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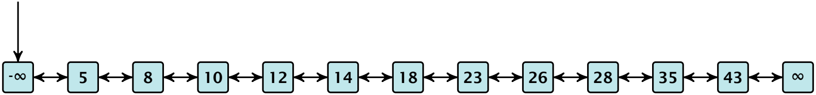

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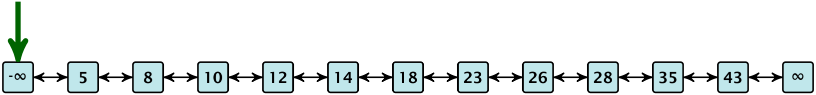

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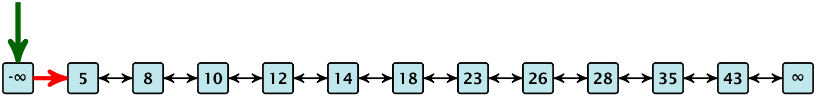

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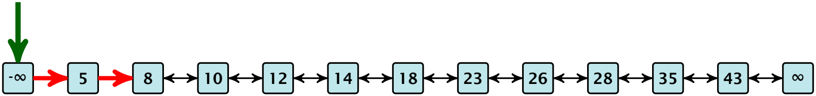

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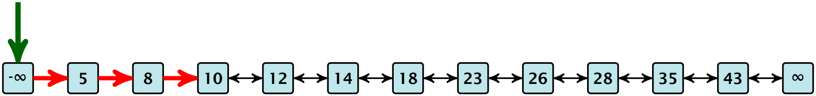

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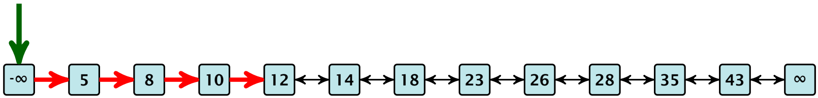

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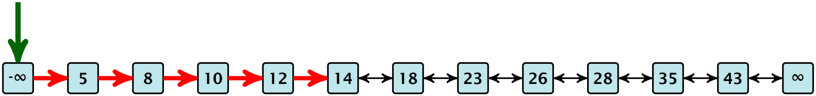

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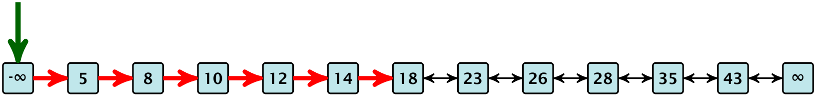

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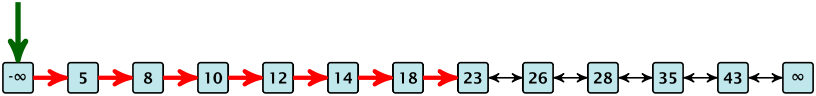

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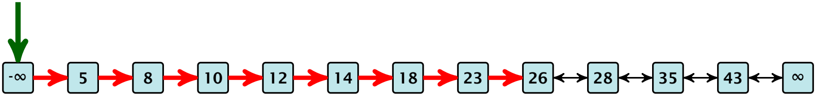

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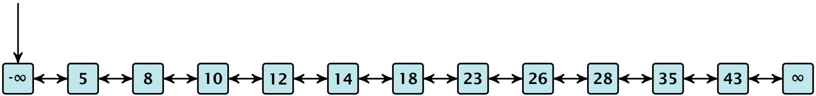

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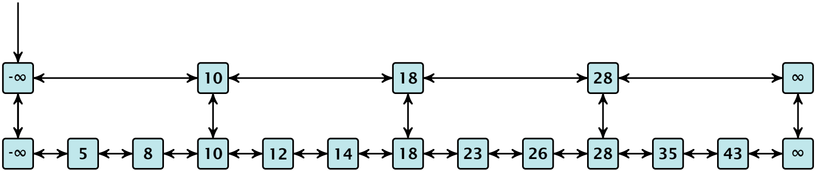

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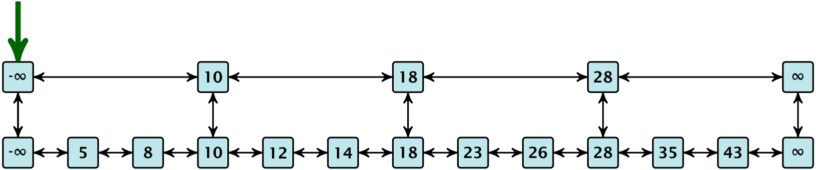

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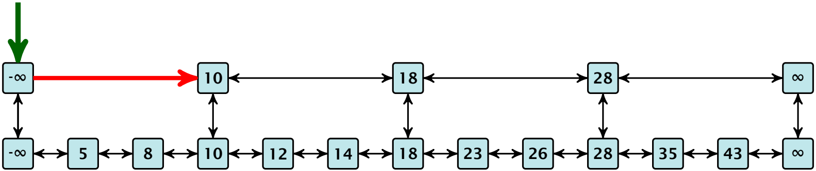

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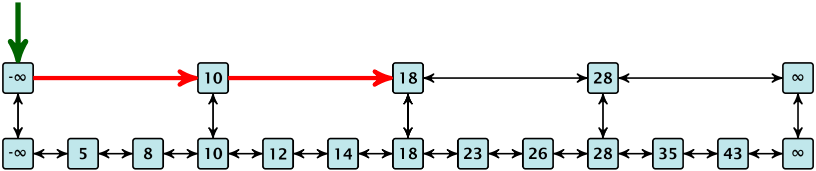

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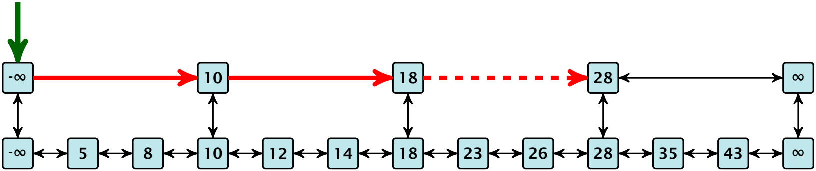

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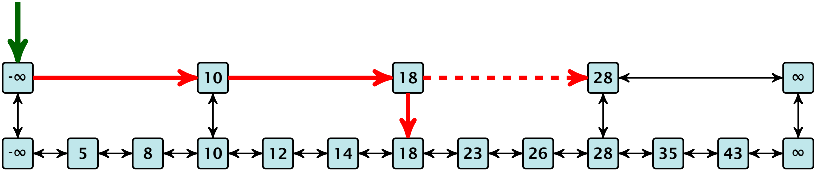

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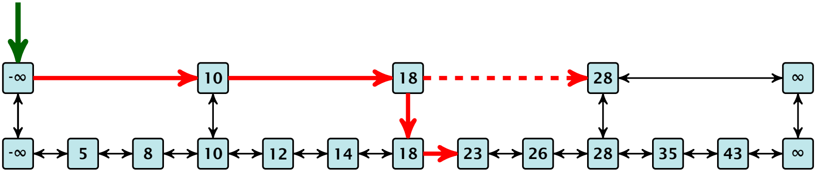

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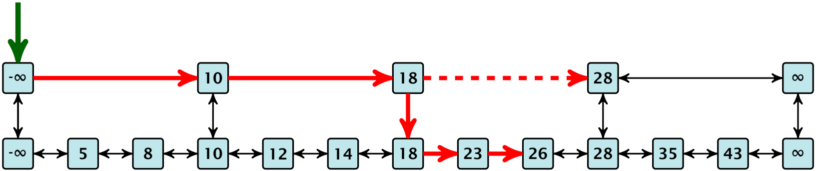

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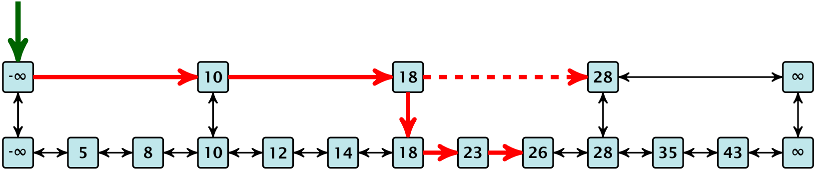

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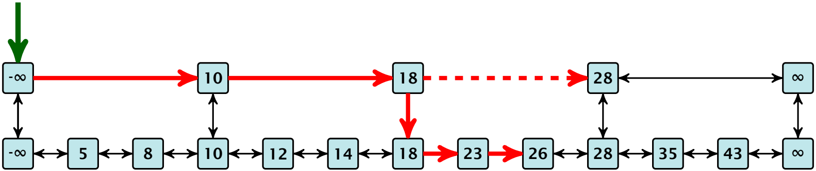

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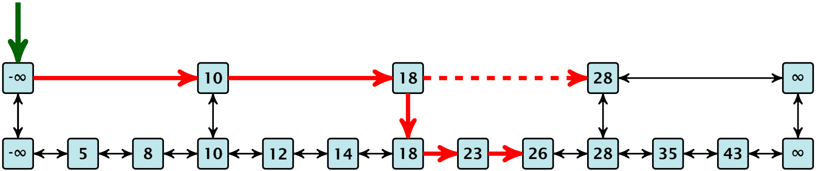

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Choose $|L_1| = \sqrt{n}$. Then search time $\Theta(\sqrt{n})$.

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Choosing $k = \Theta(\log k)$ gives a logarithmic running time.

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How to do insert and delete?

- If we want that in L_i we always skip over roughly the same number of elements in L_{i-1} an insert or delete may require a lot of re-organisation.

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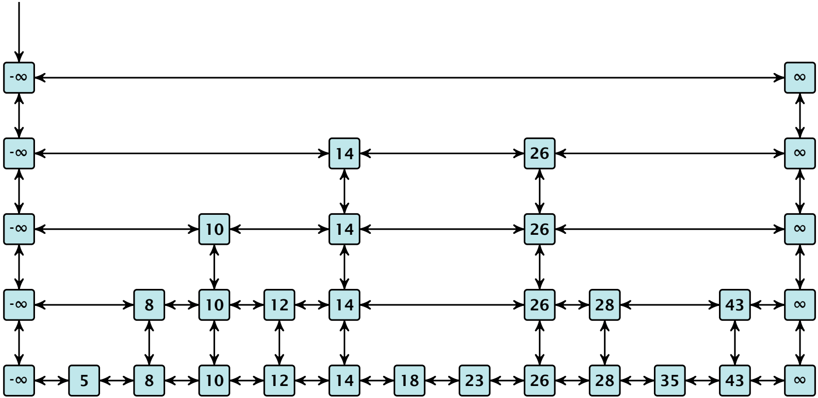

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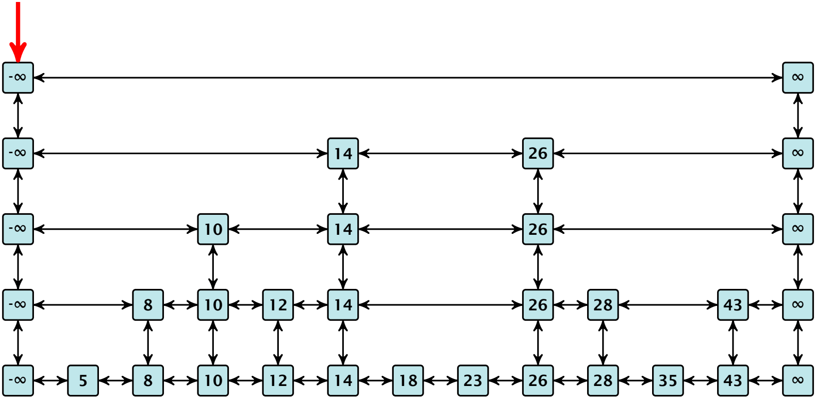

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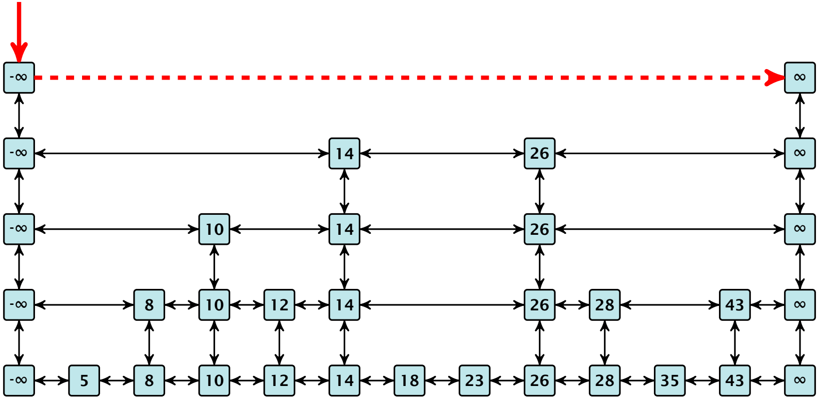

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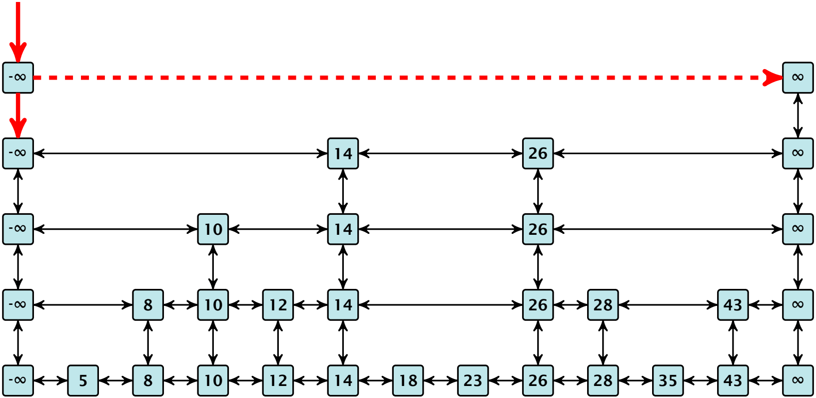

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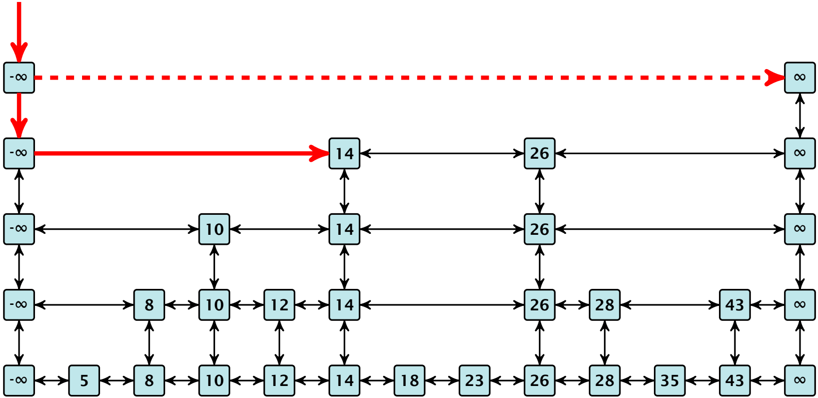

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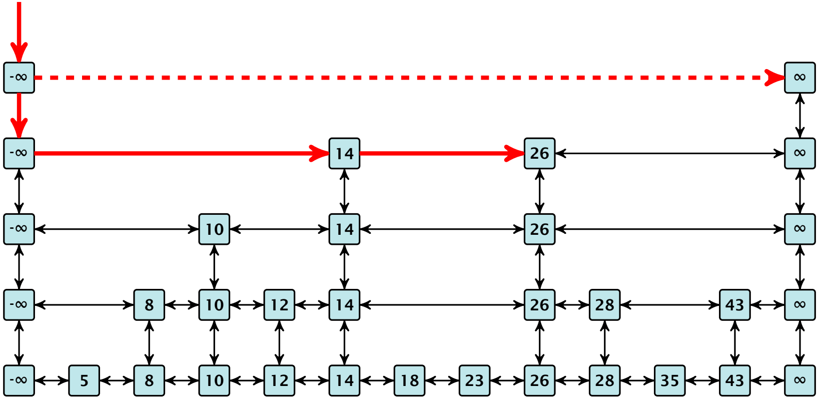

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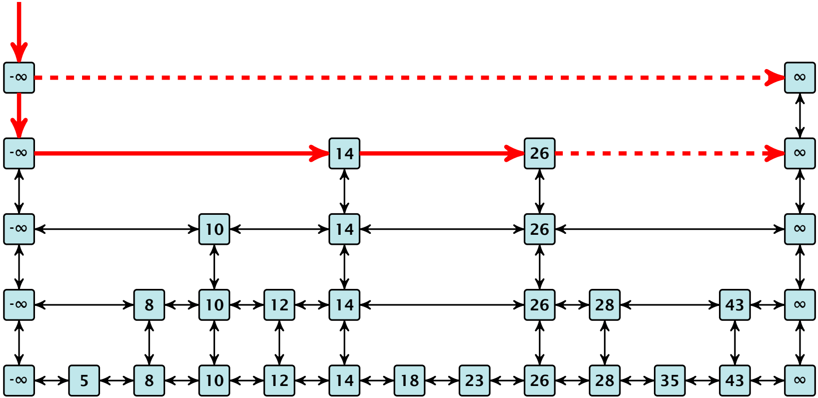

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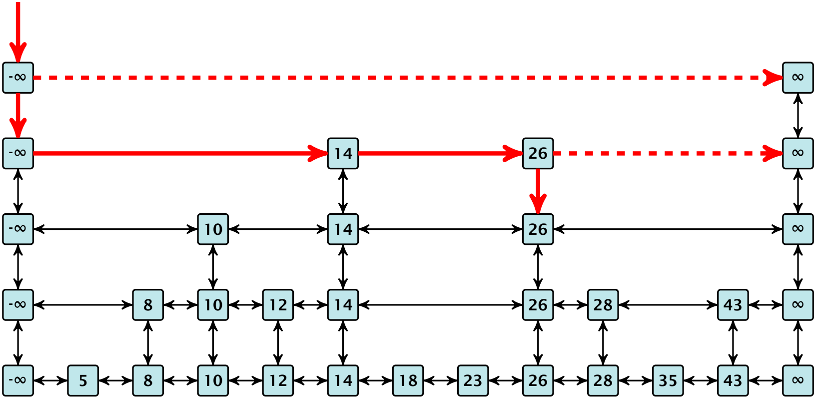

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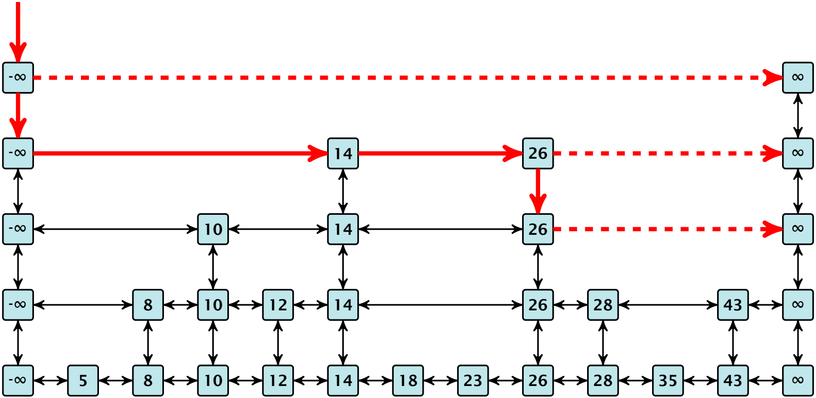

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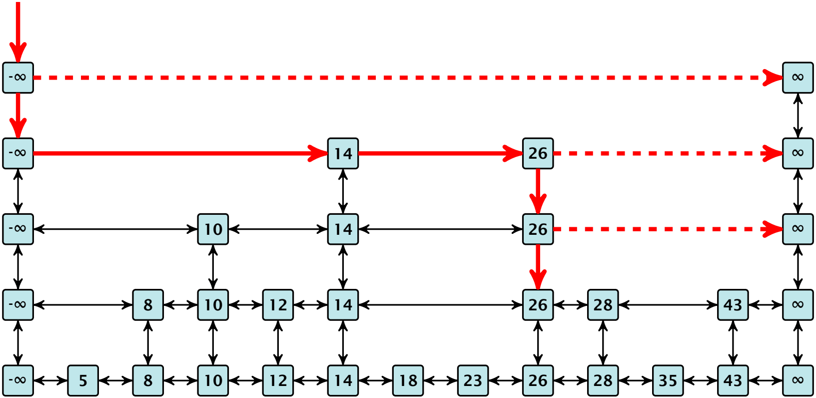

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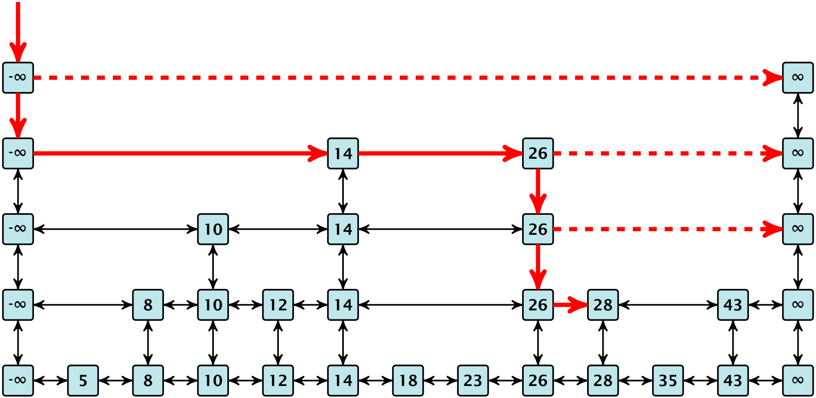

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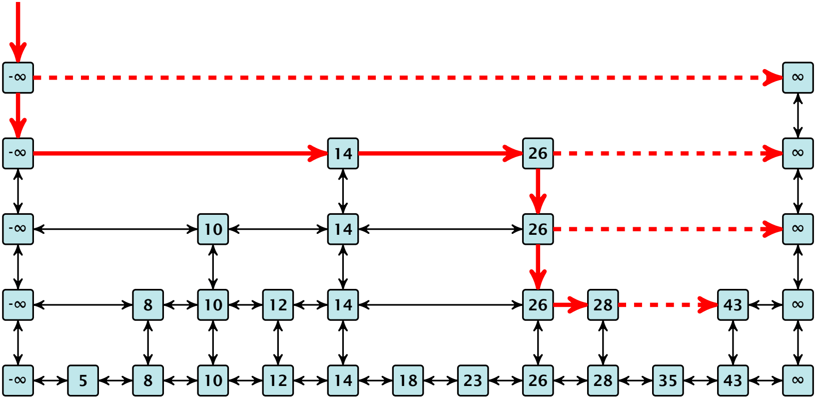

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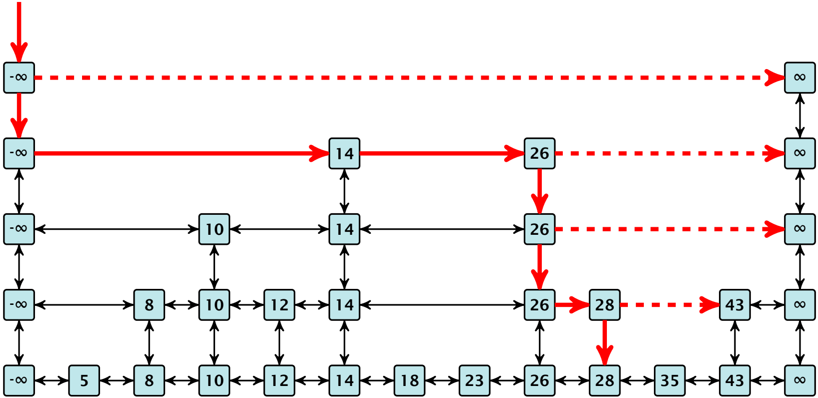

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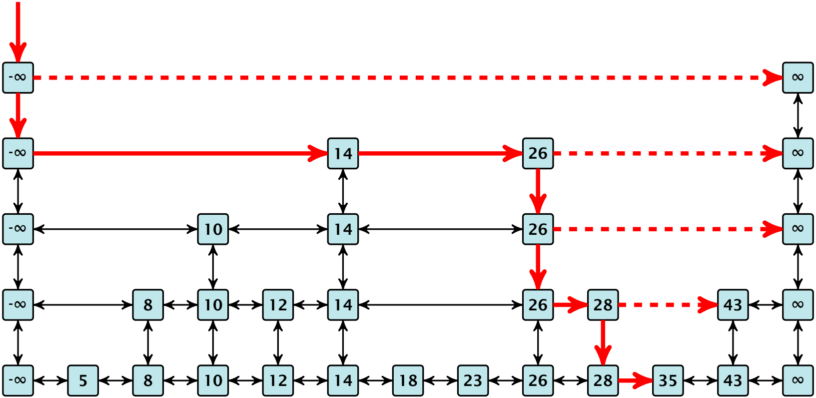

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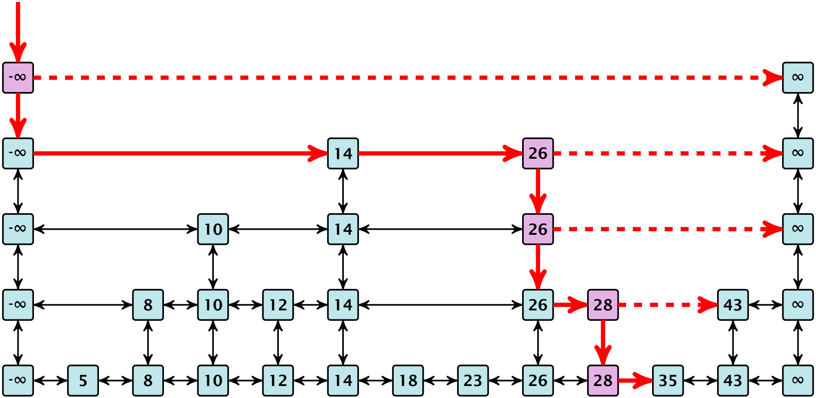

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Lemma 20

A search (and, hence, also insert and delete) in a skip list with n elements takes time $\mathcal{O}(\log n)$ with high probability (w. h. p.).

This means for any constant α the search takes time $\mathcal{O}(\log n)$ with probability at least $1 - \frac{1}{n^\alpha}$.

Note that the constant in the $\mathcal{O}$-notation may depend on α.

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High Probability

Suppose there are a polynomially many events $E_1, E_2, \ldots, E_\ell$, $\ell = n^c$ each holding with high probability (e.g. E_i may be the event that the i-th search in a skip list takes time at most $\mathcal{O}(\log n)$).

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This means $\Pr[E_1 \wedge \cdots \wedge E_\ell]$ holds with high probability.

Skip Lists

Backward analysis:

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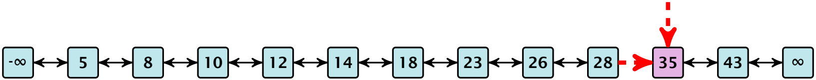

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Backward analysis:

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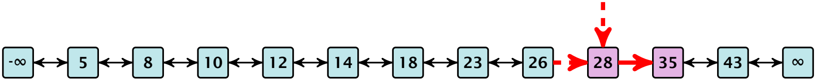

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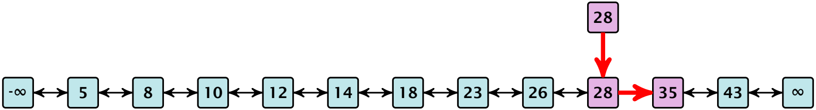

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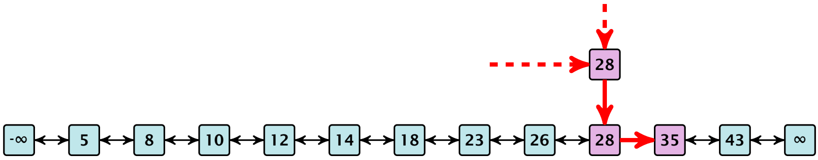

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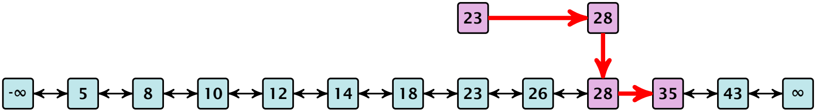

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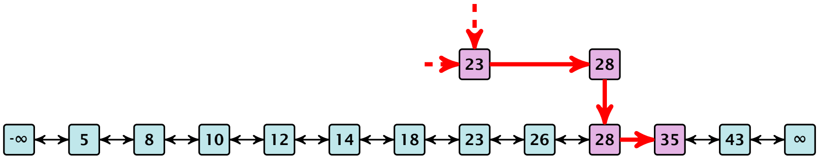

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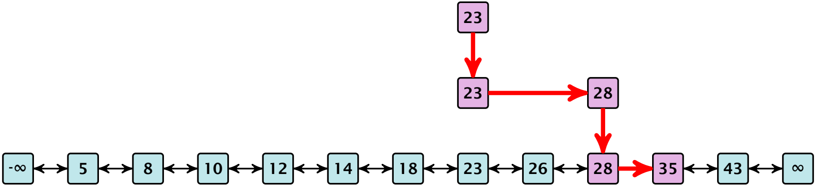

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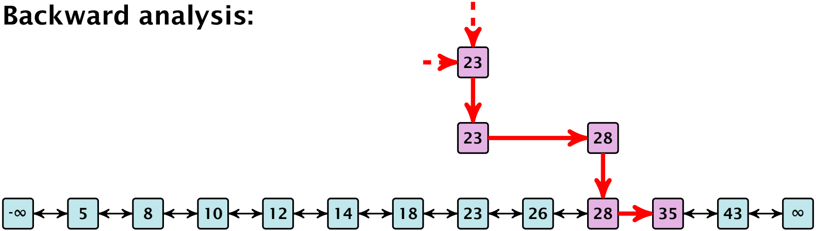

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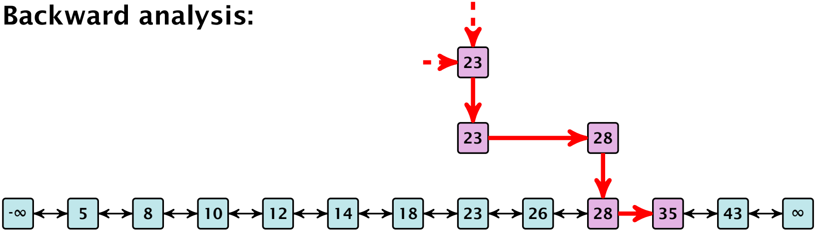

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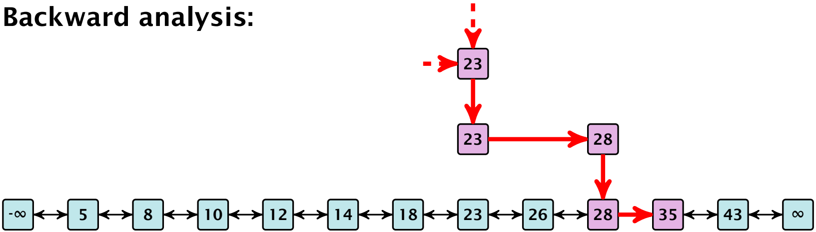

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We show that w.h.p:

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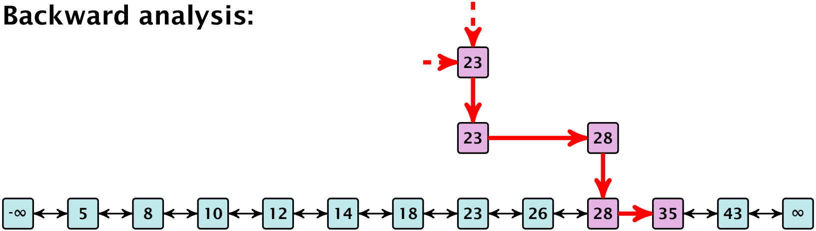

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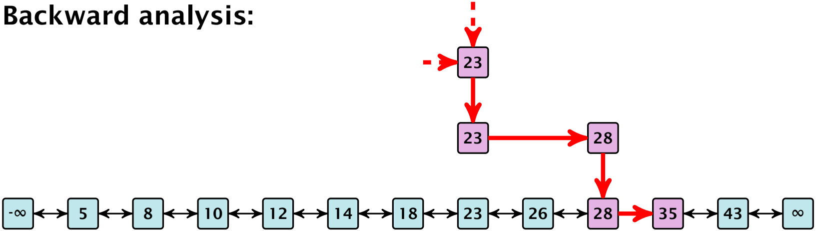

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We show that w.h.p:

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From this it follows that w.h.p. there are no long paths.

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In particular, this means that during the construction in the backward analysis we see at most k heads (i.e., coin flips that tell you to go up) in z trials.

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This means, the search requires at most z steps, w. h. p.