

# A Fast Matching Algorithm

## Algorithm 54 Bimatch-Hopcroft-Karp( $G$ )

```
1:  $M \leftarrow \emptyset$ 
2: repeat
3:   let  $\mathcal{P} = \{P_1, \dots, P_k\}$  be maximal set of
4:   vertex-disjoint, shortest augmenting path w.r.t.  $M$ .
5:    $M \leftarrow M \oplus (P_1 \cup \dots \cup P_k)$ 
6: until  $\mathcal{P} = \emptyset$ 
7: return  $M$ 
```

We call one iteration of the repeat-loop a **phase** of the algorithm.

# Analysis

## Lemma 98

Given a matching  $M$  and a maximal matching  $M^*$  there exist  $|M^*| - |M|$  *vertex-disjoint* augmenting path w.r.t.  $M$ .

### Proof:

- ▶ Similar to the proof that a matching is optimal iff it does not contain an augmenting paths.
- ▶ Consider the graph  $G = (V, M \oplus M^*)$ , and mark edges in this graph blue if they are in  $M$  and red if they are in  $M^*$ .
- ▶ The connected components of  $G$  are cycles and paths.
- ▶ The graph contains  $k \stackrel{\text{def}}{=} |M^*| - |M|$  more red edges than blue edges.
- ▶ Hence, there are at least  $k$  components that form a path starting and ending with a blue edge. These are augmenting paths w.r.t.  $M$ .

# Analysis

- ▶ Let  $P_1, \dots, P_k$  be a maximal collection of vertex-disjoint, shortest augmenting paths w.r.t.  $M$  (let  $\ell = |P_i|$ ).
- ▶  $M' \stackrel{\text{def}}{=} M \oplus (P_1 \cup \dots \cup P_k) = M \oplus P_1 \oplus \dots \oplus P_k$ .
- ▶ Let  $P$  be an augmenting path in  $M'$ .

## Lemma 99

*The set  $A \stackrel{\text{def}}{=} M \oplus (M' \oplus P) = (P_1 \cup \dots \cup P_k) \oplus P$  contains at least  $(k + 1)\ell$  edges.*

## Proof.

- ▶ The set describes exactly the symmetric difference between matchings  $M$  and  $M' \oplus P$ .
- ▶ Hence, the set contains at least  $k + 1$  vertex-disjoint augmenting paths w.r.t.  $M$  as  $|M'| = |M| + k + 1$ .
- ▶ Each of these paths is of length at least  $\ell$ .

# Analysis

## Lemma 100

*$P$  is of length at least  $\ell + 1$ . This shows that the length of a shortest augmenting path increases between two phases of the Hopcroft-Karp algorithm.*

### Proof.

- ▶ If  $P$  does not intersect any of the  $P_1, \dots, P_k$ , this follows from the maximality of the set  $\{P_1, \dots, P_k\}$ .
- ▶ Otherwise, at least one edge from  $P$  coincides with an edge from paths  $\{P_1, \dots, P_k\}$ .
- ▶ This edge is not contained in  $A$ .
- ▶ Hence,  $|A| \leq k\ell + |P| - 1$ .
- ▶ The lower bound on  $|A|$  gives  $(k + 1)\ell \leq |A| \leq k\ell + |P| - 1$ , and hence  $|P| \geq \ell + 1$ .

# Analysis

If the shortest augmenting path w.r.t. a matching  $M$  has  $\ell$  edges then the cardinality of the maximum matching is of size at most  $|M| + \lfloor \frac{|V|}{\ell+1} \rfloor$ .

## Proof.

The symmetric difference between  $M$  and  $M^*$  contains  $|M^*| - |M|$  vertex-disjoint augmenting paths. Each of these paths contains at least  $\ell + 1$  vertices. Hence, there can be at most  $\lfloor \frac{|V|}{\ell+1} \rfloor$  of them.

## Lemma 101

*The Hopcroft-Karp algorithm requires at most  $2\sqrt{|V|}$  phases.*

### Proof.

- ▶ After iteration  $\lfloor \sqrt{|V|} \rfloor$  the length of a shortest augmenting path must be at least  $\lfloor \sqrt{|V|} \rfloor + 1 \geq \sqrt{|V|}$ .
- ▶ Hence, there can be at most  $|V| / (\sqrt{|V|} + 1) \leq \sqrt{|V|}$  additional augmentations.

## Lemma 102

*One phase of the Hopcroft-Karp algorithm can be implemented in time  $\mathcal{O}(m)$ .*