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Effiziente Algorithmen und Datenstrukturen I

Aufgabe 1 (10 Punkte)

A path cover of a directed graph $G = (V, E)$ is a set P of vertex-disjoint paths such that every vertex in V is included in exactly one path in P. Paths may start and end anywhere, and they may be of any length, including 0. A minimum path cover of G is a path cover containing the fewest possible paths.

a Give an efficient algorithm to find a minimum path cover of a directed acyclic graph $G = (V, E)$. (*Hint:* Assuming that $V = \{1, 2, \ldots, n\}$, construct the graph $G' = (V', E')$, where

$$\begin{aligned} V' &= \{x_0, x_1, \ldots, x_n\} \cup \{y_0, y_1, \ldots, y_n\}, \\ E' &= \{(x_0, x_i) : i \in V\} \cup \{(y_i, y_0) : i \in V\} \cup \{(xi, yj) : (i, j) \in E\} \end{aligned}$$

and run a maximum-flow algorithm.)

b Does your algorithm work for directed graphs that contain cycles? Explain.

Aufgabe 2 (10 Punkte)

Let $G = (V, E)$ be a flow network with source s, sink t, and integer capacities. Suppose that we are given a maximum flow in G.

a Suppose that the capacity of a single edge $(u, v) \in E$ is increased by 1. Give an $O(V+E)$ time algorithm to update the maximum flow.

b Suppose that the capacity of a single edge $(u, v) \in E$ is decreased by 1. Give an $O(V+E)$ time algorithm to update the maximum flow.

Aufgabe 3 (10 Punkte)

We say that a bipartite graph $G = (V, E)$, where $V = L \cup R$, is d-regular if every vertex $v \in V$ has degree exactly d. Every d-regular bipartite graph has $|L| = |R|$. Prove that every d-regular bipartite graph has a matching of cardinality $|L|$ by arguing that a minimum cut of the corresponding flow network has capacity $|L|$.

Aufgabe 4 (10 Punkte)

In the lecture, you studied the problem of min-cost flow problem formulated as follows:

$$\min \sum_{e \in E} c(e) \cdot f(e)$$

$$\begin{aligned} \text{where } l(e) &\leq f(e) &\leq u(e) \\ f(v) &= b(v) \end{aligned}$$

where $c(e)$ and $b(v)$ can be negative, $l(e)$ can be $-\infty$ and $u(e)$ can be ∞. Show that we can reduce this to a problem where for an edge $e = (u, v)$, at least one of $l(e)$ or $u(e)$ is finite. In the lecture this reduction was already shown when $c(e) = 0$. Show the reduction when $c(e) \neq 0$.
(*Hint:* First show that it is sufficient to show this reduction for $b(u) = b(v) = 0$)

Aufgabe 5 (5 Punkte)

(*Note:*Attempt this question iff your marks in previous assignments are below the required threshold of 40%)
Let f be a flow in a network, and let α be a real number. The scalar flow product, denoted $\alpha \cdot f$, is a function from $V \times V$ to R defined by

$$(\alpha \cdot f)(u, v) = \alpha \cdot f(u, v)$$

Prove that the flows in a network form a convex set. That is, show that if f_1 and f_2 are flows, then so is $\alpha f_1 + (1 - \alpha) f_2$ for $0 \leq \alpha \leq 1$.