a weight w_v for every node. Find a vertex subset $S \subseteq V$ of minimum weight such that every edge is incident to at least one vertex in S.

IP-Formulation of Vertex Cover

$$
\begin{array}{llrcl}
\min & & \sum_{v\in V} w_v x_v & & \\
\text{s.t.} & \forall e=(i,j)\in E & x_i + x_j & \geq & 1 \\
& \forall v\in V & x_v & \in & \{0,1\}
\end{array}
$$

Maximum Weighted Matching

Given a graph $G = (V, E)$, and a weight w_e for every edge $e \in E$. Find a subset of edges of maximum weight such that no vertex is incident to more than one edge.

$$
\begin{array}{llrcl}
\max & & \sum_{e \in E} w_e x_e & & \\
\text{s.t.} & \forall v \in V & \sum_{e: v \in e} x_e & \leq & 1 \\
 & \forall e \in E & x_e & \in & \{0, 1\}
\end{array}
$$

Maximum Independent Set

Given a graph $G = (V, E)$, and a weight w_v for every node $v \in V$. Find a subset $S \subseteq V$ of nodes of maximum weight such that no two vertices in S are adjacent.

$$
\begin{array}{llrcl}
\max & & \sum_{v \in V} w_v x_v & & \\
\text{s.t.} & \forall e = (i,j) \in E & x_i + x_j & \le & 1 \\
 & \forall v \in V & x_v & \in & \{0,1\}
\end{array}
$$

Knapsack

Given a set of items $\{1,\ldots,n\}$, where the i-th item has weight w_i and profit p_i, and given a threshold K. Find a subset $I \subseteq \{1,\ldots,n\}$ of items of total weight at most K such that the profit is maximized.

$$
\begin{array}{llrcl}
\max & & \sum_{i=1}^{n} p_i x_i & & \\
\text{s.t.} & & \sum_{i=1}^{n} w_i x_i & \leq & K \\
& \forall i \in \{1,\ldots,n\} & x_i & \in & \{0,1\}
\end{array}
$$

Facility Location

Given a set L of (possible) locations for placing facilities and a set C of customers together with cost functions $s : C \times L \to \mathbb{R}^+$ and $o : L \to \mathbb{R}^+$ find a set of facility locations F together with an assignment $\phi : C \to F$ of customers to open facilities such that

$$\sum_{f \in F} o(f) + \sum_{c} s(c, \phi(c))$$

is minimized.

In the metric facility location problem we have

$$s(c, f) \le s(c, f') + s(c', f) + s(c', f') \enspace .$$

Facility Location

$$
\begin{array}{llrcl}
\min & & \multicolumn{3}{l}{\sum_f x_f o(f) + \sum_c \sum_f y_{cf} s(c,f)} \\
\text{s.t.} & \forall c \in C, f \in L & y_{cf} & \leq & x_f \\
 & \forall c \in C & \sum_f y_{cf} & \geq & 1 \\
 & \forall f \in L & x_f & \in & \{0,1\} \\
 & \forall c \in C, f \in L & y_{cf} & \in & \{0,1\}
\end{array}
$$

- $y_+cf \leq x_f$ ensures that we cannot assign customers to facilities that are not open.
- $\sum_f y_{cf} \geq 1$ ensures that every customer is assigned to a facility.

Relaxations

Definition 4
A linear program LP is a relaxation of an integer program IP if any feasible solution for IP is also feasible for LP and if the objective values of these solutions are identical in both programs.

We obtain a relaxation for all examples by writing $x_i \in [0, 1]$ instead of $x_i \in \{0, 1\}$.

By solving a relaxation we obtain an upper bound for a maximization problem and a lower bound for a minimization problem.