

# Duality

How do we get an upper bound to a maximization LP?

$$\begin{aligned} \max \quad & 13a + 23b \\ \text{s.t.} \quad & 5a + 15b \leq 480 \\ & 4a + 4b \leq 160 \\ & 35a + 20b \leq 1190 \\ & a, b \geq 0 \end{aligned}$$

Note that a lower bound is easy to derive. Every choice of  $a, b \geq 0$  gives us a lower bound (e.g.  $a = 12, b = 28$  gives us a lower bound of 800).

If you take a conic combination of the rows (multiply the  $i$ -th row with  $y_i \geq 0$ ) such that  $\sum_i y_i a_{ij} \geq c_j$  then  $\sum_i y_i b_i$  will be an upper bound.

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## Definition 2

Let  $z = \max\{c^t x \mid Ax \geq b, x \geq 0\}$  be a linear program  $P$  (called the primal linear program).

The linear program  $D$  defined by

$$w = \min\{b^t y \mid A^t y \geq c, y \geq 0\}$$

is called the **dual problem**.

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## Lemma 3

*The dual of the dual problem is the primal problem.*

Proof:

*Let  $(x^*, y^*)$  be an optimal solution of the primal problem. Then  $(x^*, y^*)$  is also an optimal solution of the dual problem.*

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- ▶  $w = \min\{b^t y \mid A^t y \geq c, y \geq 0\}$
- ▶  $w = \max\{-b^t y \mid -A^t y \leq -c, y \geq 0\}$

The dual problem is

- ▶  $\max\{c^t x \mid Ax = b, x \geq 0\}$
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# Weak Duality

Let  $z = \max\{c^t x \mid Ax \leq b, x \geq 0\}$  and  
 $w = \min\{b^t y \mid A^t y \geq c, y \geq 0\}$  be a primal dual pair.

$x$  is primal feasible iff  $x \in \{x \mid Ax \leq b, x \geq 0\}$

$y$  is dual feasible, iff  $y \in \{y \mid A^t y \geq c, y \geq 0\}$ .

## Theorem 4 (Weak Duality)

*Let  $\hat{x}$  be primal feasible and let  $\hat{y}$  be dual feasible. Then*

$$c^t \hat{x} \leq z \leq w \leq b^t \hat{y} .$$

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$$A^t \hat{y} \geq c \Rightarrow \hat{x}^t A^t \hat{y} \geq \hat{x}^t c \quad (\hat{x} \geq 0)$$

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Since, there exists primal feasible  $\hat{x}$  with  $c^t \hat{x} = z$ , and dual feasible  $\hat{y}$  with  $b^t \hat{y} = w$  we get  $z \leq w$ .

If  $P$  is unbounded then  $D$  is infeasible.

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The following linear programs form a primal dual pair:

$$z = \max\{c^t x \mid Ax = b, x \geq 0\}$$
$$w = \min\{b^t y \mid A^t y \geq c\}$$

This means for computing the dual of a standard form LP, we do not have non-negativity constraints for the dual variables.

# Proof

**Primal:**

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# Proof of Optimality Criterion for Simplex

Suppose that we have a basic feasible solution with **reduced cost**

$$\tilde{c} = c^t - c_B^t A_B^{-1} A \leq 0$$

This is equivalent to  $A^t (A_B^{-1})^t c_B \geq c$

$y^* = (A_B^{-1})^t c_B$  is solution to the **dual**  $\min\{b^t y \mid A^t y \geq c\}$ .

Hence, the solution is optimal.

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# Strong Duality

## Theorem 5 (Strong Duality)

*Let  $P$  and  $D$  be a primal dual pair of linear programs, and let  $z^*$  and  $w^*$  denote the optimal solution to  $P$  and  $D$ , respectively.*

*Then*

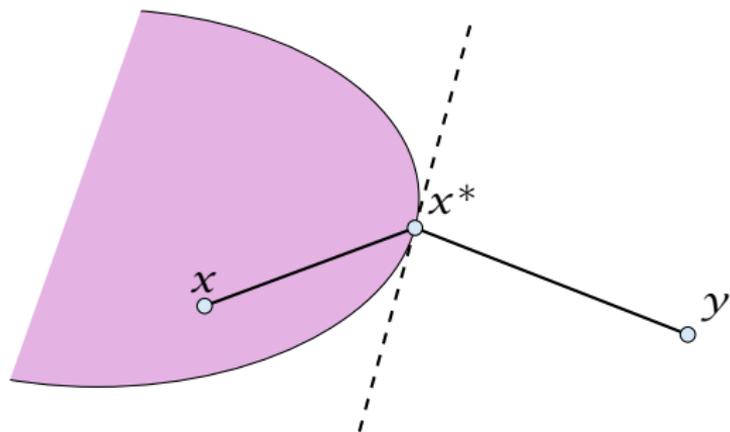
$$z^* = w^*$$

## Lemma 6 (Weierstrass)

*Let  $X$  be a compact set and let  $f(x)$  be a continuous function on  $X$ . Then  $\min\{f(x) : x \in X\}$  exists.*

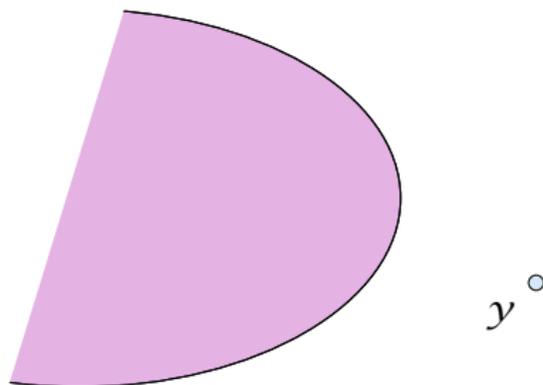
## Lemma 7 (Projection Lemma)

Let  $X \subseteq \mathbb{R}^m$  be a non-empty convex set, and let  $y \notin X$ . Then there exist  $x^* \in X$  with minimum distance from  $y$ . Moreover for all  $x \in X$  we have  $(y - x^*)^t(x - x^*) \leq 0$ .



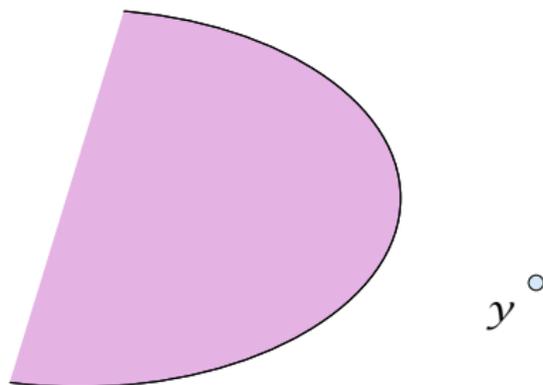
# Proof of the Projection Lemma

- ▶ Define  $f(x) = \|y - x\|$ .
- ▶ We want to apply Weierstrass but  $X$  may not be bounded.
- ▶  $X \neq \emptyset$ . Hence, there exists  $x' \in X$ .
- ▶ Define  $X' = \{x \in X \mid \|y - x\| \leq \|y - x'\|\}$ . This set is closed and bounded.
- ▶ Applying Weierstrass gives the existence.



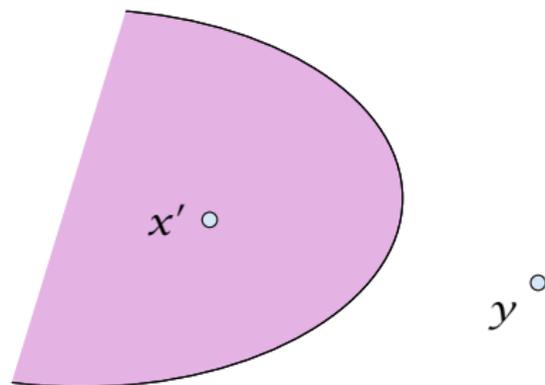
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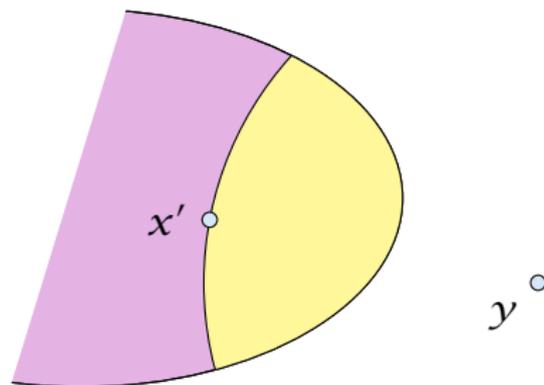
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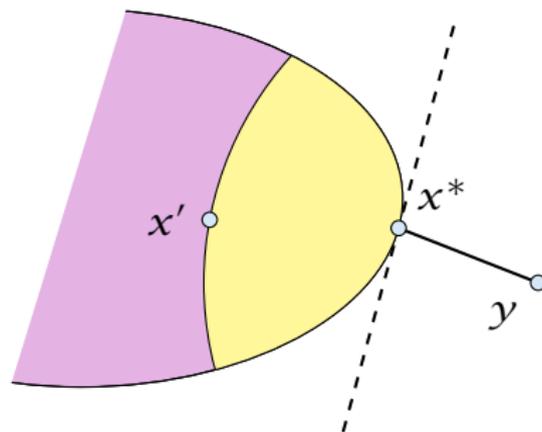
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- ▶ Define  $f(x) = \|y - x\|$ .
- ▶ We want to apply Weierstrass but  $X$  may not be bounded.
- ▶  $X \neq \emptyset$ . Hence, there exists  $x' \in X$ .
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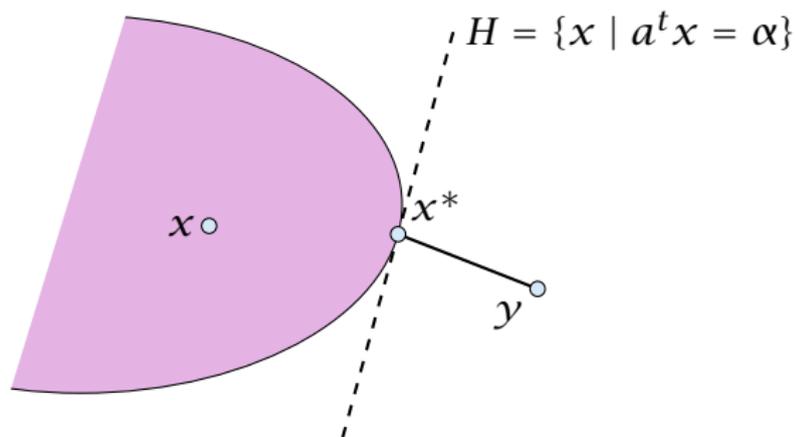
Letting  $\epsilon \rightarrow 0$  gives the result.

## Theorem 8 (Separating Hyperplane)

Let  $X \subseteq \mathbb{R}^m$  be a non-empty closed convex set, and let  $y \notin X$ . Then there exists a *separating hyperplane*  $\{x \in \mathbb{R}^m : a^t x = \alpha\}$  where  $a \in \mathbb{R}^m$ ,  $\alpha \in \mathbb{R}$  that *separates*  $y$  from  $X$ . ( $a^t y < \alpha$ ;  $a^t x \geq \alpha$  for all  $x \in X$ )

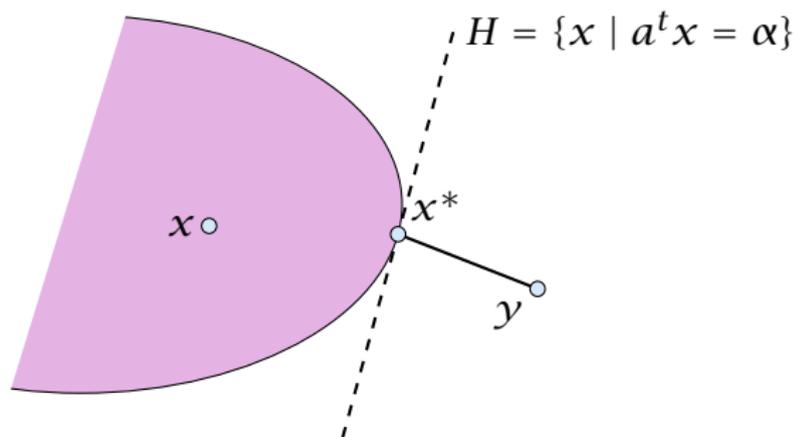
# Proof of the Hyperplane Lemma

- ▶ Let  $x^* \in X$  be closest point to  $y$  in  $X$ .
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- ▶ Choose  $a = (x^* - y)$  and  $\alpha = a^t x^*$ .
- ▶ For  $x \in X$ :  $a^t(x - x^*) \geq 0$ , and, hence,  $a^t x \geq \alpha$ .
- ▶ Also,  $a^t y = a^t(x^* - a) = \alpha - \|a\|^2 < \alpha$



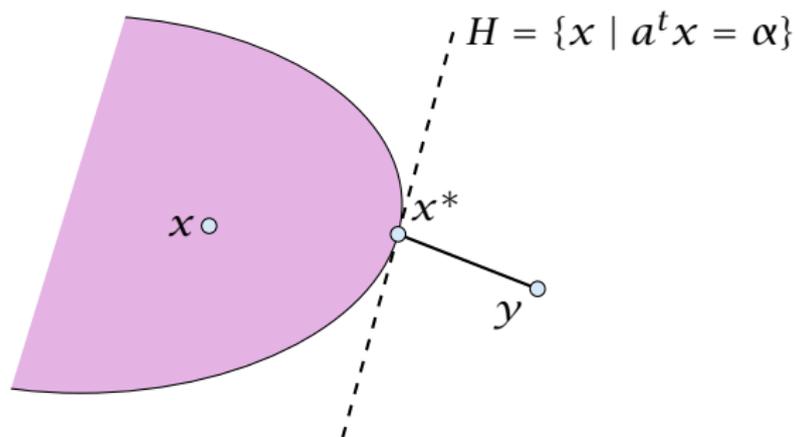
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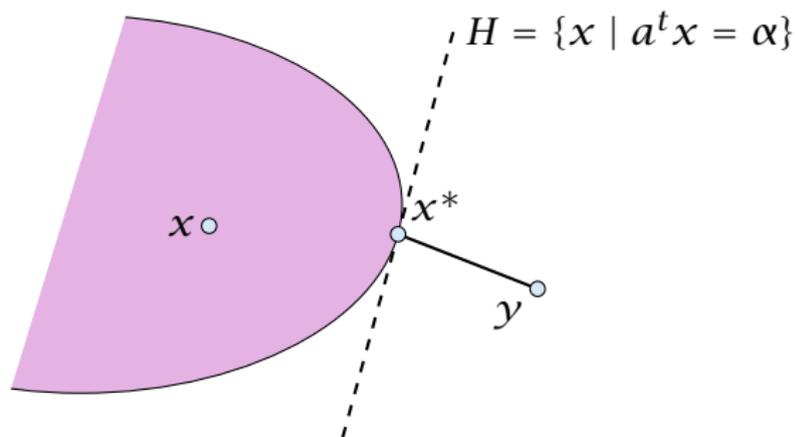
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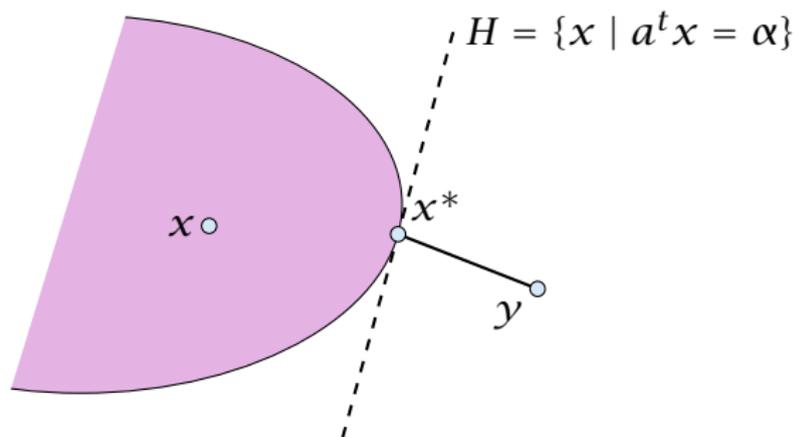
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## Lemma 9 (Farkas Lemma)

Let  $A$  be an  $m \times n$  matrix,  $b \in \mathbb{R}^m$ . Then *exactly one* of the following statements holds.

1.  $\exists x \in \mathbb{R}^n$  with  $Ax = b$ ,  $x \geq 0$
2.  $\exists y \in \mathbb{R}^m$  with  $A^t y \geq 0$ ,  $b^t y < 0$

Assume  $\hat{x}$  satisfies 1. and  $\hat{y}$  satisfies 2. Then

$$0 > \hat{y}^t b = \hat{y}^t A \hat{x} \geq 0$$

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Now, assume that 1. does not hold.

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## Lemma 10 (Farkas Lemma; different version)

Let  $A$  be an  $m \times n$  matrix,  $b \in \mathbb{R}^m$ . Then exactly one of the following statements holds.

1.  $\exists x \in \mathbb{R}^n$  with  $Ax \leq b, x \geq 0$
2.  $\exists y \in \mathbb{R}^m$  with  $A^t y \geq 0, b^t y < 0, y \geq 0$

Rewrite the conditions:

1.  $\exists x \in \mathbb{R}^n$  with  $\begin{bmatrix} A & I \end{bmatrix} \cdot \begin{bmatrix} x \\ s \end{bmatrix} = b, x \geq 0, s \geq 0$
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# Proof of Strong Duality

$$P: z = \max\{c^t x \mid Ax \leq b, x \geq 0\}$$

$$D: w = \min\{b^t y \mid A^t y \geq c, y \geq 0\}$$

## Theorem 11 (Strong Duality)

*Let  $P$  and  $D$  be a primal dual pair of linear programs, and let  $z$  and  $w$  denote the optimal solution to  $P$  and  $D$ , respectively (i.e.,  $P$  and  $D$  are non-empty). Then*

$$z = w .$$

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$$\begin{array}{l} \exists x \in \mathbb{R}^n \\ \text{s.t.} \quad Ax \leq b \\ \quad \quad -c^t x \leq -\alpha \\ \quad \quad x \geq 0 \end{array}$$

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From the definition of  $\alpha$  we know that the first system is infeasible; hence the second must be feasible.

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If the solution  $y, z$  has  $z = 0$  we have that

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is feasible. By Farkas lemma this gives that LP  $P$  is infeasible. Contradiction to the assumption of the lemma.

# Proof of Strong Duality

Hence, there exists a solution  $y, z$  with  $z > 0$ .

We can rescale this solution (scaling both  $y$  and  $z$ ) s.t.  $z = 1$ .

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# Fundamental Questions

## Definition 12 (Linear Programming Problem (LP))

Let  $A \in \mathbb{Q}^{m \times n}$ ,  $b \in \mathbb{Q}^m$ ,  $c \in \mathbb{Q}^n$ ,  $\alpha \in \mathbb{Q}$ . Does there exist  $x \in \mathbb{Q}^n$  s.t.  $Ax = b$ ,  $x \geq 0$ ,  $c^t x \geq \alpha$ ?

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- ▶ Is LP in NP?
- ▶ Is LP in co-NP? yes!
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- ▶ Given a primal maximization problem  $P$  and a parameter  $\alpha$ . Suppose that  $\alpha > \text{opt}(P)$ .
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- ▶ Is LP in NP?
- ▶ Is LP in co-NP? yes!
- ▶ Is LP in P?

### Proof:

- ▶ Given a primal maximization problem  $P$  and a parameter  $\alpha$ . Suppose that  $\alpha > \text{opt}(P)$ .
- ▶ We can prove this by providing an optimal basis for the dual.
- ▶ A verifier can check that the associated dual solution fulfills all dual constraint and that it has dual cost  $< \alpha$ .

# Complementary Slackness

## Lemma 13

Assume a linear program  $P = \max\{c^t x \mid Ax \leq b; x \geq 0\}$  has solution  $x^*$  and its dual  $D = \min\{b^t y \mid A^t y \geq c; y \geq 0\}$  has solution  $y^*$ .

1. If  $x_j^* > 0$  then the  $j$ -th constraint in  $D$  is tight.
2. If the  $j$ -th constraint in  $D$  is not tight then  $x_j^* = 0$ .
3. If  $y_i^* > 0$  then the  $i$ -th constraint in  $P$  is tight.
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If we say that a variable  $x_j^*$  ( $y_i^*$ ) has slack if  $x_j^* > 0$  ( $y_i^* > 0$ ), (i.e., the corresponding variable restriction is not tight) and a constraint has slack if it is not tight, then the above says that for a primal-dual solution pair it is not possible that a constraint **and** its corresponding (dual) variable has slack.

## Proof: Complementary Slackness

Analogous to the proof of weak duality we obtain

$$c^t x^* \leq y^{*t} A x^* \leq b^t y^*$$

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From the constraint of the dual it follows that  $y^t A \geq c^t$ . Hence the left hand side is a sum over the product of non-negative numbers. Hence, if e.g.  $(y^t A - c^t)_j > 0$  (the  $j$ -th constraint in the dual is not tight) then  $x_j = 0$  (2.). The result for (1./3./4.) follows similarly.

## Interpretation of Dual Variables

- ▶ Brewer: find mix of ale and beer that maximizes profits

$$\begin{aligned} \max \quad & 13a + 23b \\ \text{s.t.} \quad & 5a + 15b \leq 480 \\ & 4a + 4b \leq 160 \\ & 35a + 20b \leq 1190 \\ & a, b \geq 0 \end{aligned}$$

- ▶ Entrepreneur: buy resources from brewer at minimum cost  
 $C, H, M$ : unit price for corn, hops and malt.

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Note that brewer won't sell (at least not all) if e.g.  
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- ▶ We are interested in the marginal price, i.e., what happens if we increase the amount of Corn, Hops, and Malt by  $\varepsilon_C$ ,  $\varepsilon_H$ , and  $\varepsilon_M$ , respectively.

The profit increases to  $\max\{c^t x \mid Ax \leq b + \varepsilon; x \geq 0\}$ . Because of strong duality this is equal to

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If  $\epsilon$  is “small” enough then the optimum dual solution  $y^*$  might not change. Therefore the profit increases by  $\sum_i \epsilon_i y_i^*$ .

Therefore we can interpret the dual variables as **marginal prices**.

Note that with this interpretation, complementary slackness becomes obvious.

If the **lowner** has **slack** of some resource (i.e.  $x_i < b_i$ ) then he is **not willing to pay anything** for it (corresponding dual variable is zero).

If the dual variable for some resource is non-zero, then an **increase** of this resource **increases** the profit of the **lowner**. Hence, it makes no sense to have **in excess** of this resource. Therefore **slack must be zero**.

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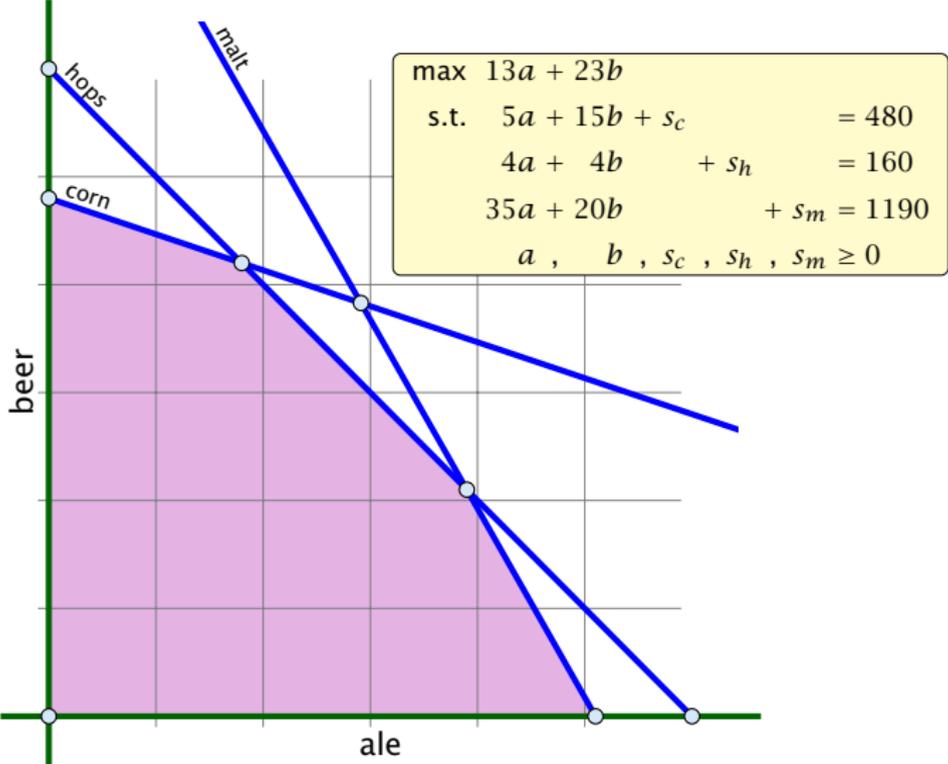
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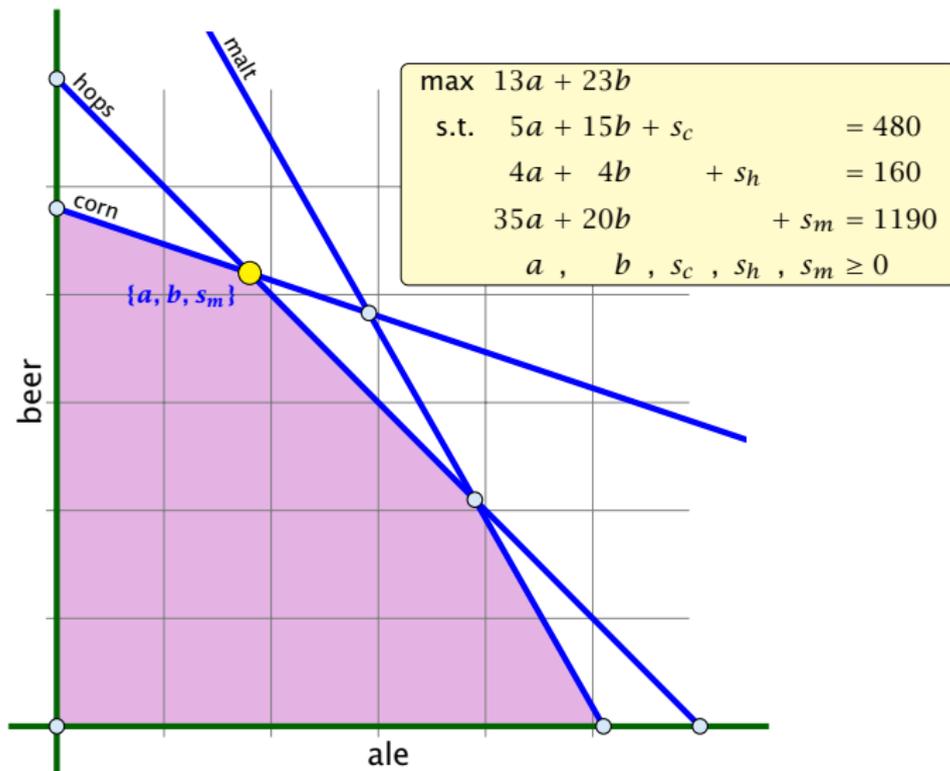
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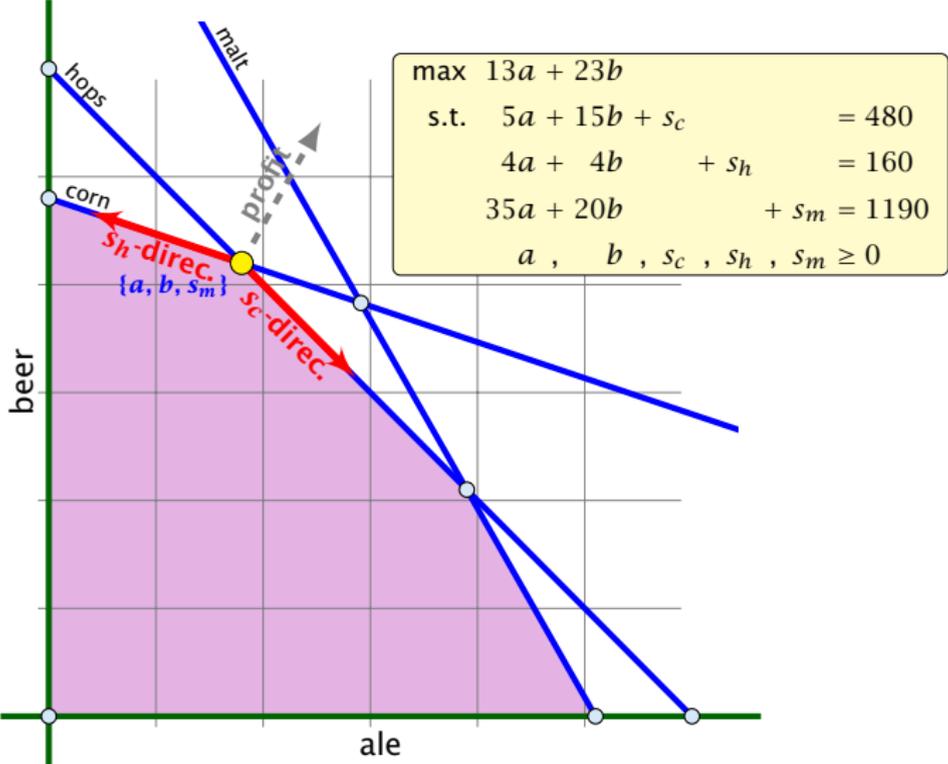
# Example



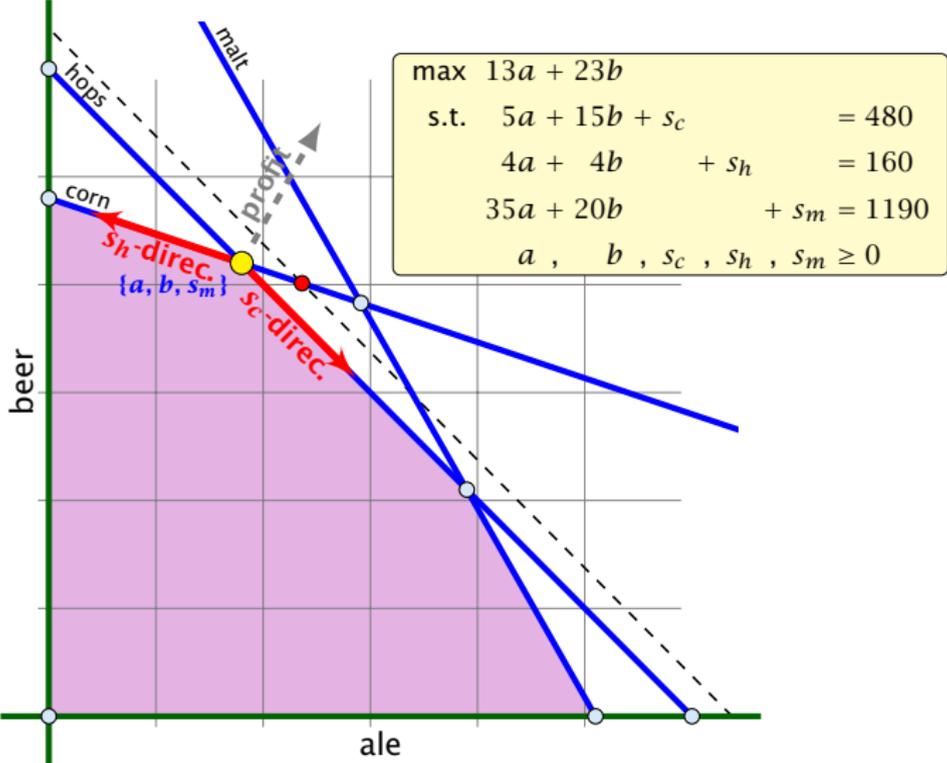
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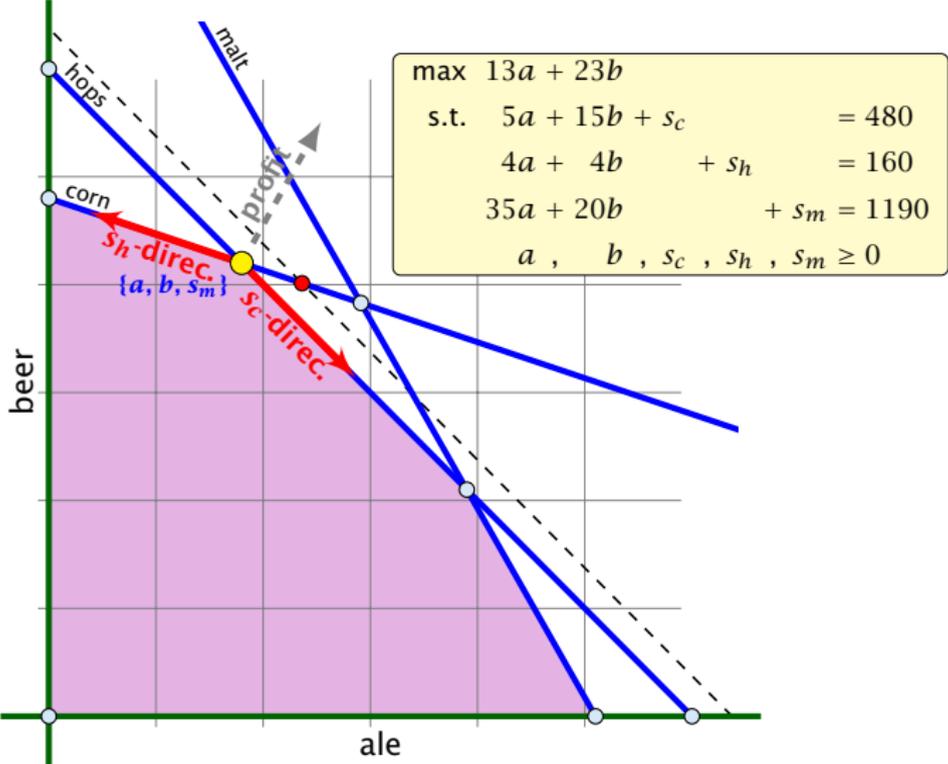
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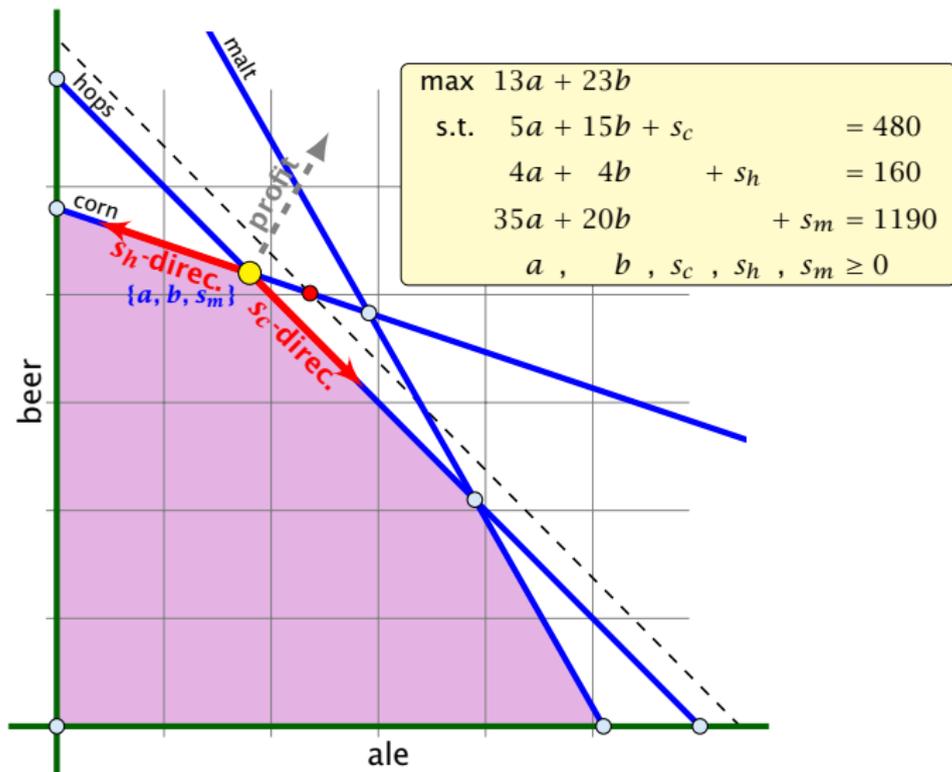
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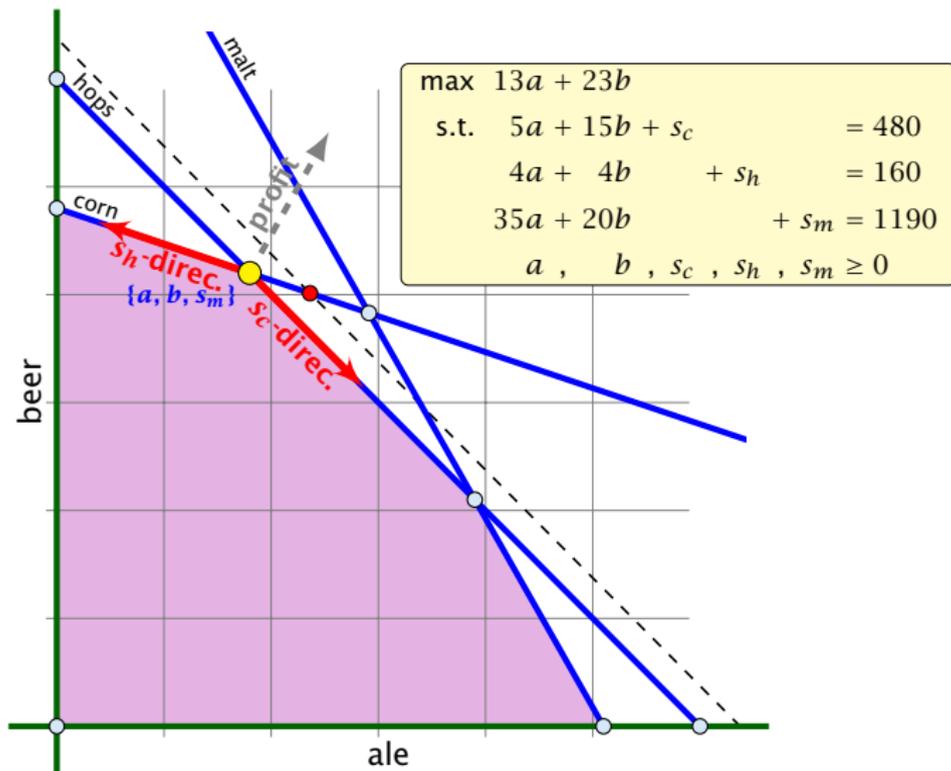


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The change in profit when increasing hops by one unit is  $-\tilde{c}_h$

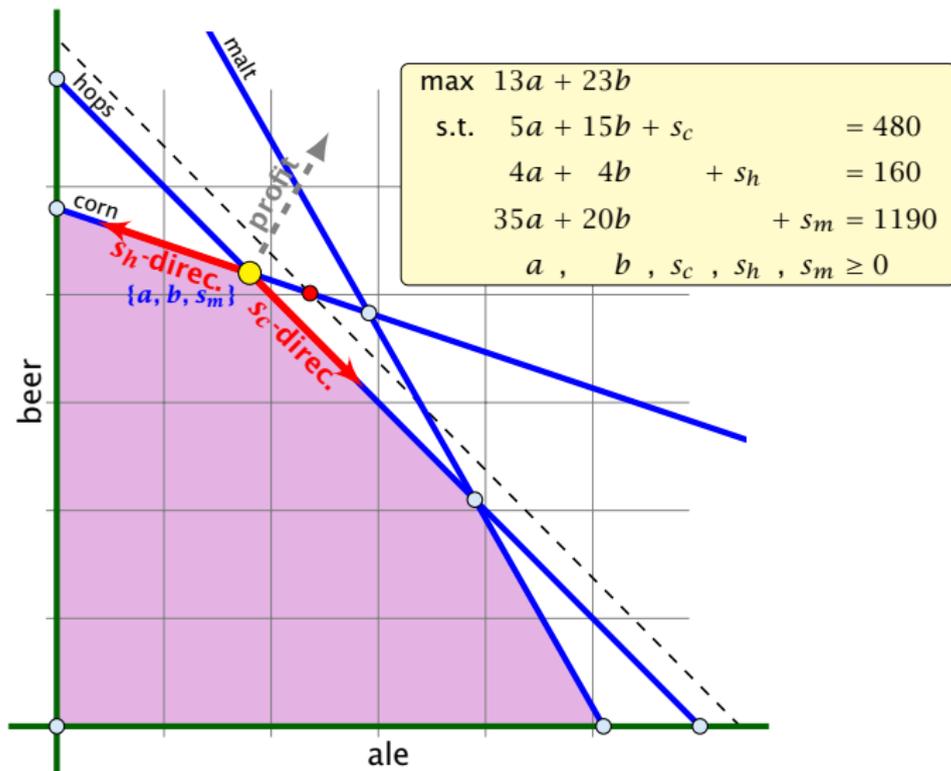
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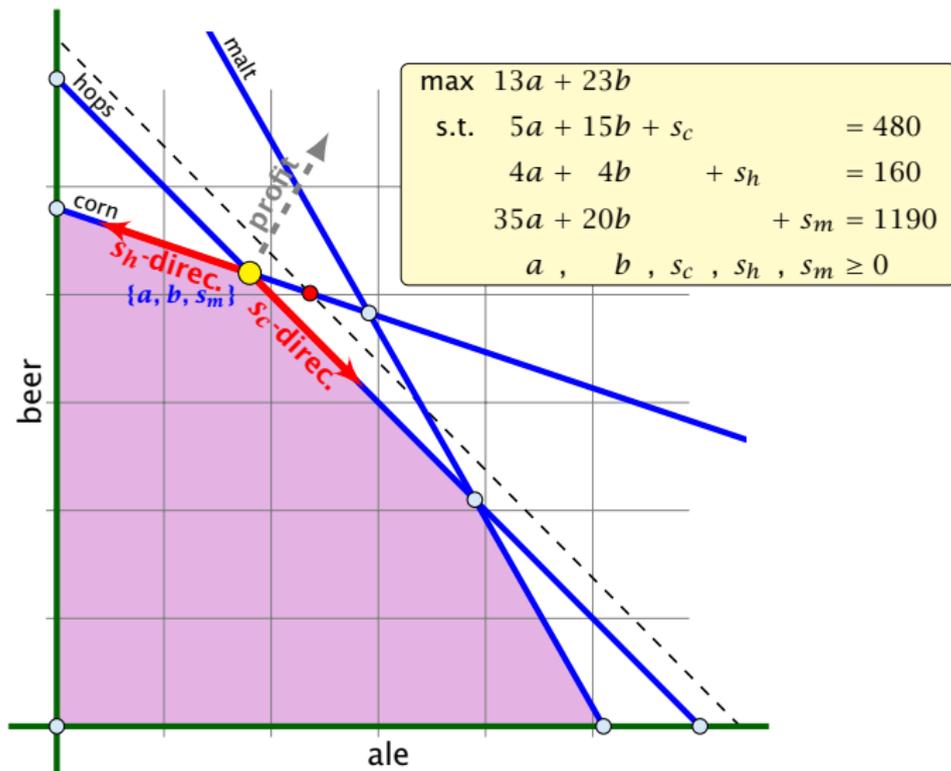
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Of course, the previous argument about the increase in the primal objective only holds for the non-degenerate case.

If the optimum basis is degenerate then increasing the supply of one resource may not allow the objective value to increase.

## Definition 14

An  $(s, t)$ -flow in a (complete) directed graph  $G = (V, V \times V, c)$  is a function  $f : V \times V \mapsto \mathbb{R}_0^+$  that satisfies

1. For each edge  $(x, y)$

$$0 \leq f_{xy} \leq c_{xy} .$$

(capacity constraints)

2. For each  $v \in V \setminus \{s, t\}$

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$$\text{val}(f) = \sum_x f_{sx} - \sum_x f_{xs} .$$

Maximum Flow Problem:

Find an  $(s, t)$ -flow with maximum value.

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Find an  $(s, t)$ -flow with maximum value.

# LP-Formulation of Maxflow

$$\begin{array}{ll} \max & \sum_z f_{sz} - \sum_z f_{zs} \\ \text{s.t.} & \forall (z, w) \in V \times V \quad f_{zw} \leq c_{zw} \quad \ell_{zw} \\ & \forall w \neq s, t \quad \sum_z f_{zw} - \sum_z f_{wz} = 0 \quad p_w \\ & f_{zw} \geq 0 \end{array}$$

$$\begin{array}{ll} \min & \sum_{(xy)} c_{xy} \ell_{xy} \\ \text{s.t.} & f_{xy} \ (x, y \neq s, t) : \quad 1\ell_{xy} - 1p_x + 1p_y \geq 0 \\ & f_{sy} \ (y \neq s, t) : \quad 1\ell_{sy} \quad + 1p_y \geq 1 \\ & f_{xs} \ (x \neq s, t) : \quad 1\ell_{xs} - 1p_x \geq -1 \\ & f_{ty} \ (y \neq s, t) : \quad 1\ell_{ty} \quad + 1p_y \geq 0 \\ & f_{xt} \ (x \neq s, t) : \quad 1\ell_{xt} - 1p_x \geq 0 \\ & f_{st} : \quad 1\ell_{st} \geq 1 \\ & f_{ts} : \quad 1\ell_{ts} \geq -1 \\ & \ell_{xy} \geq 0 \end{array}$$

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$$\begin{array}{ll} \min & \sum_{(x,y)} c_{xy} l_{xy} \\ \text{s.t.} & f_{xy} \ (x, y \neq s, t) : \quad 1l_{xy} - 1p_x + 1p_y \geq 0 \\ & f_{sy} \ (y \neq s, t) : \quad 1l_{sy} - \quad 1 + 1p_y \geq 0 \\ & f_{xs} \ (x \neq s, t) : \quad 1l_{xs} - 1p_x + \quad 1 \geq 0 \\ & f_{ty} \ (y \neq s, t) : \quad 1l_{ty} - \quad 0 + 1p_y \geq 0 \\ & f_{xt} \ (x \neq s, t) : \quad 1l_{xt} - 1p_x + \quad 0 \geq 0 \\ & f_{st} : \quad 1l_{st} - \quad 1 + \quad 0 \geq 0 \\ & f_{ts} : \quad 1l_{ts} - \quad 0 + \quad 1 \geq 0 \\ & & l_{xy} \geq 0 \end{array}$$

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with  $p_t = 0$  and  $p_s = 1$ .

# LP-Formulation of Maxflow

$$\begin{array}{ll} \min & \sum_{(xy)} c_{xy} \ell_{xy} \\ \text{s.t.} & f_{xy} : 1\ell_{xy} - 1p_x + 1p_y \geq 0 \\ & \ell_{xy} \geq 0 \\ & p_s = 1 \\ & p_t = 0 \end{array}$$

We can interpret the  $\ell_{xy}$  value as assigning a length to every edge.

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This means  $p_x = 1$  or  $p_x = 0$  for our case. This gives rise to a cut in the graph with vertices having value 1 on one side and the other vertices on the other side. The objective function then evaluates the capacity of this cut.

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