6.5 Transformation of the Recurrence

Example 6

$$
\begin{aligned}
f_0 &= 1 \\
f_1 &= 2 \\
f_n &= f_{n-1} \cdot f_{n-2} \text{ for } n \geq 2 \,.
\end{aligned}
$$

Define

$$g_n := \log f_n \,.$$

Then

$$
\begin{aligned}
g_n &= g_{n-1} + g_{n-2} \text{ for } n \geq 2 \\
g_1 &= \log 2 = 1,\ g_0 = 0 \text{ (für } \log = \log_2 \text{)} \\
g_n &= F_n \text{ (}n\text{-th Fibonacci number)} \\
f_n &= 2^{F_n}
\end{aligned}
$$

6.5 Transformation of the Recurrence

Example 7

$$f_1 = 1$$
$$f_n = 3f_{\frac{n}{2}} + n; \text{ for } n = 2^k, \ k \geq 1 \ ;$$

Define

$$g_k := f_{2^k} \ .$$

Then:

$$g_0 = 1$$
$$g_k = 3g_{k-1} + 2^k, \ k \geq 1$$

6 Recurrences

We get

$$
\begin{aligned}
g_k &= 3\left[g_{k-1}\right] + 2^k \\
&= 3\left[3g_{k-2} + 2^{k-1}\right] + 2^k \\
&= 3^2\left[g_{k-2}\right] + 32^{k-1} + 2^k \\
&= 3^2\left[3g_{k-3} + 2^{k-2}\right] + 32^{k-1} + 2^k \\
&= 3^3 g_{k-3} + 3^2 2^{k-2} + 32^{k-1} + 2^k \\
&= 2^k \cdot \sum_{i=0}^{k} \left(\frac{3}{2}\right)^i \\
&= 2^k \cdot \frac{(\frac{3}{2})^{k+1} - 1}{1/2} = 3^{k+1} - 2^{k+1}
\end{aligned}
$$

6 Recurrences

Let $n = 2^k$:

$$\begin{aligned} g_k &= 3^{k+1} - 2^{k+1}, \text{ hence} \\ f_n &= 3 \cdot 3^k - 2 \cdot 2^k \\ &= 3(2^{\log 3})^k - 2 \cdot 2^k \\ &= 3(2^k)^{\log 3} - 2 \cdot 2^k \\ &= 3n^{\log 3} - 2n \enspace . \end{aligned}$$