

8 Priority Queues

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- ▶ $S.\text{build}(x_1, \dots, x_n)$: Creates a data-structure that contains just the elements x_1, \dots, x_n .
- ▶ $S.\text{insert}(x)$: Adds element x to the data-structure.
- ▶ Element $S.\text{minimum}()$: Returns an element $x \in S$ with minimum key-value $\text{key}[x]$.
- ▶ $S.\text{delete-min}()$: Deletes the element with minimum key-value from S and returns it.
- ▶ Boolean $S.\text{empty}()$: Returns true if the data-structure is empty and false otherwise.

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An **addressable Priority Queue** also supports:

- ▶ **Handle $S.insert(x)$** : Adds element x to the data-structure, and returns a **handle** to the object for future reference.
- ▶ **$S.delete(h)$** : Deletes element specified through handle h .
- ▶ **$S.decrease-key(h, k)$** : Decreases the key of the element specified by handle h to k . Assumes that the key is at least k before the operation.

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Dijkstra's Shortest Path Algorithm

Algorithm 17 Shortest-Path($G = (V, E, d), s \in V$)

```
1: Input: weighted graph  $G = (V, E, d)$ ; start vertex  $s$ ;  
2: Output: key-field of every node contains distance from  $s$ ;  
3:  $S.build()$ ; // build empty priority queue  
4: for all  $v \in V \setminus \{s\}$  do  
5:    $v.key \leftarrow \infty$ ;  
6:    $h_v \leftarrow S.insert(v)$ ;  
7:  $s.key \leftarrow 0$ ;  $S.insert(s)$ ;  
8: while  $S.empty() = \text{false}$  do  
9:    $v \leftarrow S.delete-min()$ ;  
10:  for all  $x \in V$  s.t.  $(v, x) \in E$  do  
11:    if  $x.key > v.key + d(v, x)$  then  
12:       $S.decrease-key(h_x, v.key + d(v, x))$ ;  
13:       $x.key \leftarrow v.key + d(v, x)$ ;
```

Prim's Minimum Spanning Tree Algorithm

Algorithm 18 Prim-MST($G = (V, E, d), s \in V$)

```
1: Input: weighted graph  $G = (V, E, d)$ ; start vertex  $s$ ;  
2: Output: pred-fields encode MST;  
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14:             $x.pred \leftarrow v$ ;
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Analysis of Dijkstra and Prim

Both algorithms require:

- ▶ 1 build() operation
- ▶ $|V|$ insert() operations
- ▶ $|V|$ delete-min() operations
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How good a running time can we obtain?

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minimum	1	$\log n$	$\log n$	1
is-empty	1	1	1	1
insert	$\log n$	$\log n$	$\log n$	1
delete	$\log n^{**}$	$\log n$	$\log n$	$\log n$
delete-min	$\log n$	$\log n$	$\log n$	$\log n$
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merge	n	$n \log n$	$\log n$	1

Note that most applications use `build()` only to create an empty heap which then costs time 1.

The standard version of binary heaps is not addressable, and hence does not support a delete operation.

Fibonacci heaps only give an **amortized** guarantee.

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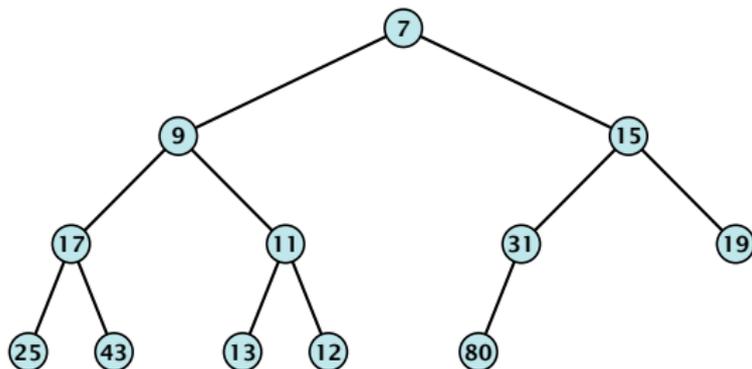
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Using Binary Heaps, Prim and Dijkstra run in time $\mathcal{O}((|V| + |E|) \log |V|)$.

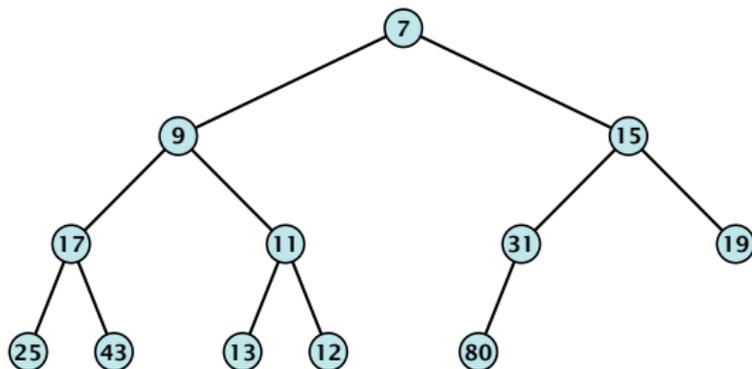
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8.1 Binary Heaps



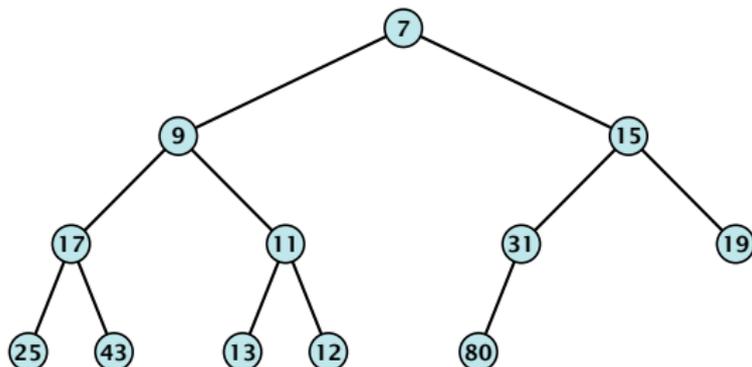
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- ▶ Nearly complete binary tree; only the last level is not full, and this one is filled from left to right.
- ▶ **Heap property:** A node's key is not larger than the key of one of its children.



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Operations:

- ▶ `minimum()`: return the root-element. Time $\mathcal{O}(1)$.
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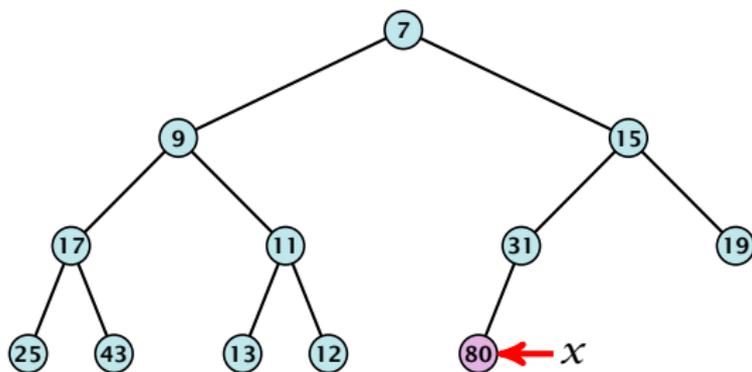
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Maintain a pointer to the **last element** x .

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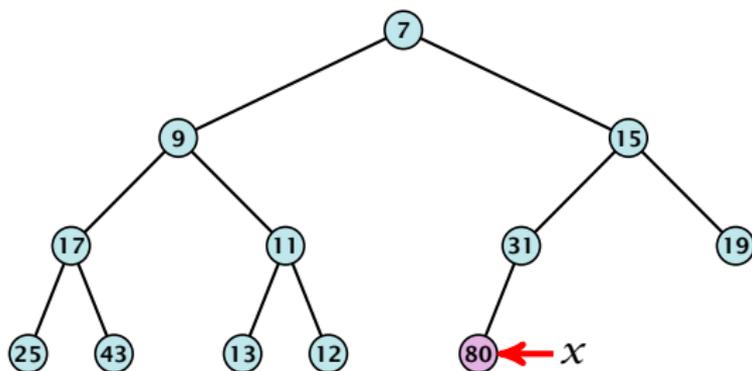
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go up until the last edge used was a right edge.

go left; go right until you reach a leaf

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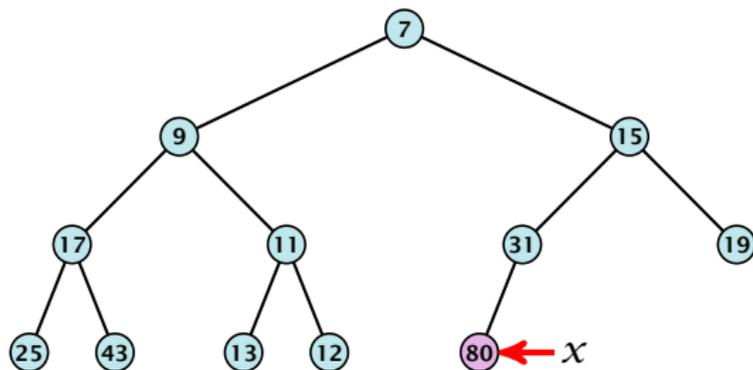
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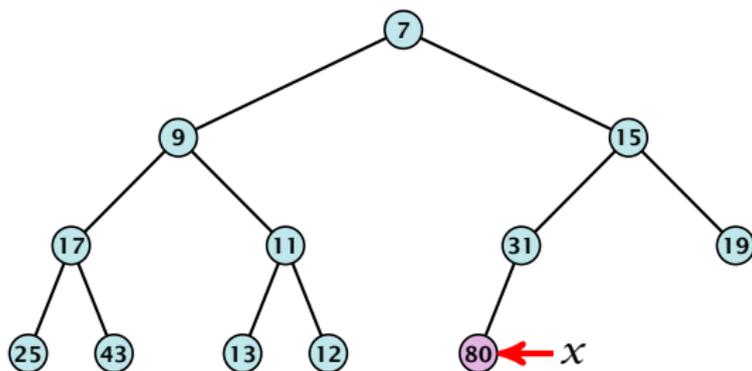
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insert a new element as a left child;



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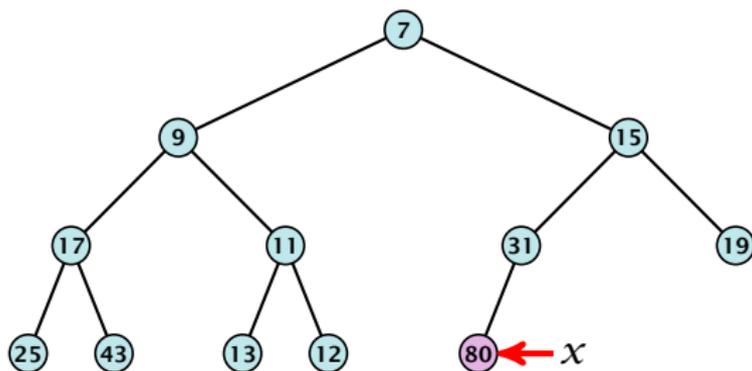
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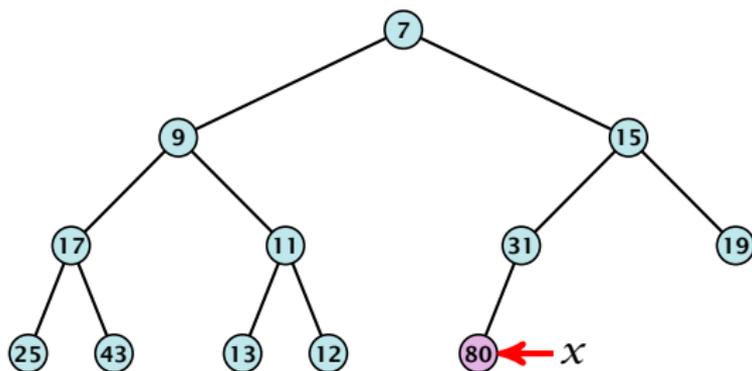
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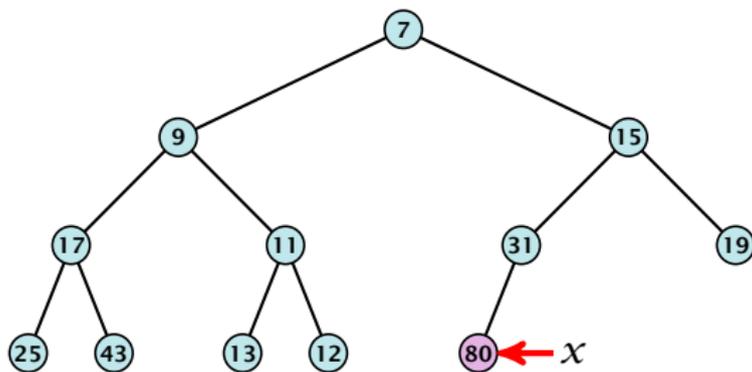
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Insert

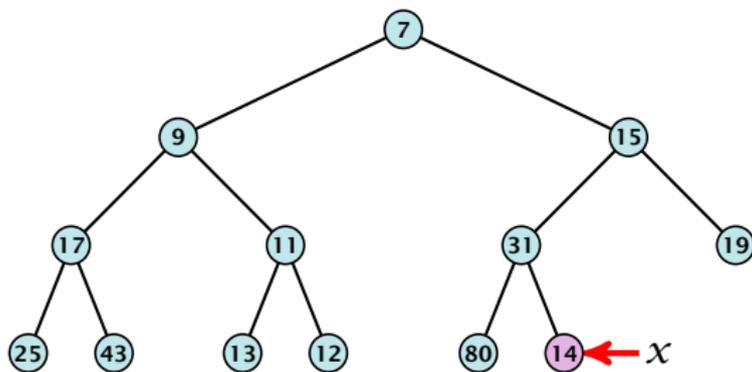
1. Insert element at successor of x .
2. Exchange with parent until heap property is fulfilled.



Note that an exchange can either be done by moving the data or by changing pointers. The latter method leads to an addressable priority queue.

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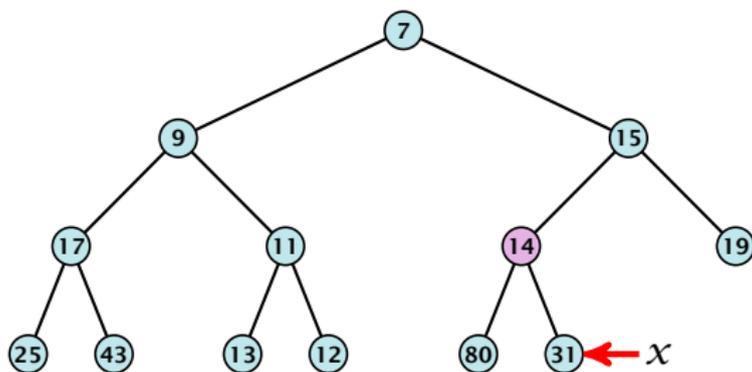
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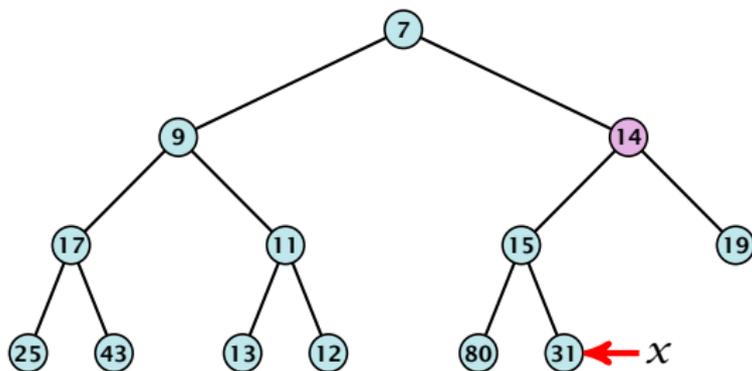
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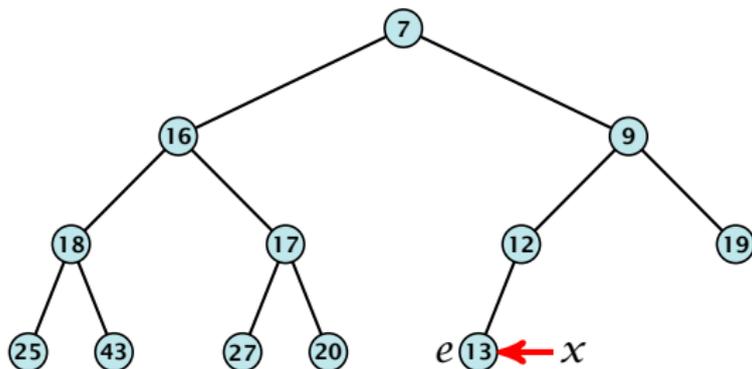
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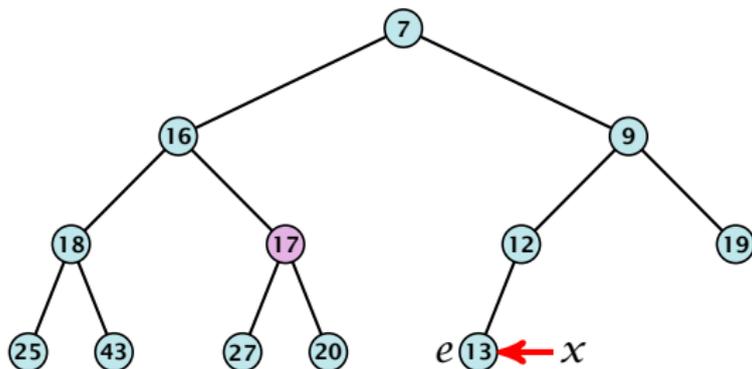
1. Exchange the element to be deleted with the element e pointed to by x .
2. Restore the heap-property for the element e .



At its new position e may either travel up or down in the tree (but not both directions).

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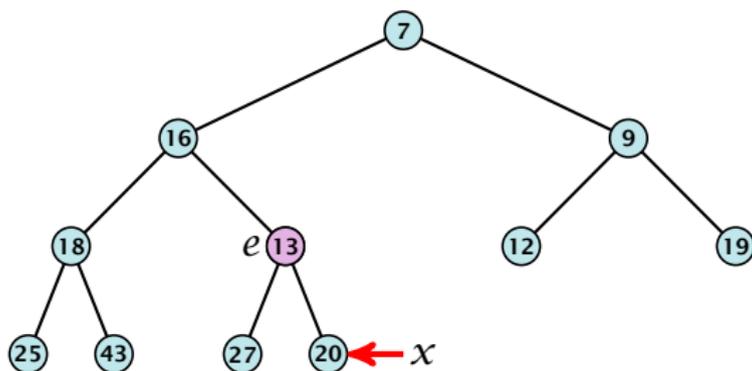
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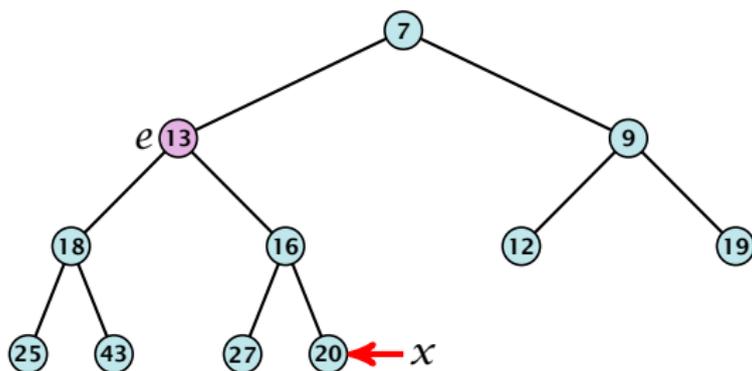
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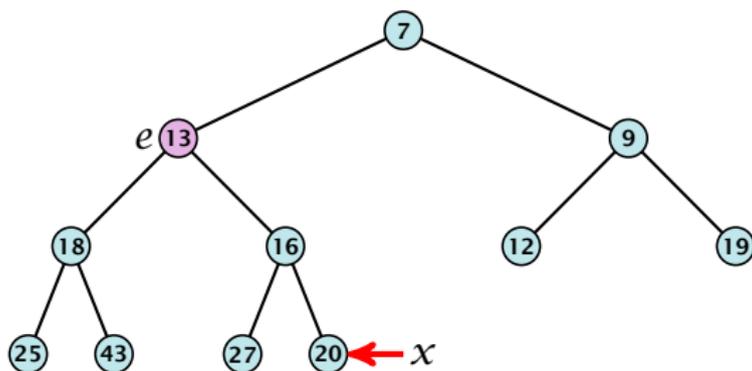
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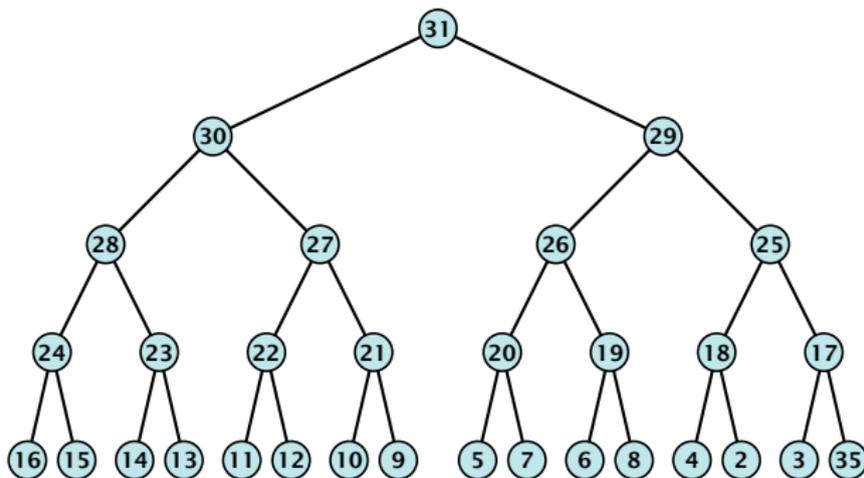
Binary Heaps

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- ▶ **minimum()**: return the root-element. Time $\mathcal{O}(1)$.
- ▶ **is-empty()**: check whether root-pointer is null. Time $\mathcal{O}(1)$.
- ▶ **insert(k)**: insert at x and bubble up. Time $\mathcal{O}(\log n)$.
- ▶ **delete(h)**: swap with x and bubble up or sift-down. Time $\mathcal{O}(\log n)$.

Build Heap

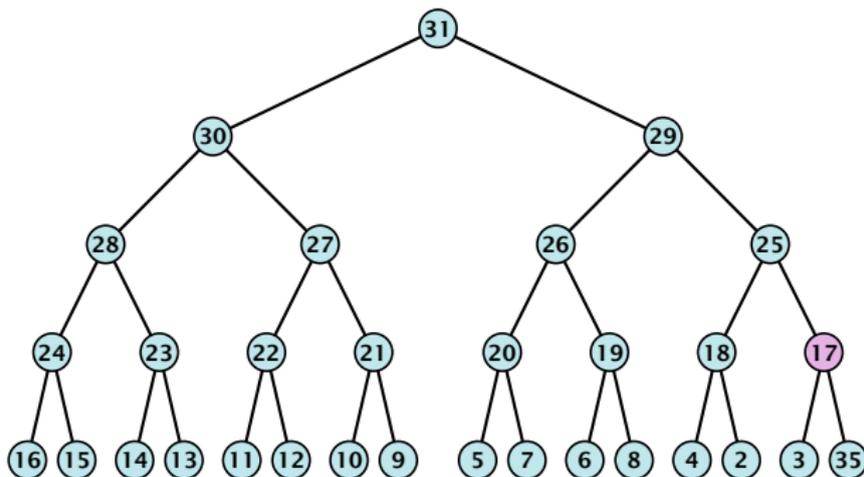
We can build a heap in linear time:



$$\sum_{\text{levels } \ell} 2^\ell \cdot (h - \ell) = \mathcal{O}(2^h) = \mathcal{O}(n)$$

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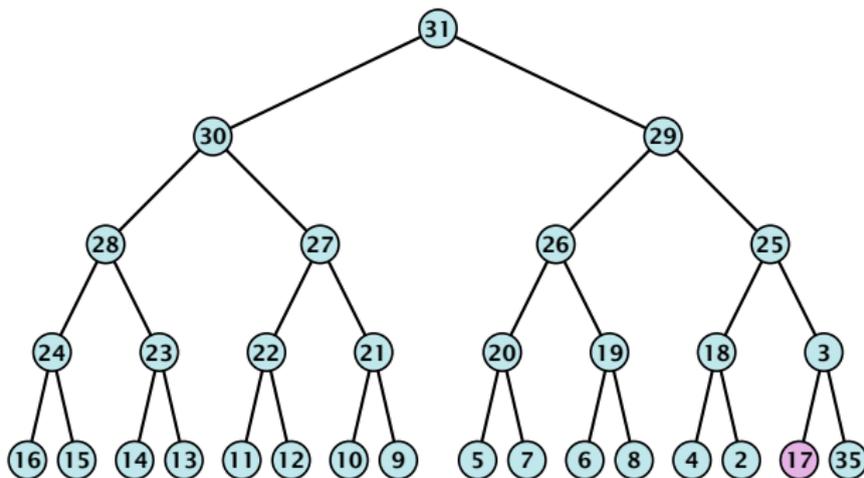
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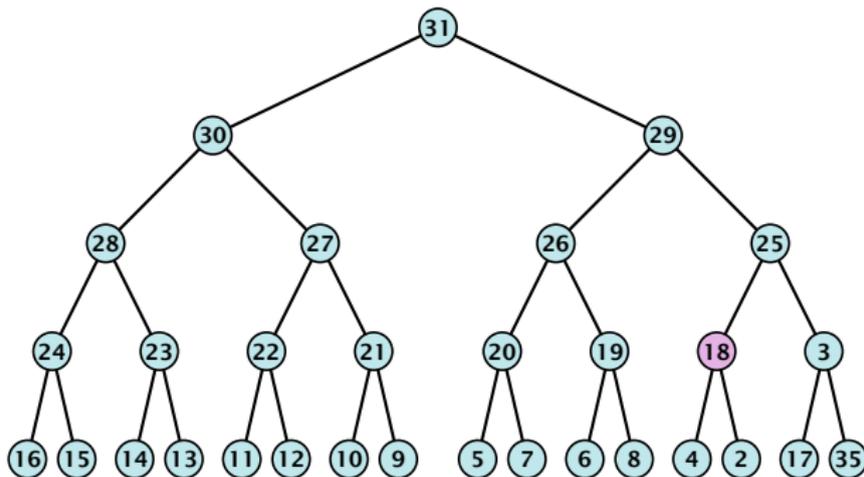
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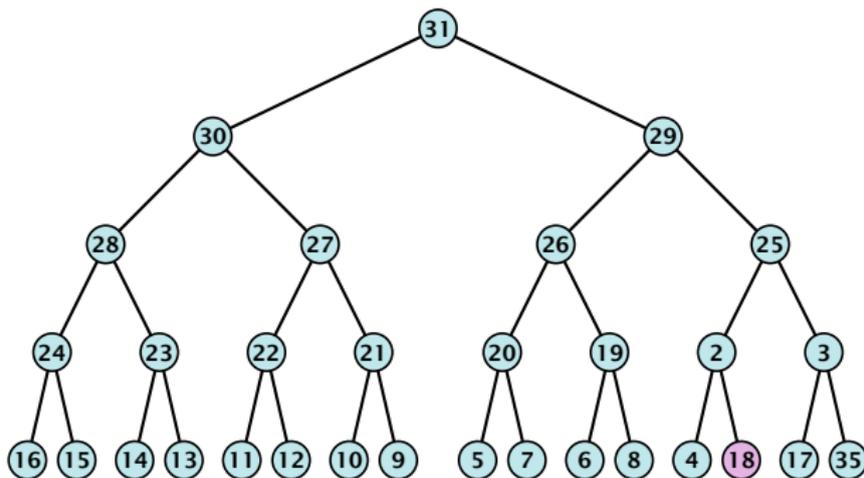
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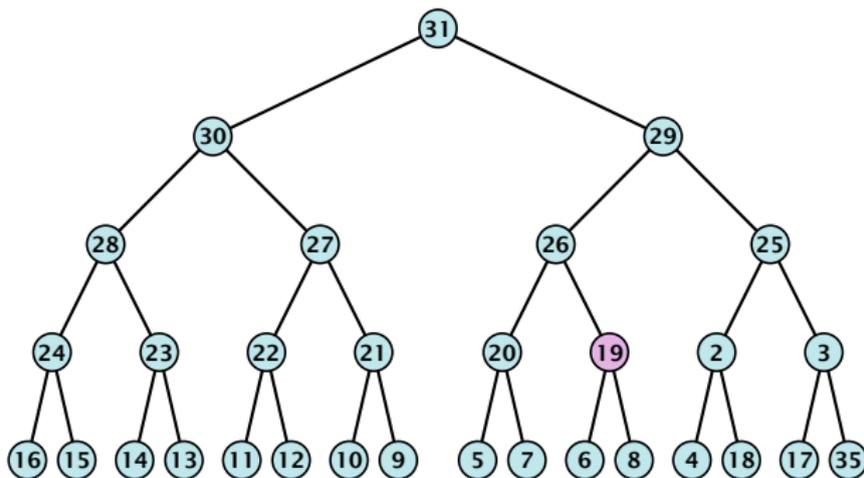
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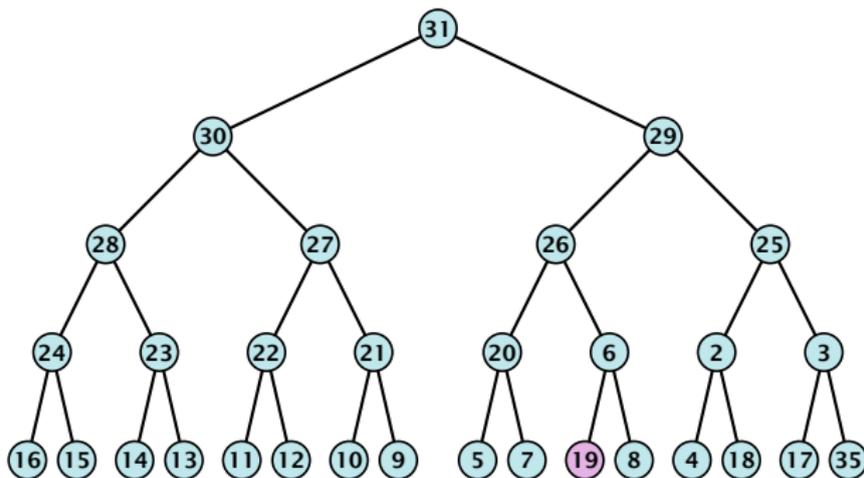
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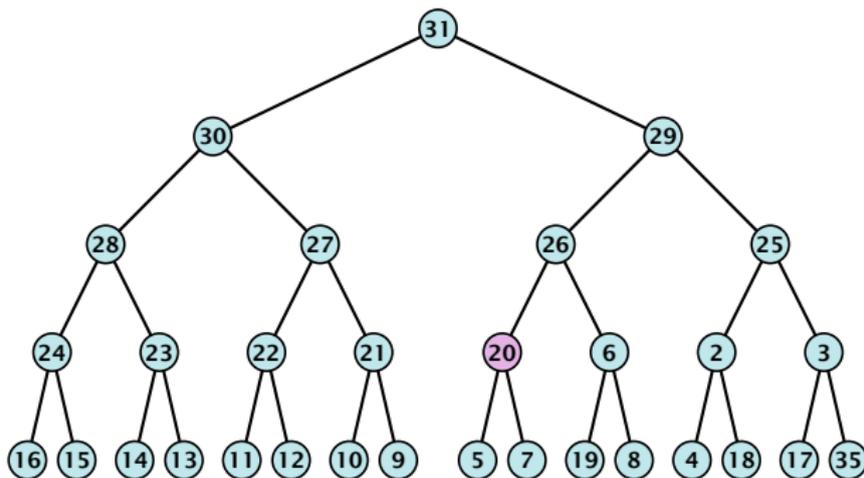
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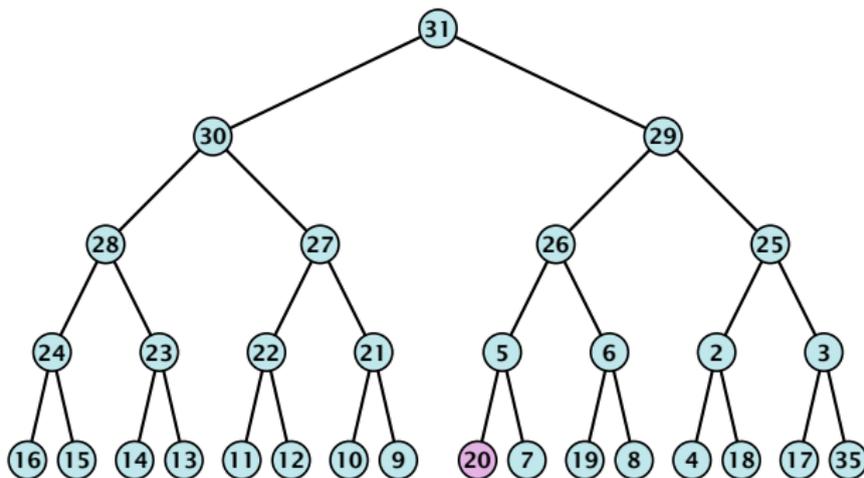
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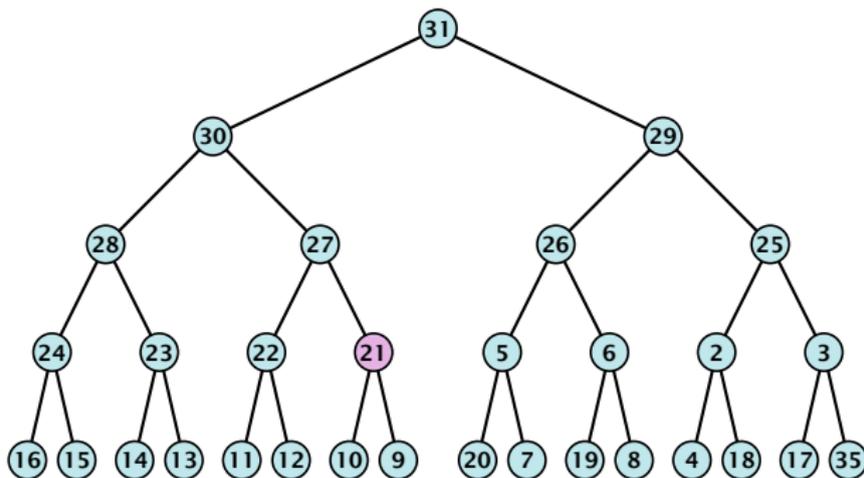
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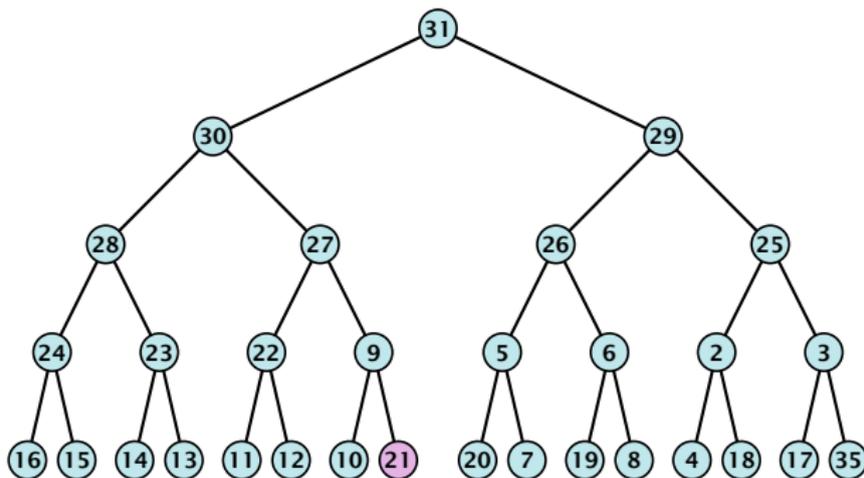
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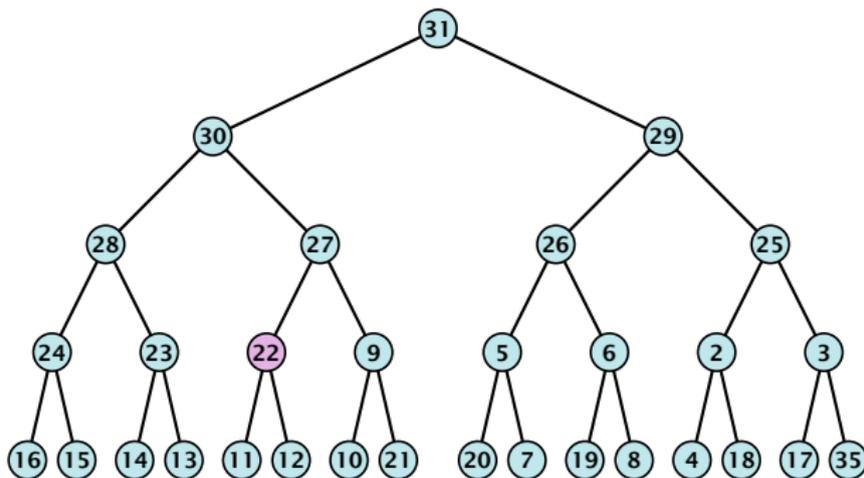
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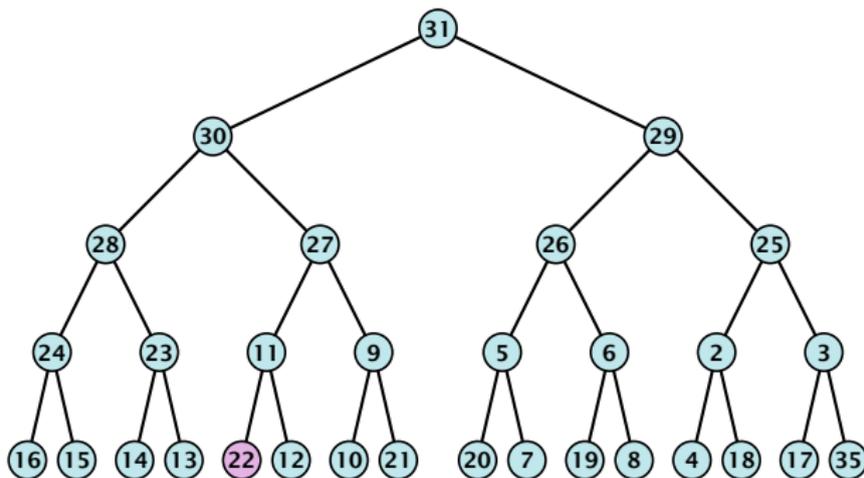
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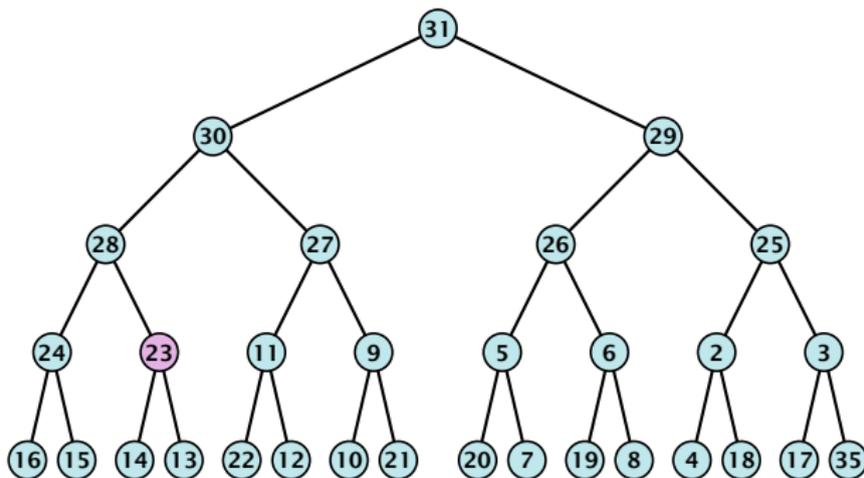
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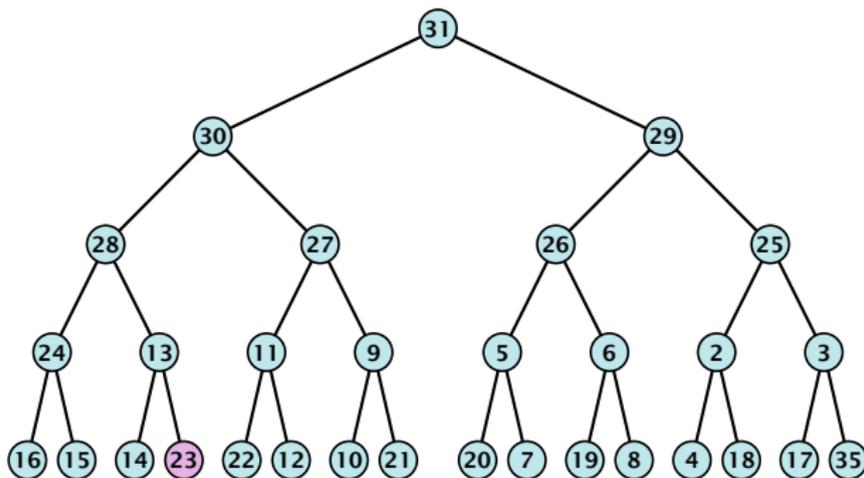
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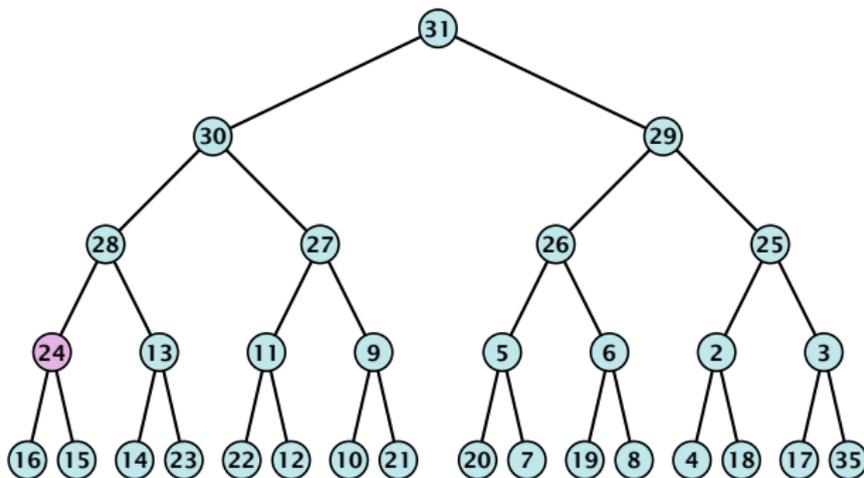
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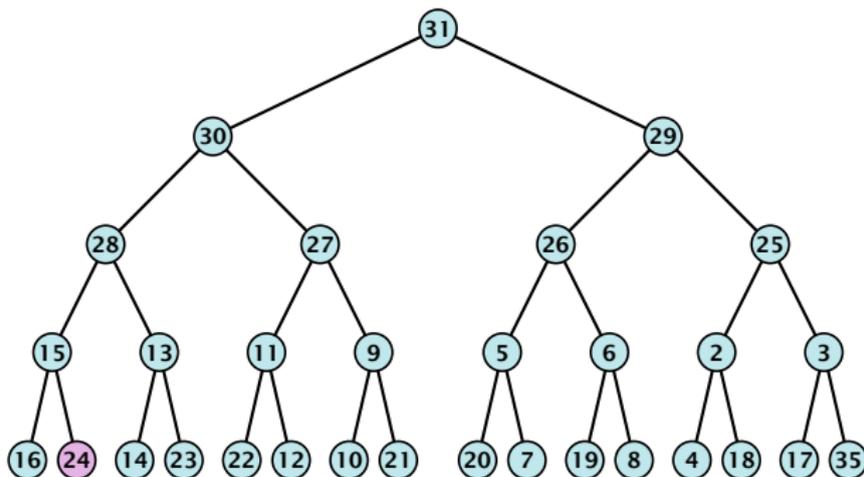
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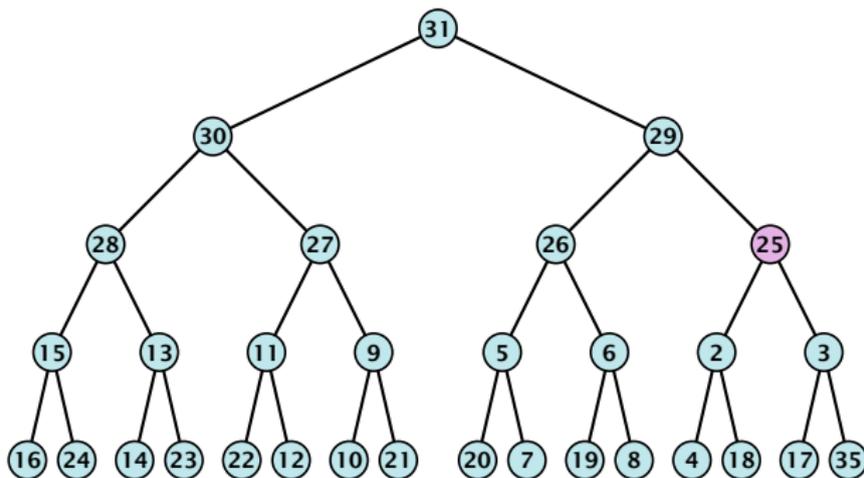
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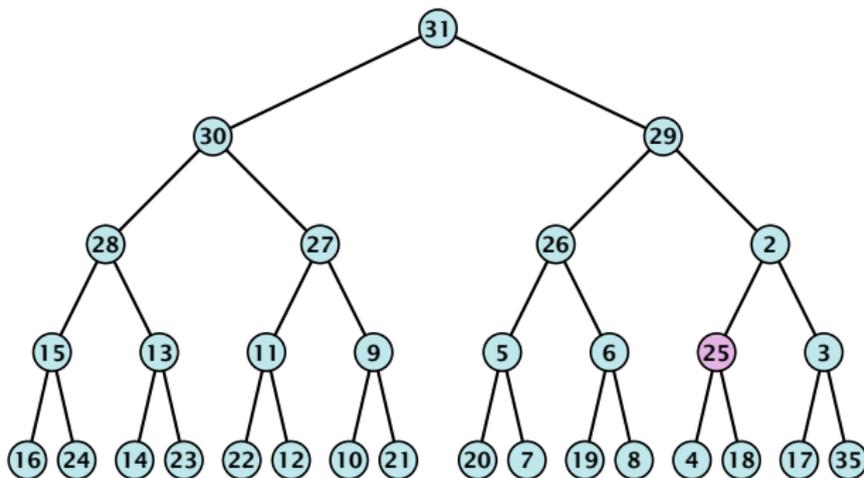
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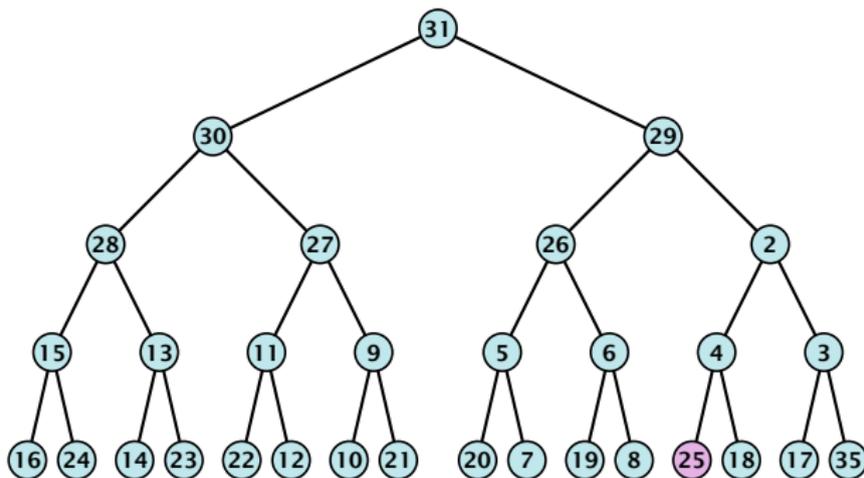
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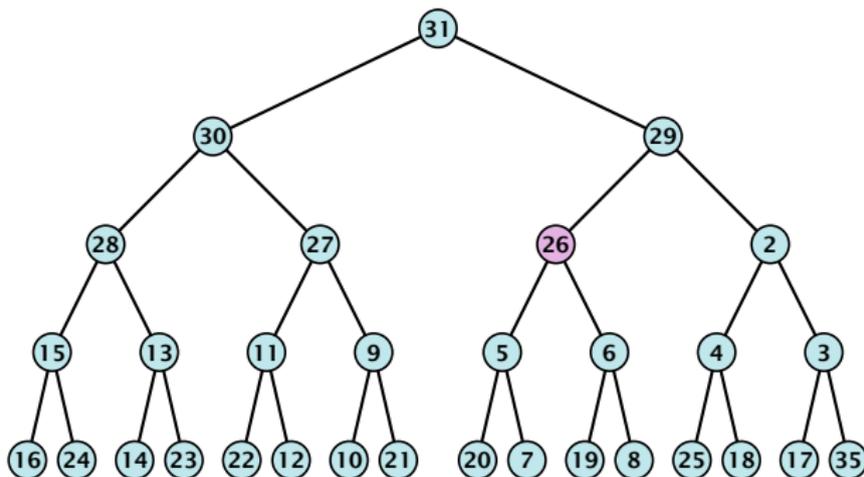
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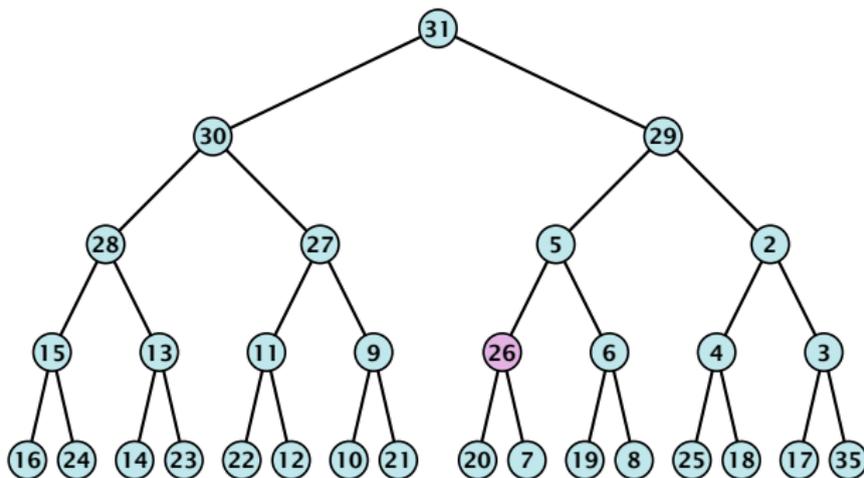
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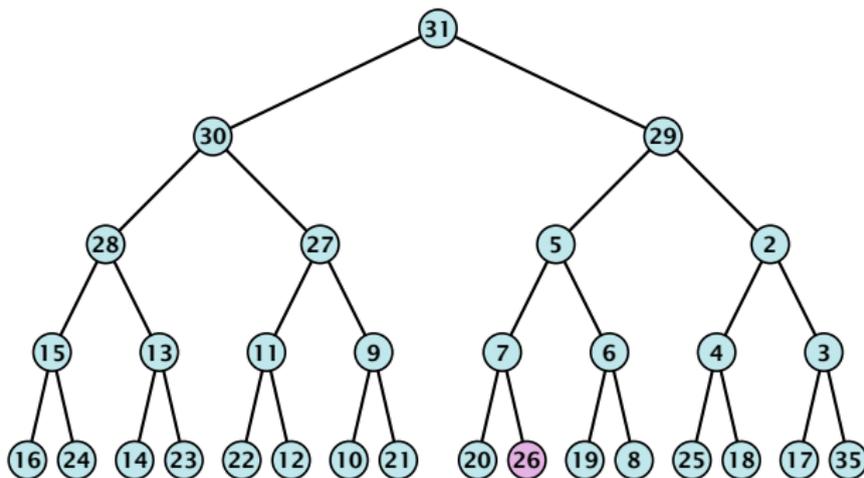
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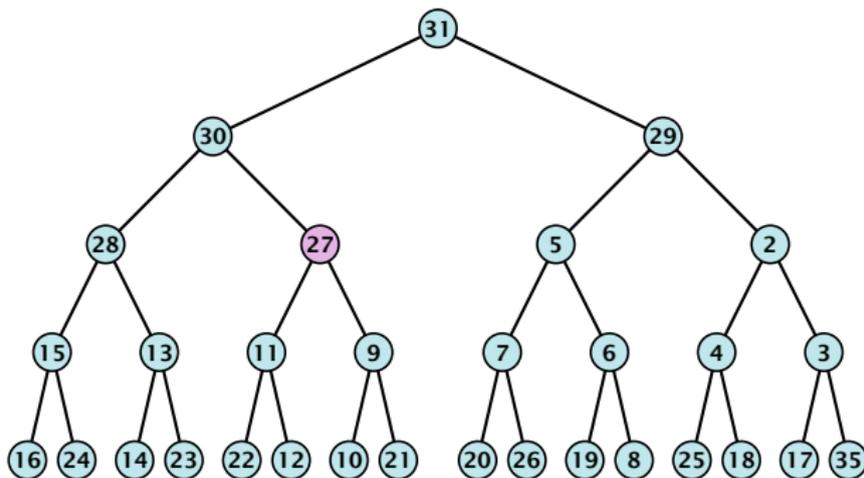
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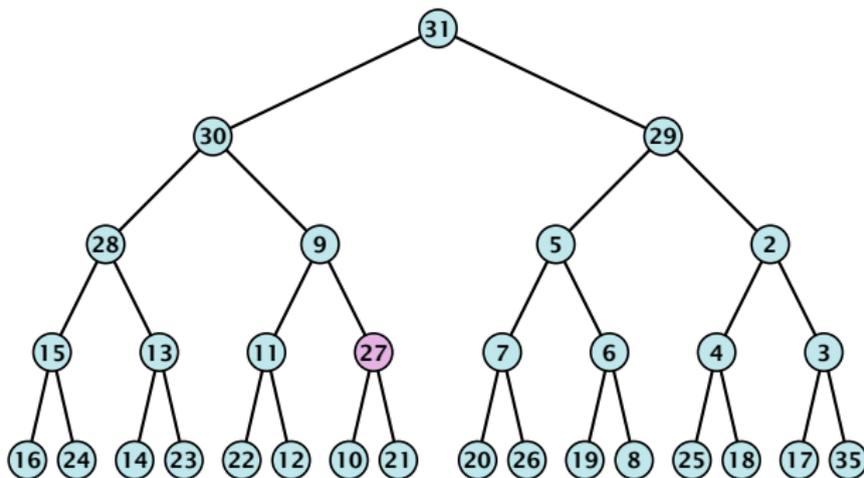
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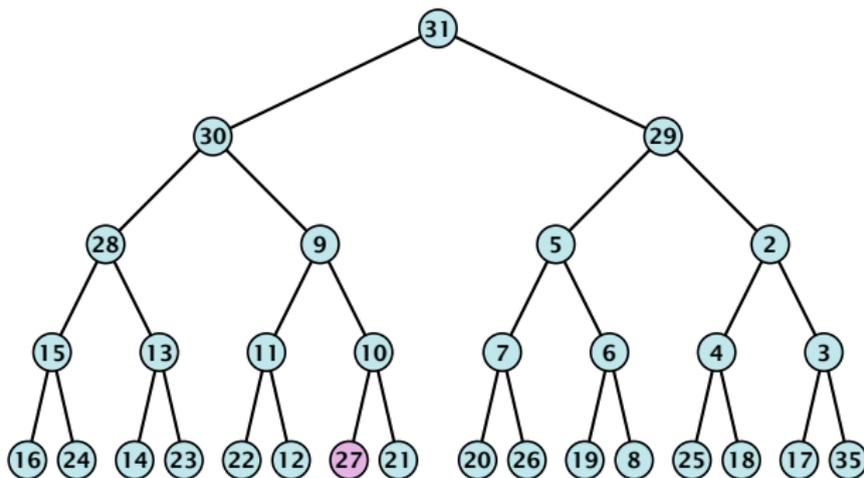
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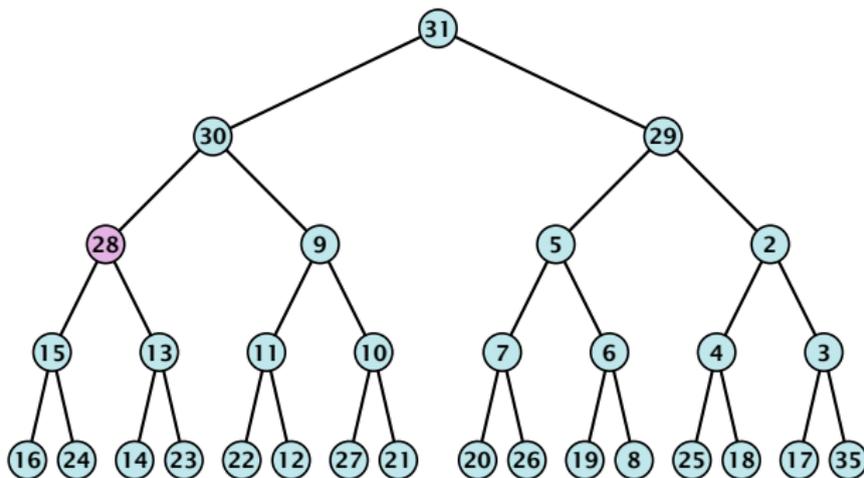
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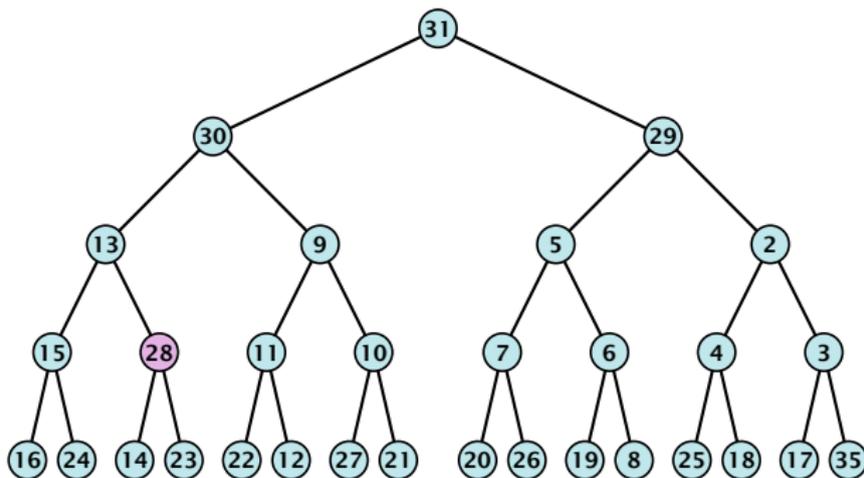
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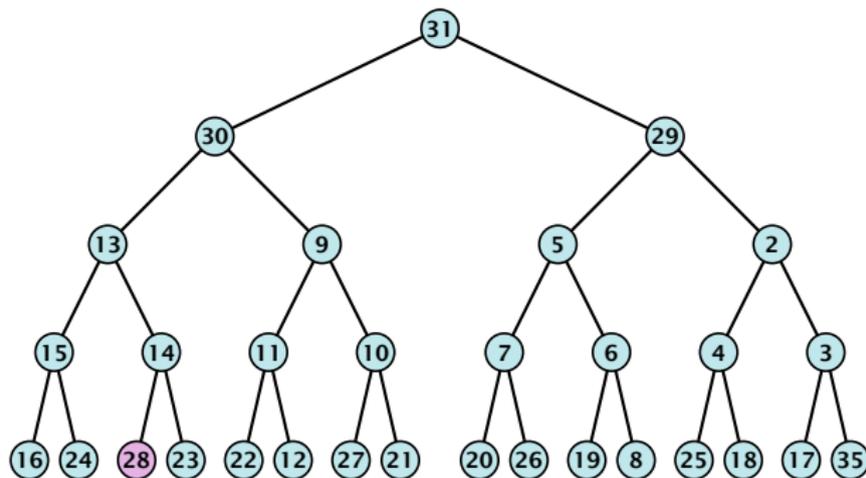
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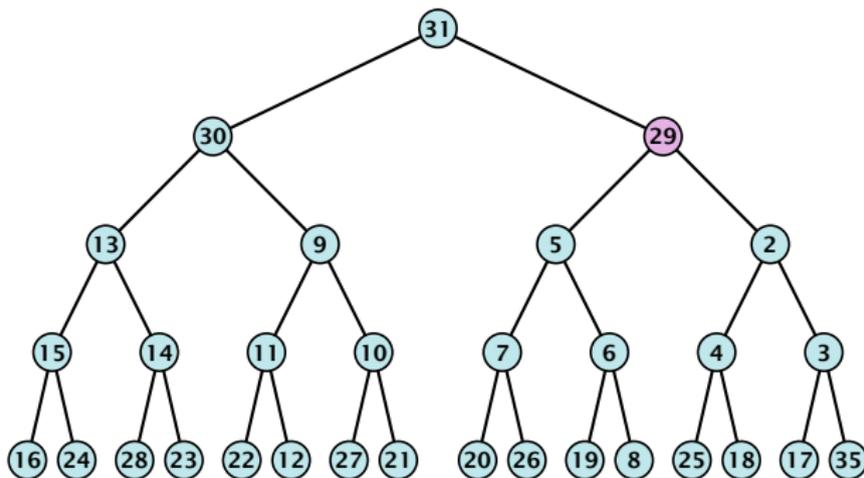
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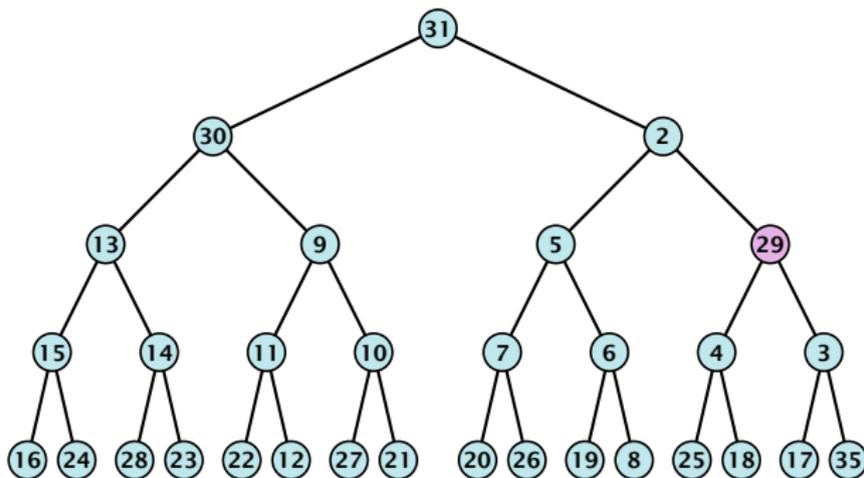
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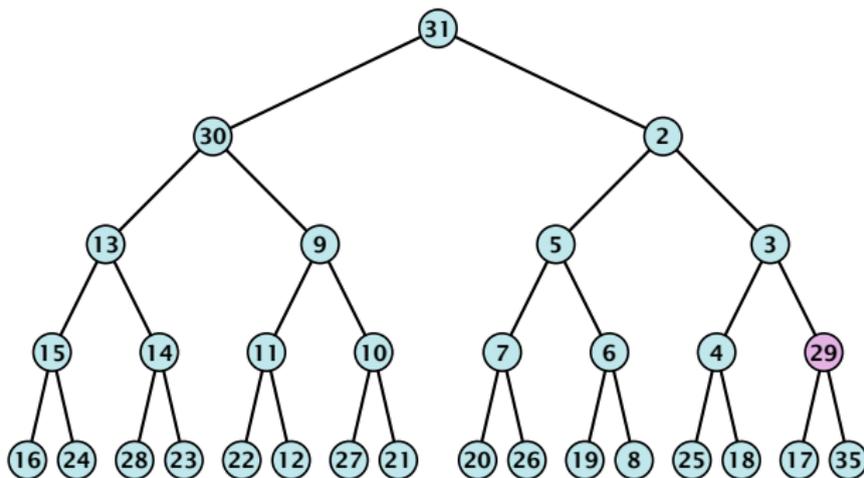
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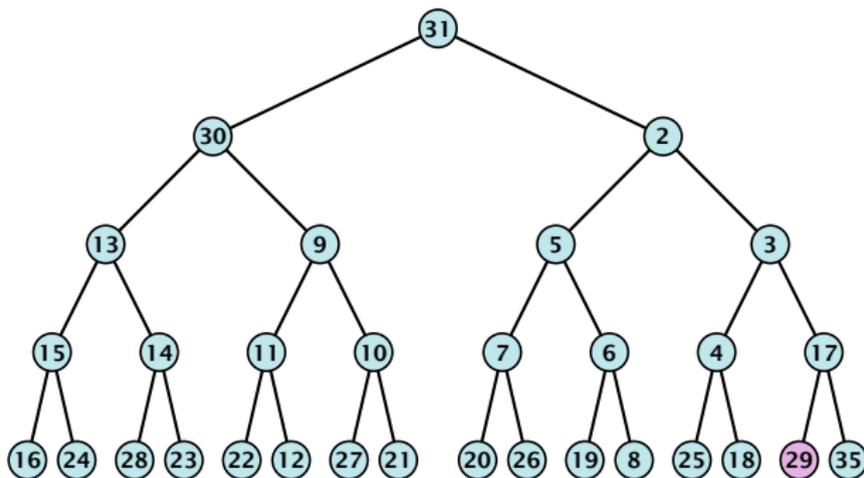
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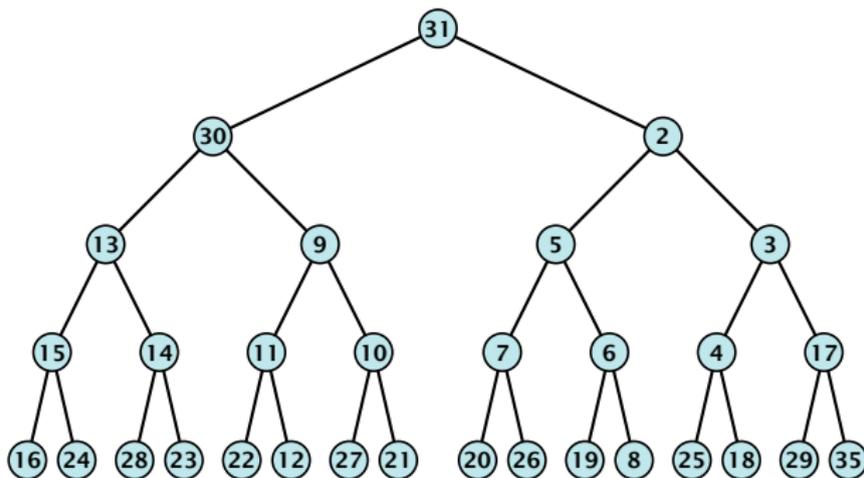
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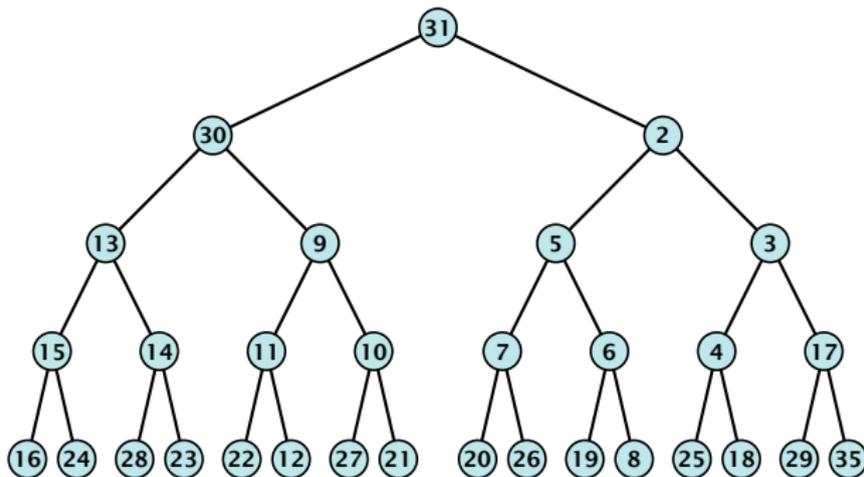
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Binary Heaps

Operations:

- ▶ **minimum()**: Return the root-element. Time $\mathcal{O}(1)$.
- ▶ **is-empty()**: Check whether root-pointer is null. Time $\mathcal{O}(1)$.
- ▶ **insert(k)**: Insert at x and bubble up. Time $\mathcal{O}(\log n)$.
- ▶ **delete(h)**: Swap with x and bubble up or sift-down. Time $\mathcal{O}(\log n)$.
- ▶ **build(x_1, \dots, x_n)**: Insert elements arbitrarily; then do sift-down operations starting with the lowest layer in the tree. Time $\mathcal{O}(n)$.

Binary Heaps

The standard implementation of binary heaps is via arrays. Let $A[0, \dots, n - 1]$ be an array

- ▶ The parent of i -th element is at position $\lfloor \frac{i-1}{2} \rfloor$.
- ▶ The left child of i -th element is at position $2i + 1$.
- ▶ The right child of i -th element is at position $2i + 2$.

Finding the successor of x is much easier than in the description on the previous slide. Simply increase or decrease x .

The resulting binary heap is not addressable. The elements don't maintain their positions and therefore there are not stable handles.

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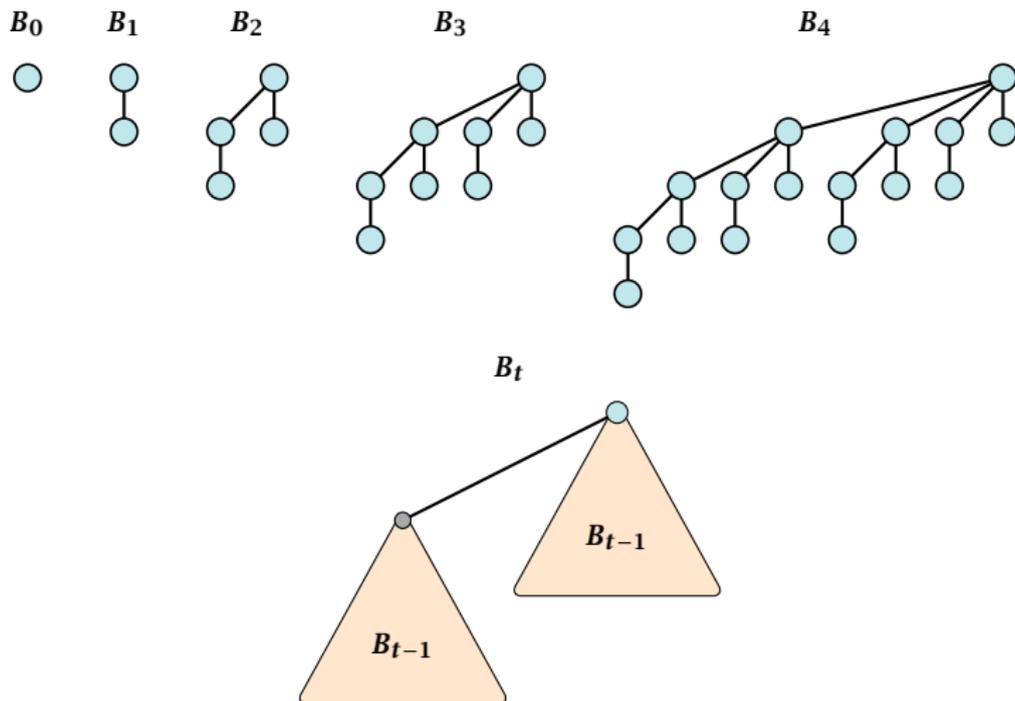
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8.2 Binomial Heaps

Operation	Binary Heap	BST	Binomial Heap	Fibonacci Heap*
build	n	$n \log n$	$n \log n$	n
minimum	1	$\log n$	$\log n$	1
is-empty	1	1	1	1
insert	$\log n$	$\log n$	$\log n$	1
delete	$\log n^{**}$	$\log n$	$\log n$	$\log n$
delete-min	$\log n$	$\log n$	$\log n$	$\log n$
decrease-key	$\log n$	$\log n$	$\log n$	1
merge	n	$n \log n$	$\log n$	1

Binomial Trees



Properties of Binomial Trees

- ▶ B_k has 2^k nodes.
- ▶ B_k has height k .
- ▶ The root of B_k has degree k .
- ▶ B_k has $\binom{k}{\ell}$ nodes on level ℓ .
- ▶ Deleting the root of B_k gives trees B_0, B_1, \dots, B_{k-1} .

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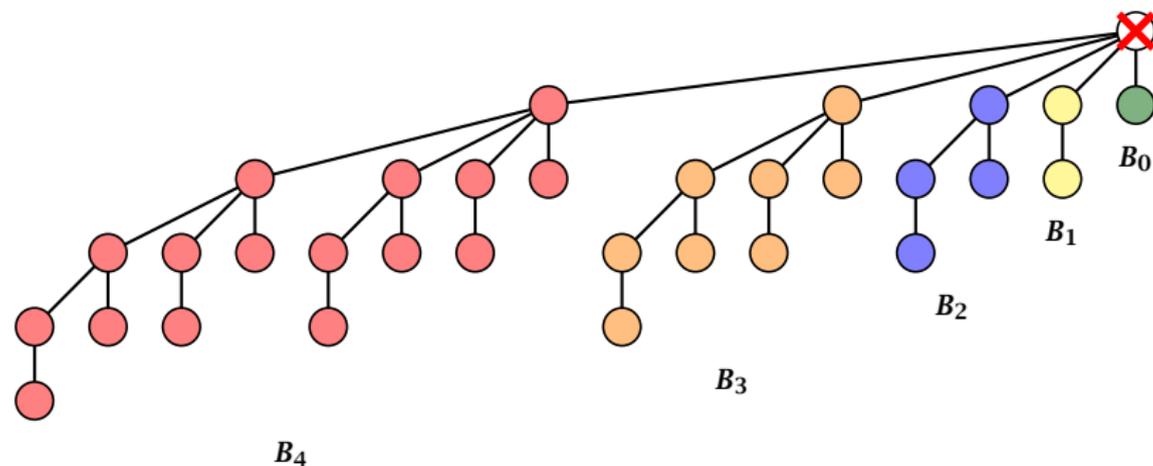
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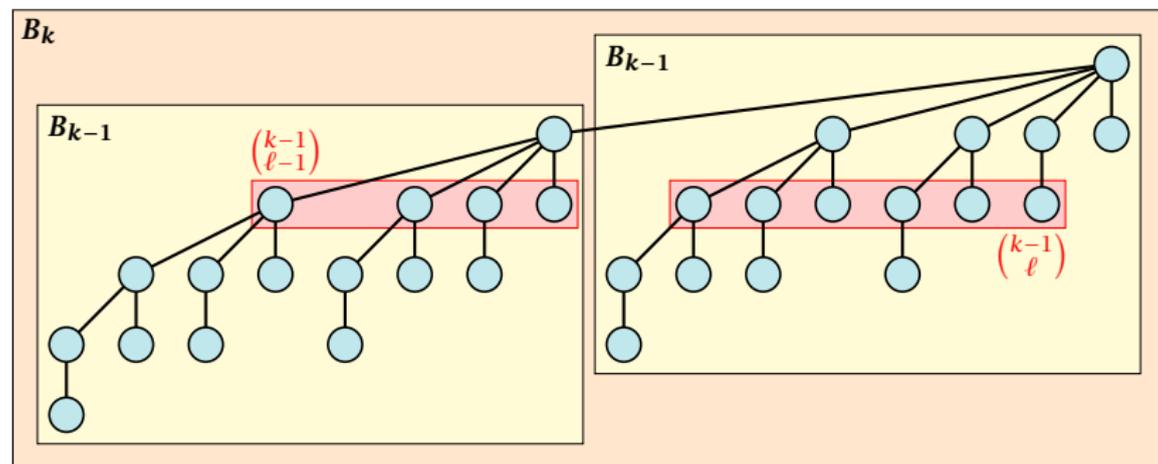
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Binomial Trees



Deleting the root of B_5 leaves sub-trees B_4, B_3, B_2 , and B_1 .

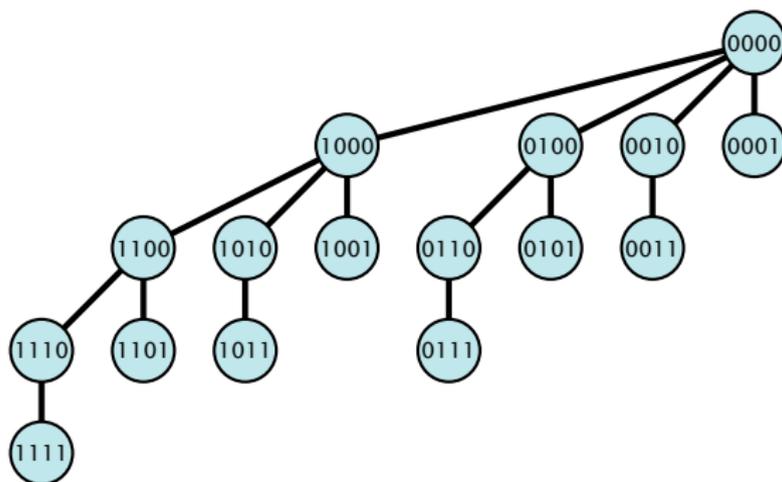
Binomial Trees



The number of nodes on level ℓ in tree B_k is therefore

$$\binom{k-1}{\ell-1} + \binom{k-1}{\ell} = \binom{k}{\ell}$$

Binomial Trees

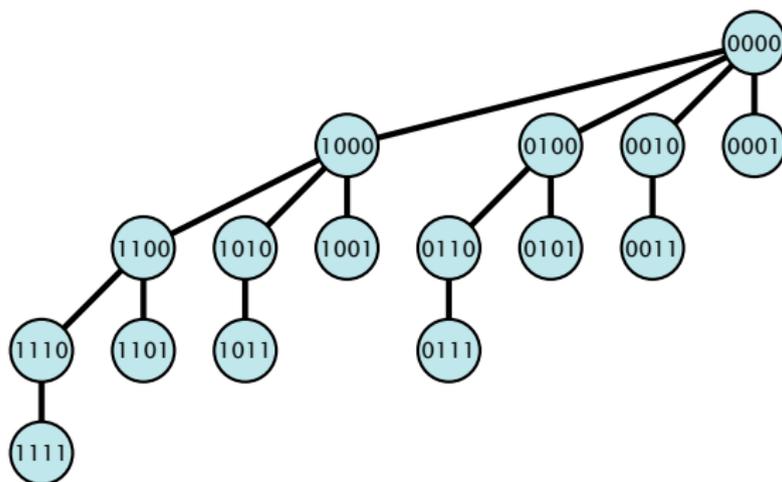


The binomial tree B_k is a sub-graph of the hypercube H_k .

The parent of a node with label b_n, \dots, b_1, b_0 is obtained by setting the least significant 1-bit to 0.

The ℓ -th level contains nodes that have ℓ 1's in their label.

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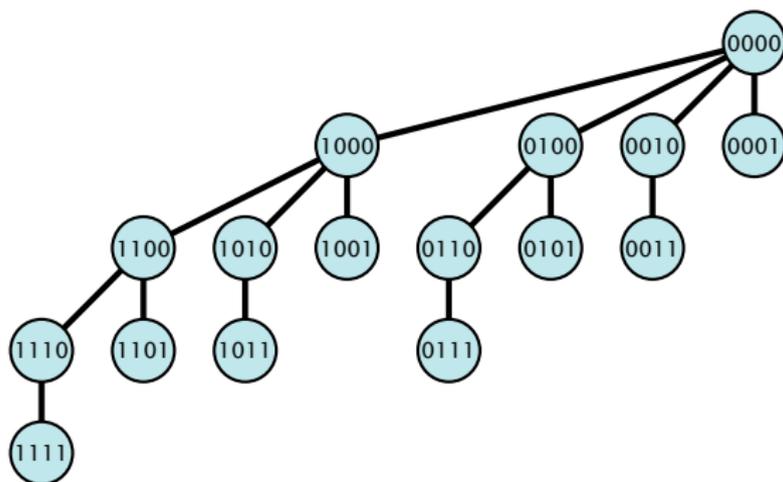


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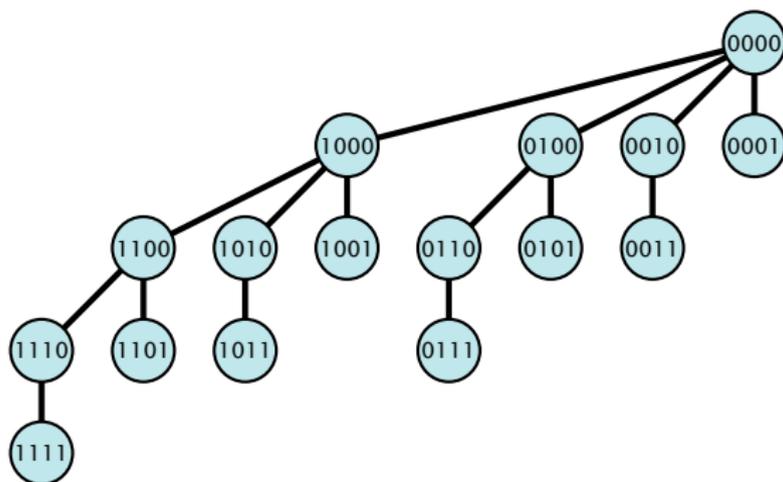


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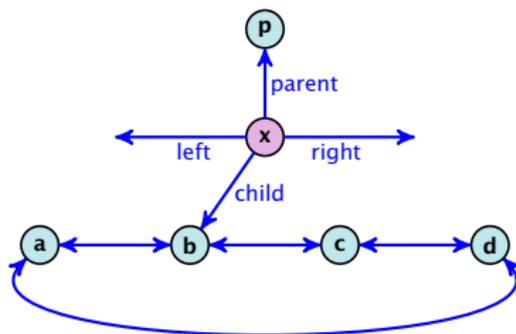
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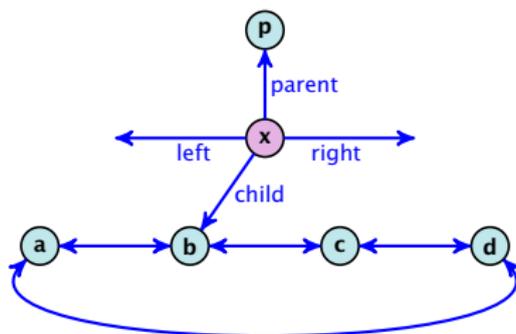
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- ▶ Pointers $x.\text{left}$ and $x.\text{right}$ point to the left and right sibling of x (if x does not have children then $x.\text{left} = x.\text{right} = x$).



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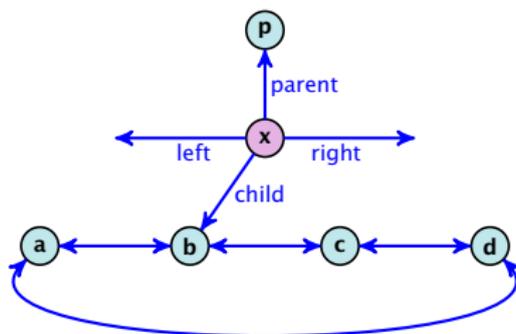
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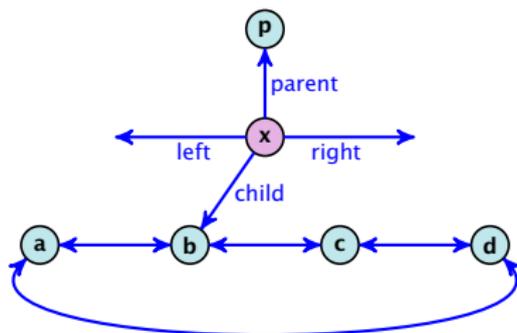
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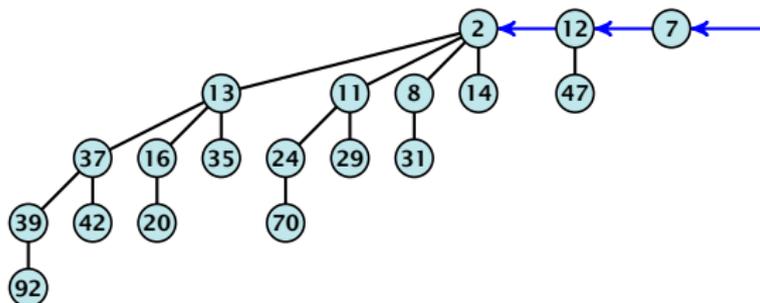
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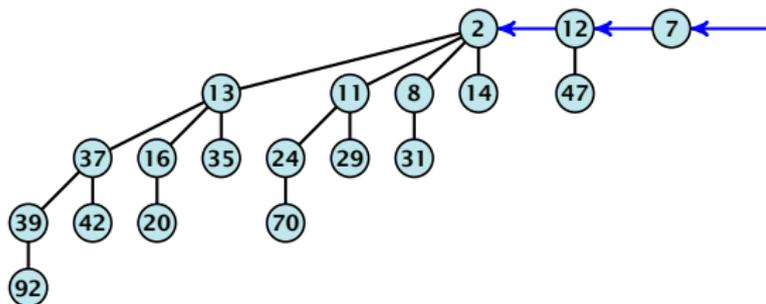


In a binomial heap the keys are arranged in a collection of binomial trees.

Every tree fulfills the heap-property

There is at most one tree for every dimension/order. For example the above heap contains trees B_0 , B_1 , and B_4 .

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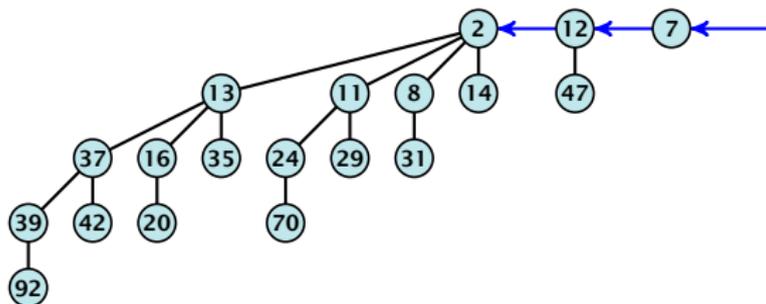


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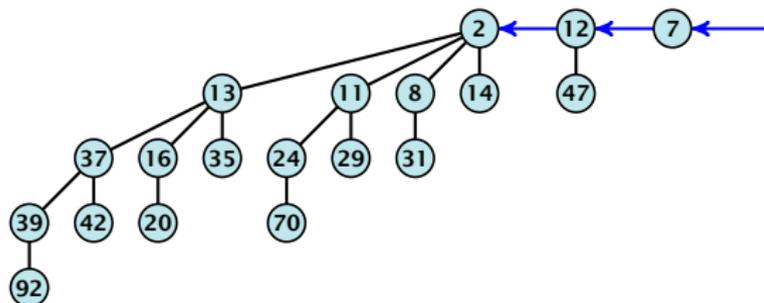


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Binomial Heap: Merge

Given the number n of keys to be stored in a binomial heap we can deduce the binomial trees that will be contained in the collection.

Let $B_{k_1}, B_{k_2}, B_{k_3}, k_i < k_{i+1}$ denote the binomial trees in the collection and recall that every tree may be contained at most once.

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Given the number n of keys to be stored in a binomial heap we can deduce the binomial trees that will be contained in the collection.

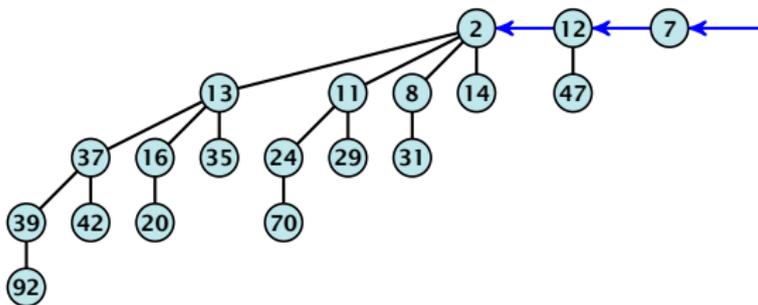
Let $B_{k_1}, B_{k_2}, B_{k_3}, k_i < k_{i+1}$ denote the binomial trees in the collection and recall that every tree may be contained at most once.

Then $n = \sum_i 2^{k_i}$ must hold. But since the k_i are all distinct this means that the k_i define the non-zero bit-positions in the dual representation of n .

Binomial Heap

Properties of a heap with n keys:

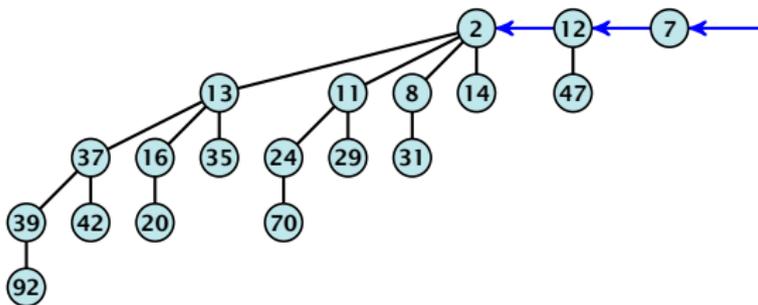
- ▶ Let $n = b_d b_{d-1} \dots b_0$ denote the dual representation of n .
- ▶ The heap contains tree B_i iff $b_i = 1$.
- ▶ Hence, at most $\lfloor \log n \rfloor + 1$ trees.
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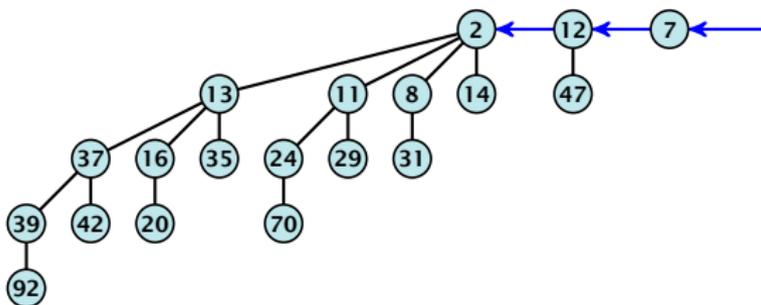
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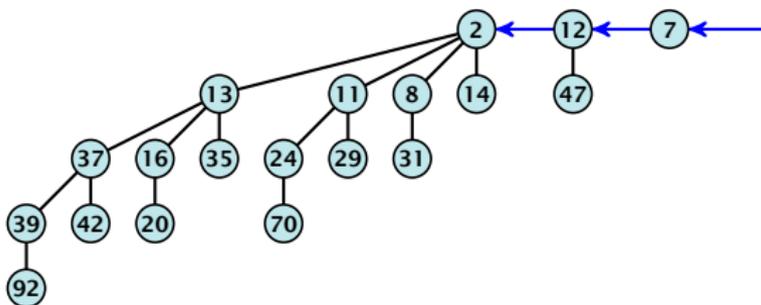
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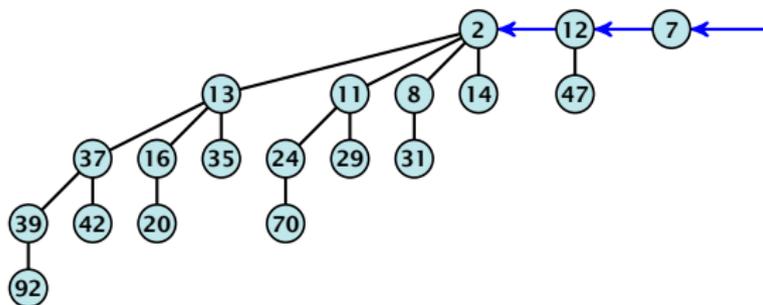
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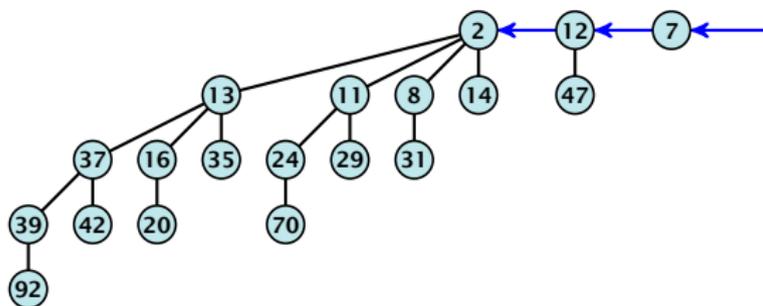
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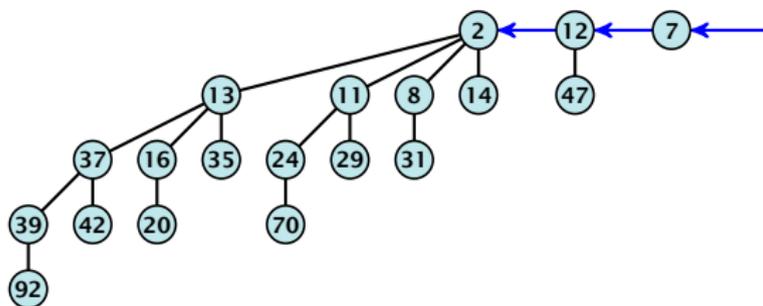
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The merge-operation is instrumental for binomial heaps.

A merge is easy if we have two heaps with different binomial trees. We can simply merge the tree-lists.

Otherwise, we cannot do this because the merged heap is not allowed to contain two trees of the same order.

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For more trees the technique is analogous to binary addition.



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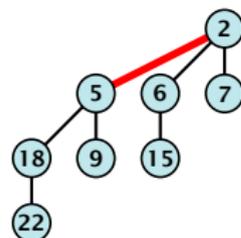
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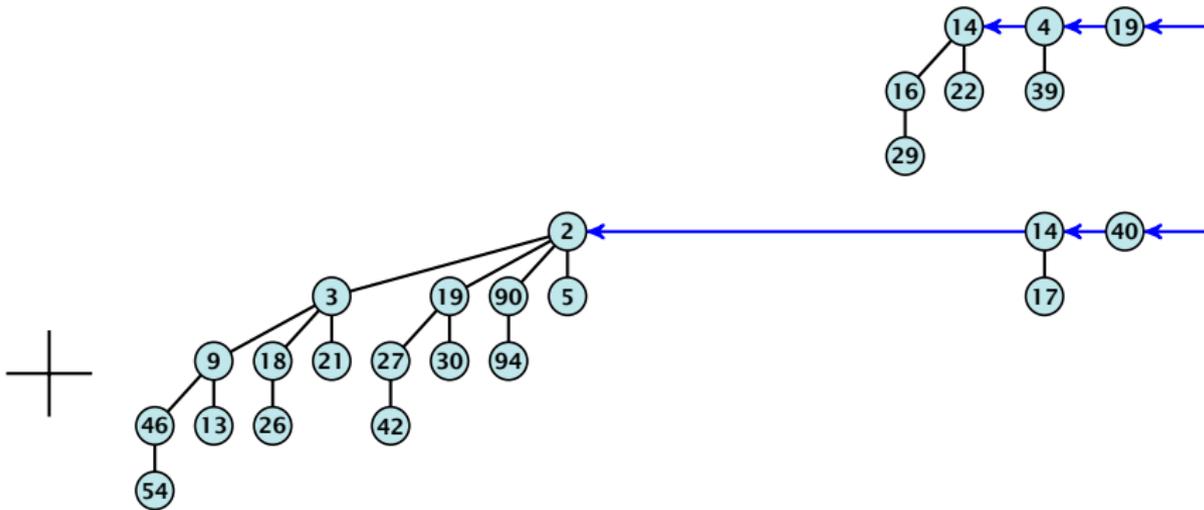
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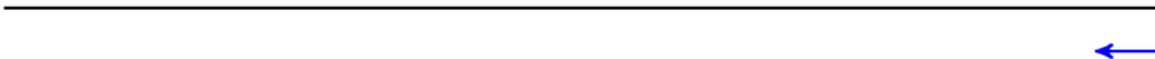
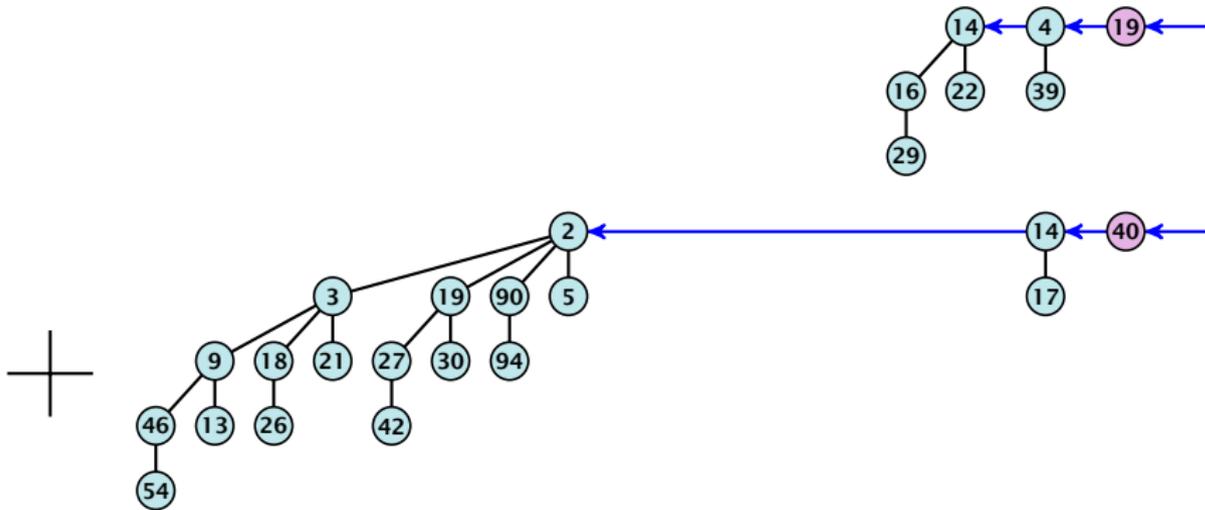
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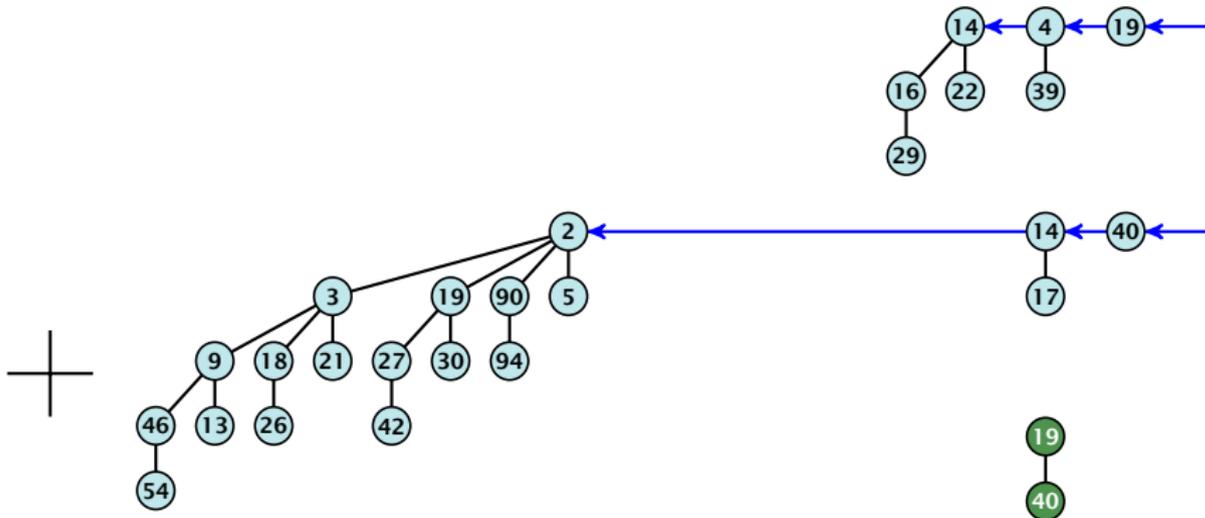
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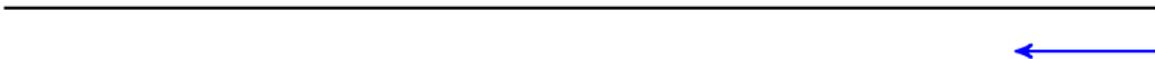
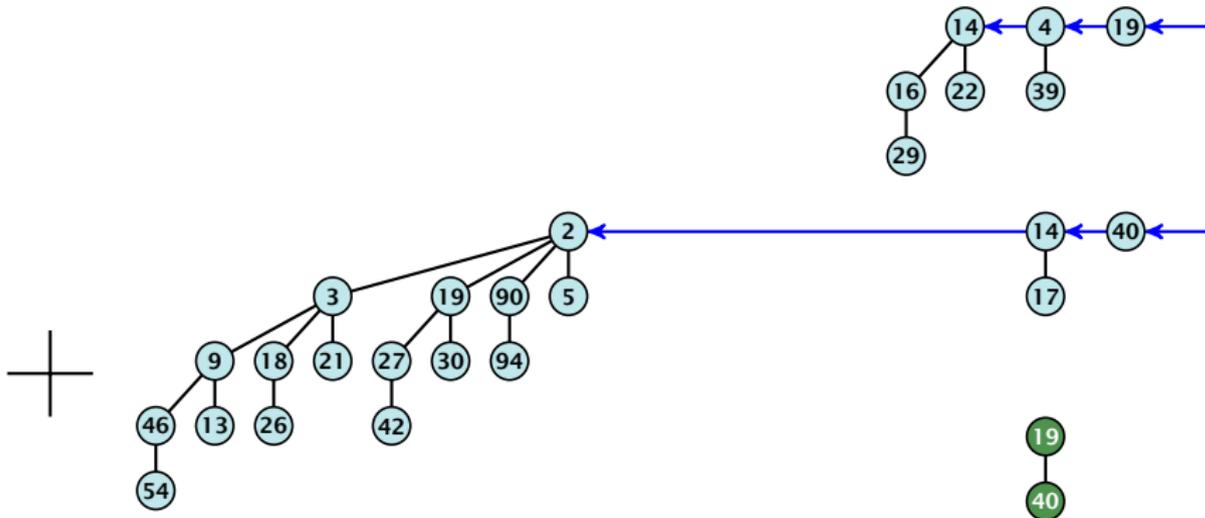
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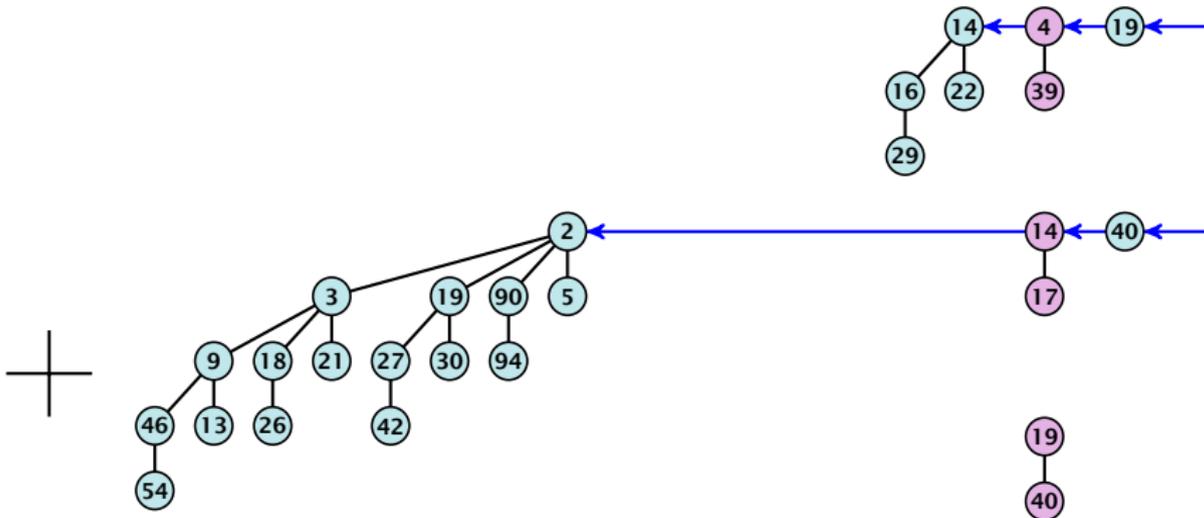


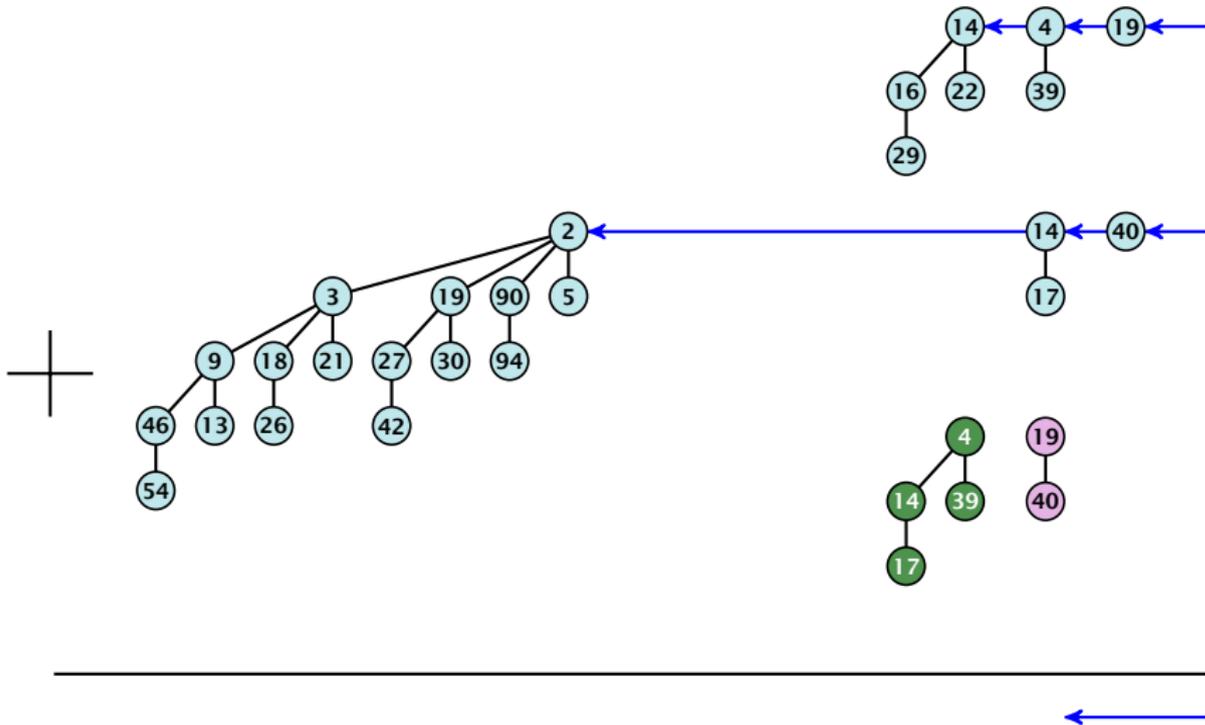


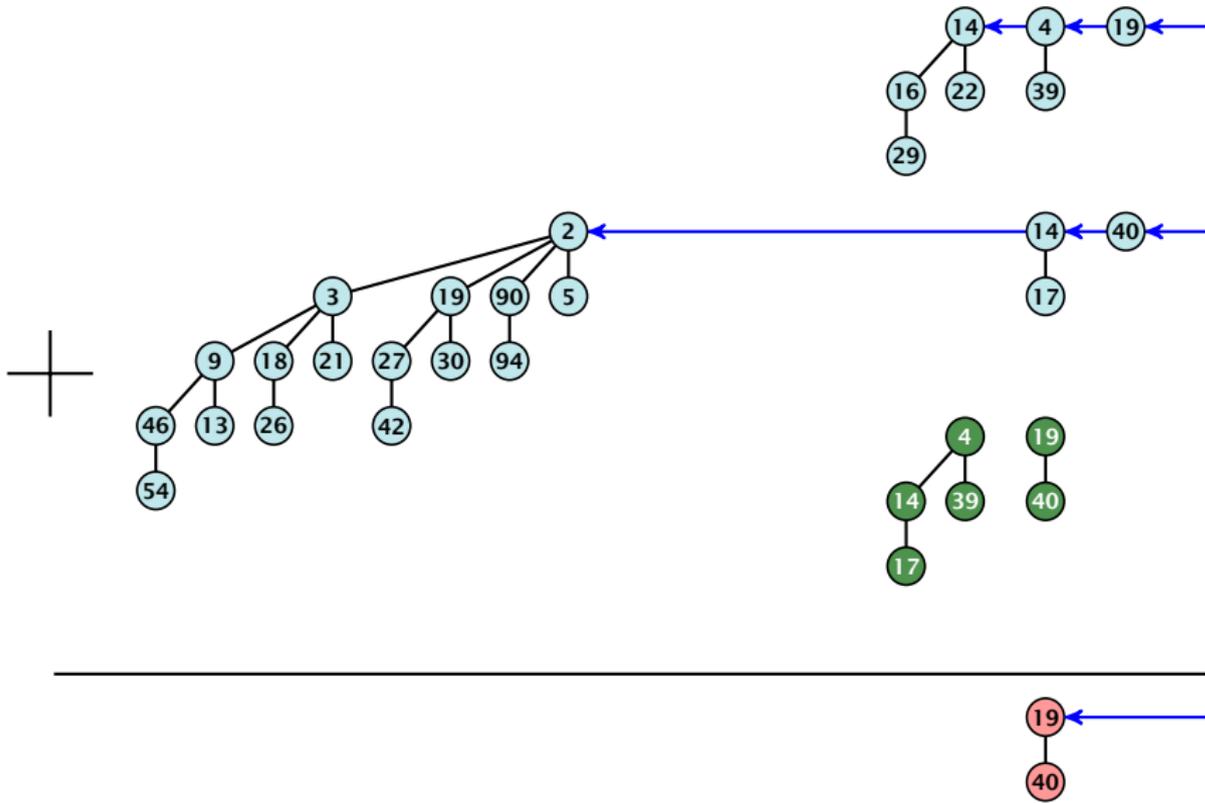


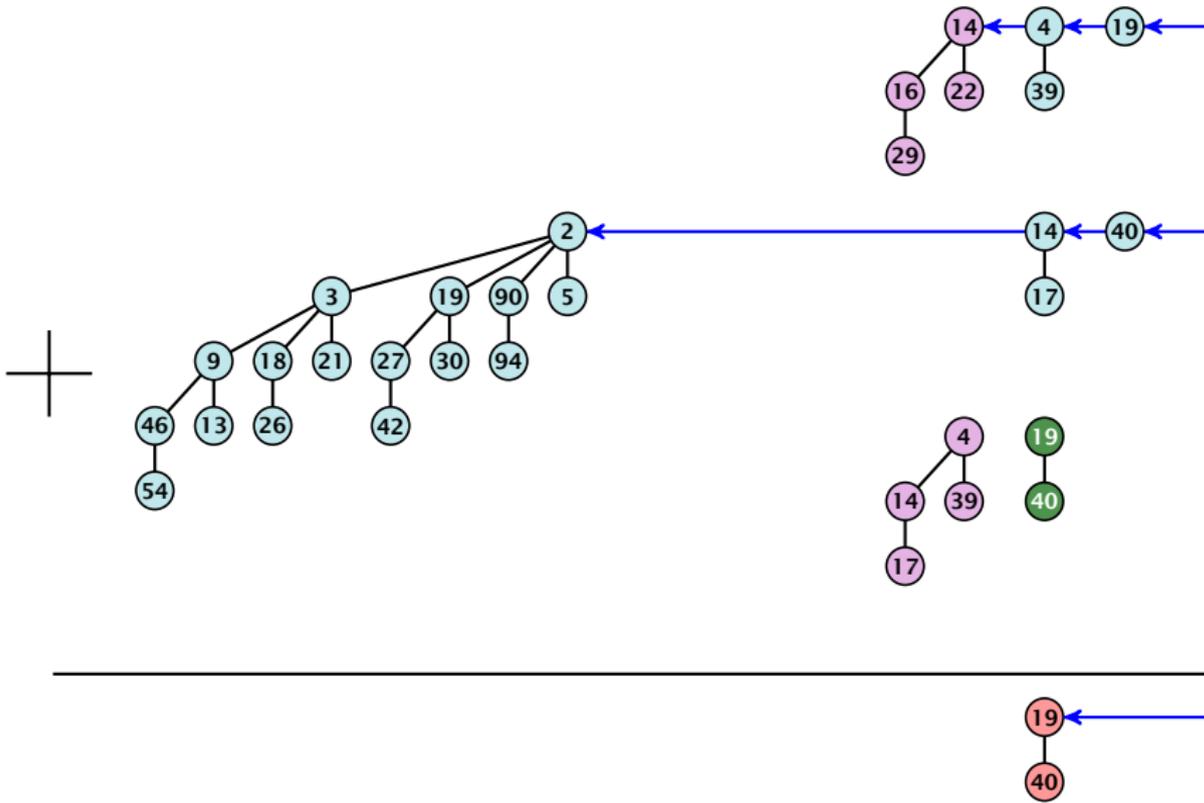


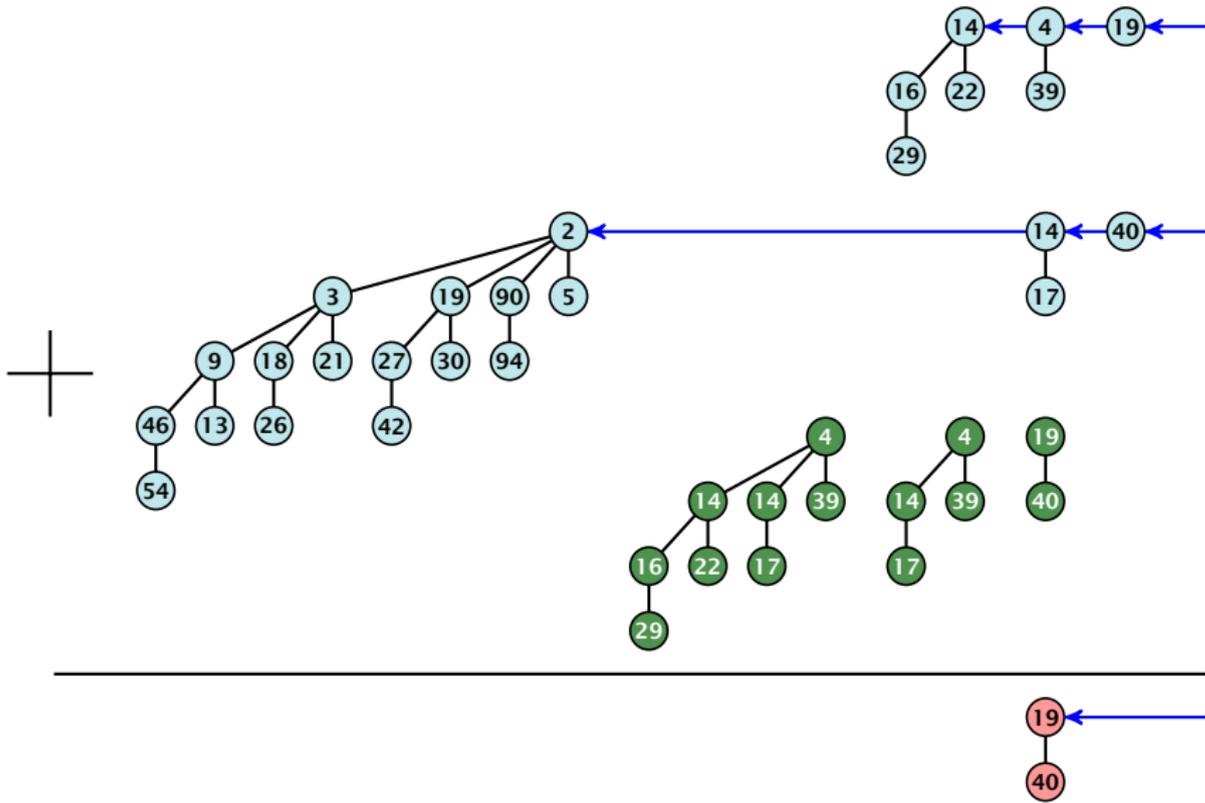


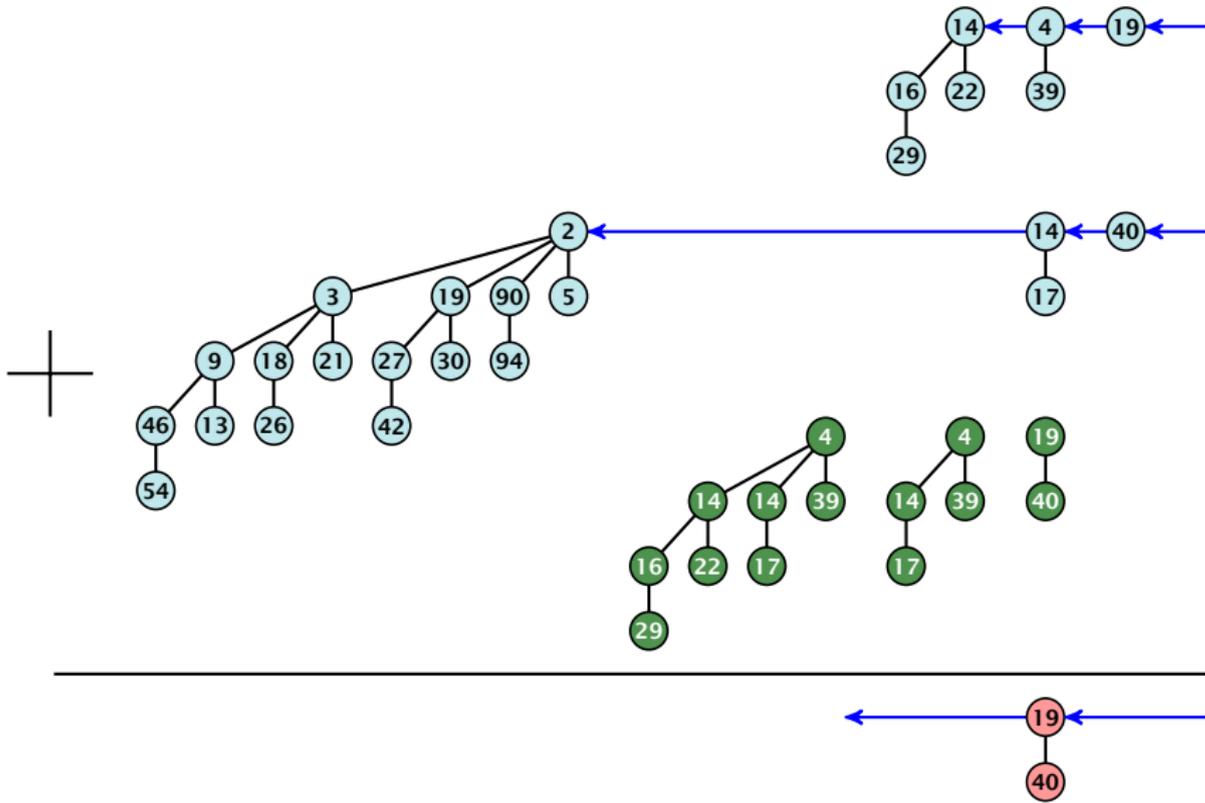


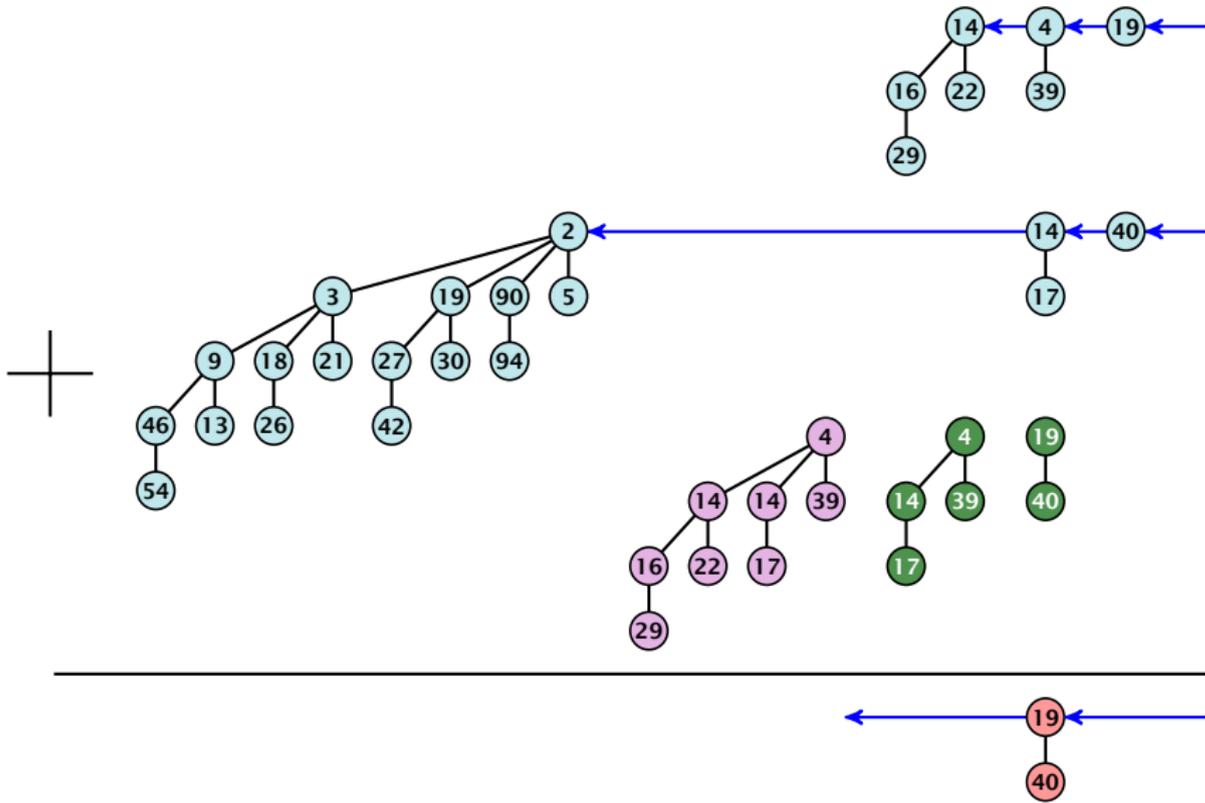


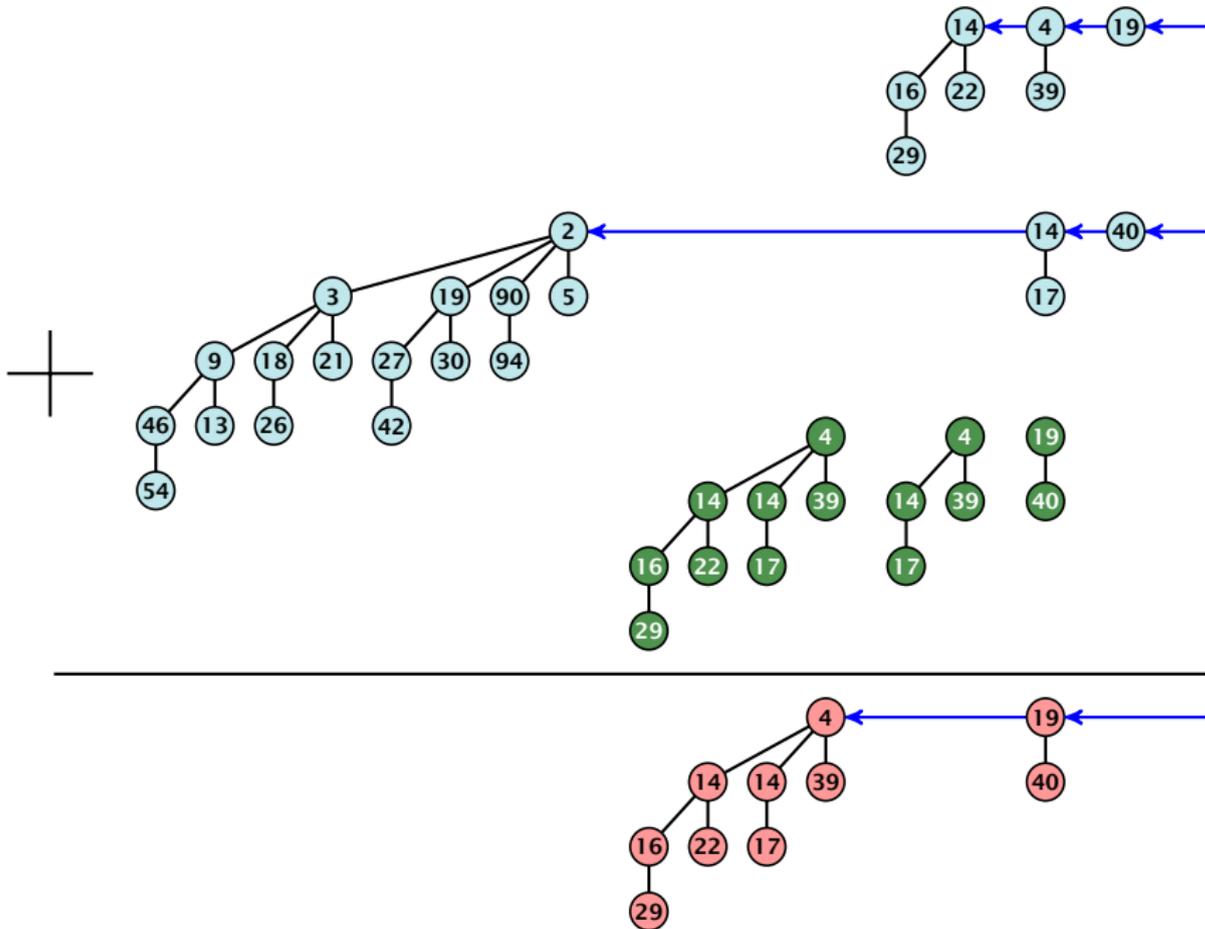


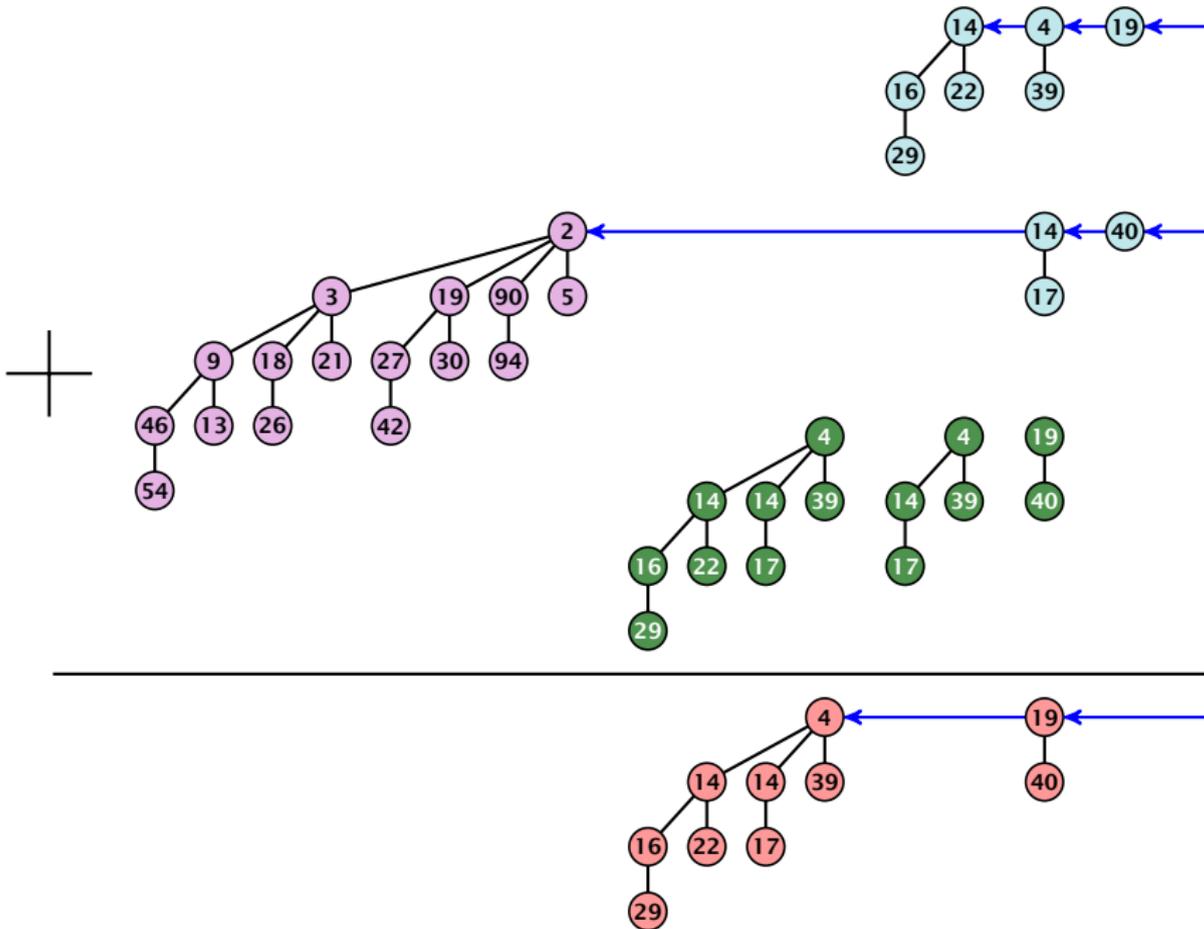


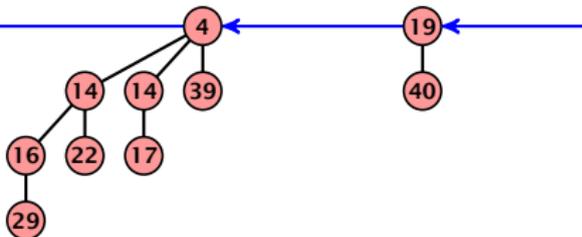
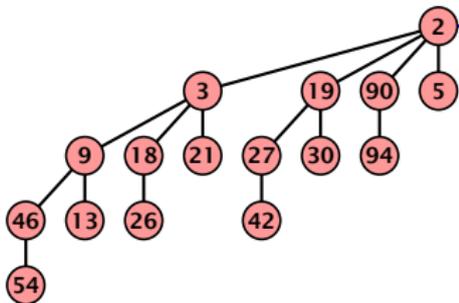
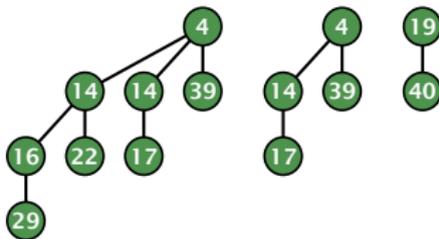
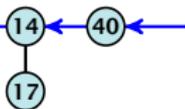
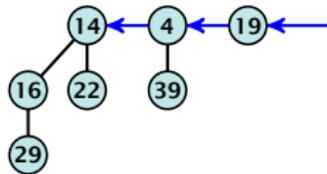
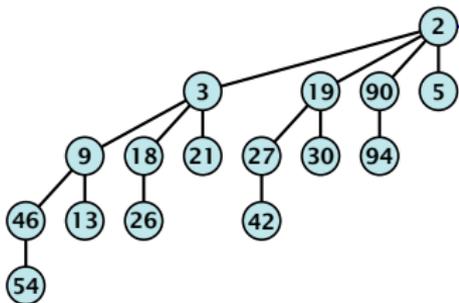


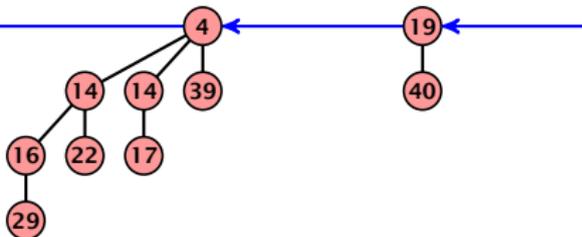
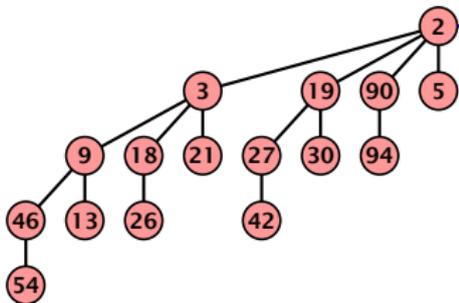
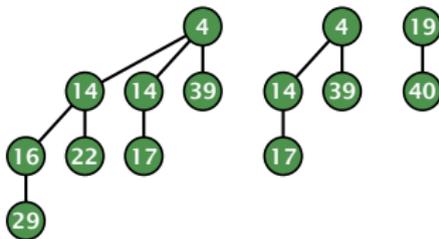
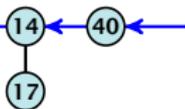
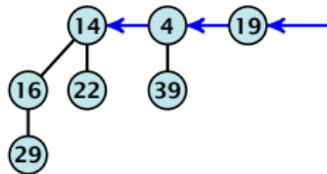
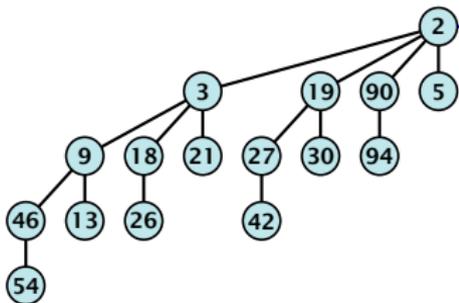












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- ▶ Analogous to binary addition.
- ▶ Time is proportional to the number of trees in both heaps.
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S.delete-min():

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A data structure with operations $op_1(), \dots, op_k()$ has amortized running times t_1, \dots, t_k for these operations if the following holds.

Suppose you are given a sequence of operations (starting with an empty data-structure) that operate on at most n elements, and let k_i denote the number of occurrences of $op_i()$ within this sequence. Then the actual running time must be at most $\sum_i k_i t_i(n)$.

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Potential Method

Introduce a potential for the data structure.

Let $\Phi(D)$ be the potential of the data structure D .

Let c_i be the cost of the i -th operation.

$$c_i = \Phi(D_{i+1}) - \Phi(D_i) + \hat{c}_i$$

Summing up $\Phi(D_{i+1}) - \Phi(D_i)$

Then

$$\sum_{i=1}^k c_i \leq \sum_{i=1}^k c_i + \Phi(D_k) - \Phi(D_0) = \sum_{i=1}^k \hat{c}_i$$

This means the amortized costs can be used to derive a bound on the total cost.

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$$\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1}) .$$

- ▶ Show that $\Phi(D_i) \geq \Phi(D_0)$.

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Stack

- ▶ $S.$ push()
- ▶ $S.$ pop()
- ▶ $S.$ multipop(k): removes k items from the stack. If the stack currently contains less than k items it empties the stack.

Actual cost:

- ▶ $S.$ push(): cost 1.
- ▶ $S.$ pop(): cost 1.
- ▶ $S.$ multipop(k): cost $\min\{\text{size}, k\}$.

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- ▶ $S.$ multipop(k): removes k items from the stack. If the stack currently contains less than k items it empties the stack.

Actual cost:

- ▶ $S.$ push(): cost 1.
- ▶ $S.$ pop(): cost 1.
- ▶ $S.$ multipop(k): cost $\min\{\text{size}, k\}$.

Example: Stack

Use potential function $\Phi(S) = \text{number of elements on the stack}$.

Amortized cost:

• $S.push()$: cost

$$C_{push} = C_{push} + \Delta\Phi = 1 + 1 = 2$$

• $S.pop()$: cost

$$C_{pop} = C_{pop} + \Delta\Phi = 1 - 1 = 0$$

• $S.empty() \text{ or } !S.empty()$: cost

$$C_{empty} = C_{empty} + \Delta\Phi = \Phi(\text{stack}) - \Phi(\text{stack}) = 0$$

Example: Stack

Use potential function $\Phi(S) = \text{number of elements on the stack}$.

Amortized cost:

- ▶ **$S.\text{push}()$** : cost

$$\hat{C}_{\text{push}} = C_{\text{push}} + \Delta\Phi = 1 + 1 \leq 2 .$$

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- ▶ $S.\text{multipop}(k)$: cost

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Example: Binary Counter

Incrementing a binary counter:

Consider a computational model where each bit-operation costs one time-unit.

Incrementing an n -bit binary counter may require to examine n -bits, and maybe change them.

Actual cost:

- ▶ Changing bit from 0 to 1: cost 1.
- ▶ Changing bit from 1 to 0: cost 1.
- ▶ Increment: cost is $k + 1$, where k is the number of consecutive ones in the least significant bit-positions (e.g., 001101 has $k = 1$).

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Example: Binary Counter

Choose potential function $\Phi(x) = k$, where k denotes the number of ones in the binary representation of x .

Amortized cost:

• Changing bit from 0 to 1: cost

$$C_{i+1} - C_i + \Delta\Phi = 1 - 1 \leq 0$$

• Changing bit from 1 to 0: cost

$$C_{i+1} - C_i + \Delta\Phi = 1 - 1 \leq 0$$

• Bonus: Let l denote the number of consecutive ones in the first i significant bit-positions. An increment applies l $0 \rightarrow 1$ -operations, and one $1 \rightarrow 0$ -operation.

• Hence, the amortized cost is $C_{i+1} - C_i \leq 2$.

Example: Binary Counter

Choose potential function $\Phi(x) = k$, where k denotes the number of ones in the binary representation of x .

Amortized cost:

- ▶ Changing bit from 0 to 1: cost

$$\hat{C}_{0 \rightarrow 1} = C_{0 \rightarrow 1} + \Delta\Phi = 1 + 1 \leq 2 .$$

- ▶ Changing bit from 1 to 0: cost 0.

$$\hat{C}_{1 \rightarrow 0} = C_{1 \rightarrow 0} + \Delta\Phi = 1 - 1 \leq 0 .$$

- ▶ Increment. Let k denotes the number of consecutive ones in the least significant bit-positions. An increment involves k (1 \rightarrow 0)-operations, and one (0 \rightarrow 1)-operation.

Hence, the amortized cost is $k\hat{C}_{1 \rightarrow 0} + \hat{C}_{0 \rightarrow 1} \leq 2$.

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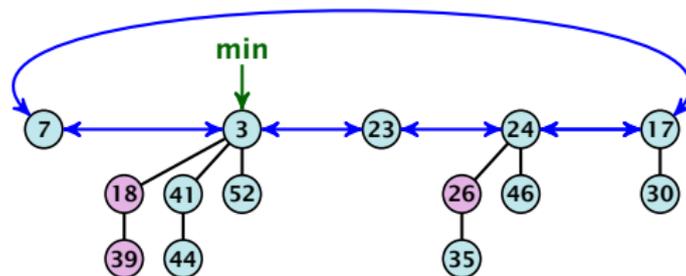
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8.3 Fibonacci Heaps

Collection of trees that fulfill the heap property.

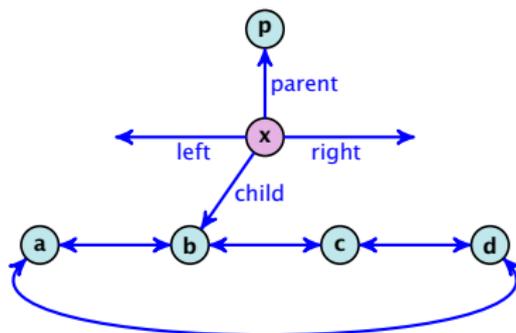
Structure is much more relaxed than binomial heaps.



8.3 Fibonacci Heaps

How do we implement trees with non-constant degree?

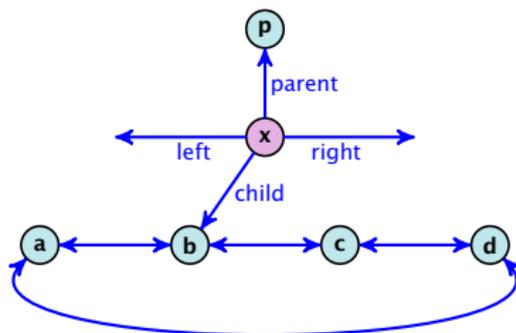
- ▶ The children of a node are arranged in a **circular linked list**.
- ▶ A child-pointer points to an arbitrary node within the list.
- ▶ A parent-pointer points to the parent node.
- ▶ Pointers $x.\text{left}$ and $x.\text{right}$ point to the left and right sibling of x (if x does not have siblings then $x.\text{left} = x.\text{right} = x$).



8.3 Fibonacci Heaps

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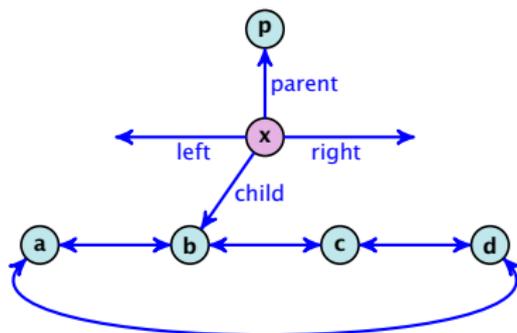
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8.3 Fibonacci Heaps

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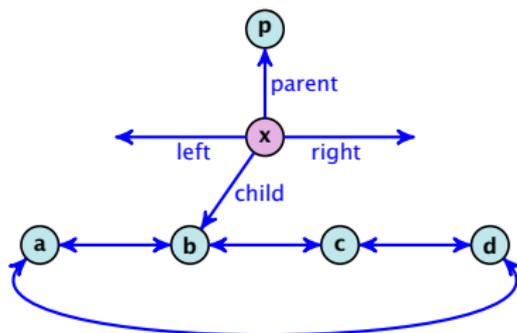
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8.3 Fibonacci Heaps

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8.3 Fibonacci Heaps

- ▶ Given a pointer to a node x we can splice out the sub-tree rooted at x in constant time.
- ▶ We can add a child-tree T to a node x in constant time if we are given a pointer to x and a pointer to the root of T .

8.3 Fibonacci Heaps

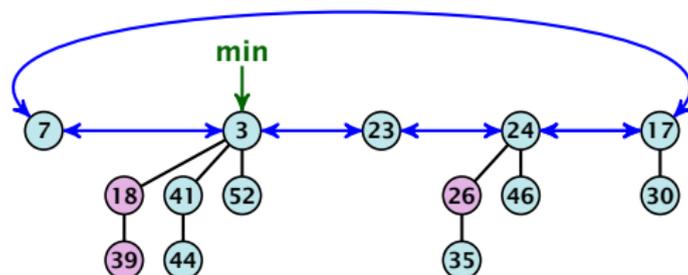
Additional implementation details:

- ▶ Every node x stores its degree in a field $x.degree$. Note that this can be updated in constant time when adding a child to x .
- ▶ Every node stores a boolean value $x.marked$ that specifies whether x is **marked** or not.

8.3 Fibonacci Heaps

The potential function:

- ▶ $t(S)$ denotes the number of trees in the heap.
- ▶ $m(S)$ denotes the number of marked nodes.
- ▶ We use the potential function $\Phi(S) = t(S) + 2m(S)$.



The potential is $\Phi(S) = 5 + 2 \cdot 3 = 11$.

8.3 Fibonacci Heaps

We assume that one unit of potential can pay for a constant amount of work, where the constant is chosen “big enough” (to take care of the constants that occur).

To make this more explicit we use c to denote the amount of work that a unit of potential can pay for.

8.3 Fibonacci Heaps

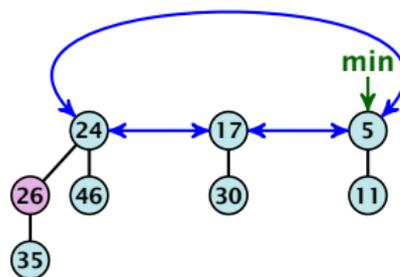
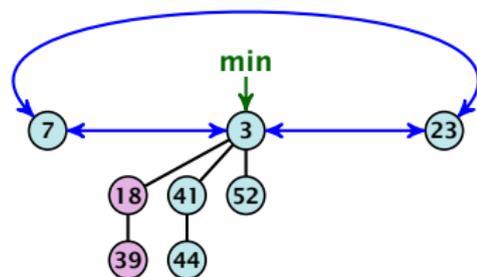
S. minimum()

- ▶ Access through the min-pointer.
- ▶ Actual cost $\mathcal{O}(1)$.
- ▶ No change in potential.
- ▶ Amortized cost $\mathcal{O}(1)$.

8.3 Fibonacci Heaps

S . merge(S')

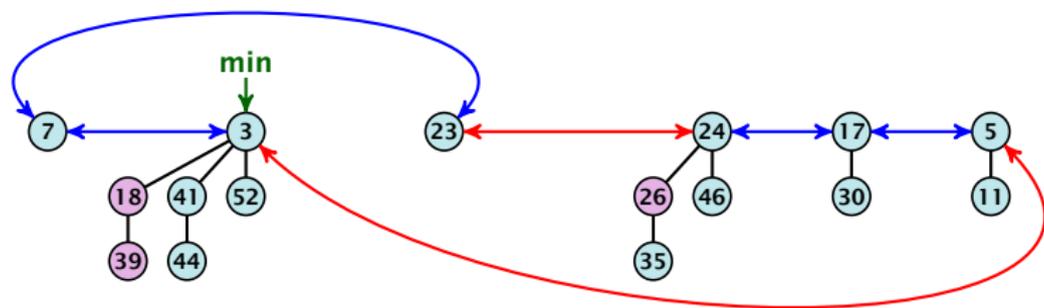
- ▶ Merge the root lists.
- ▶ Adjust the min-pointer



8.3 Fibonacci Heaps

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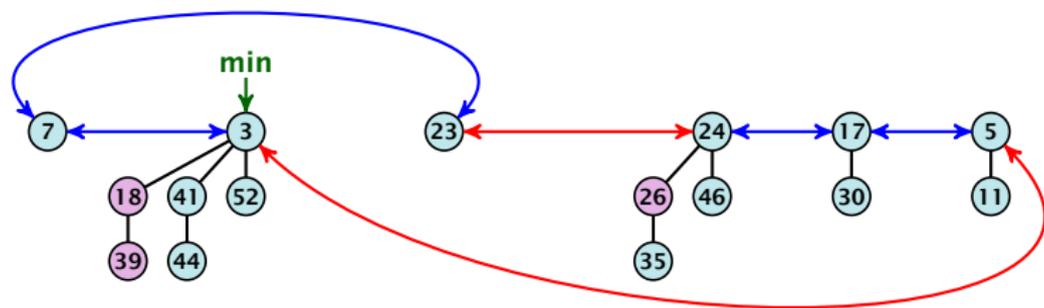
Running time:

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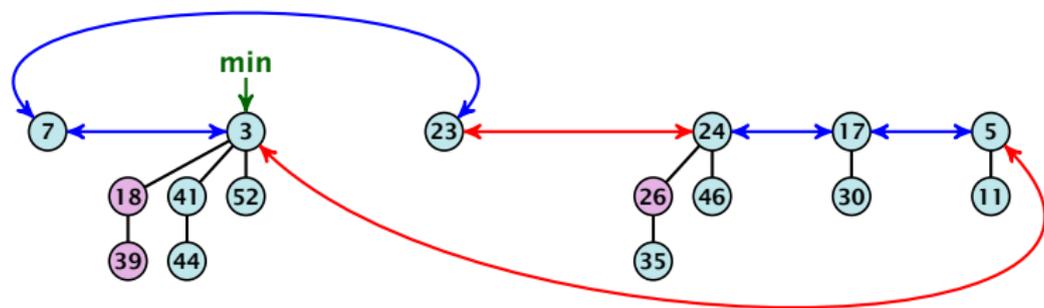
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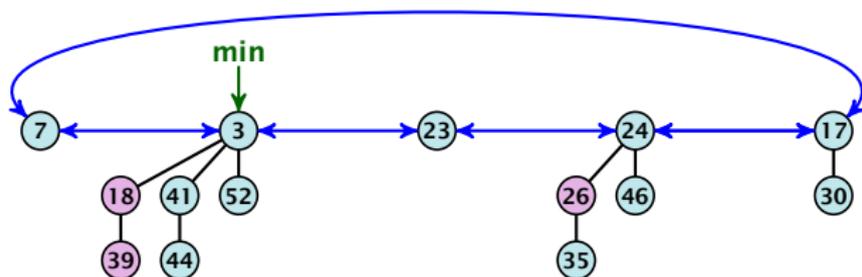
Running time:

- ▶ Actual cost $\mathcal{O}(1)$.
- ▶ No change in potential.
- ▶ Hence, amortized cost is $\mathcal{O}(1)$.

8.3 Fibonacci Heaps

S. insert(x)

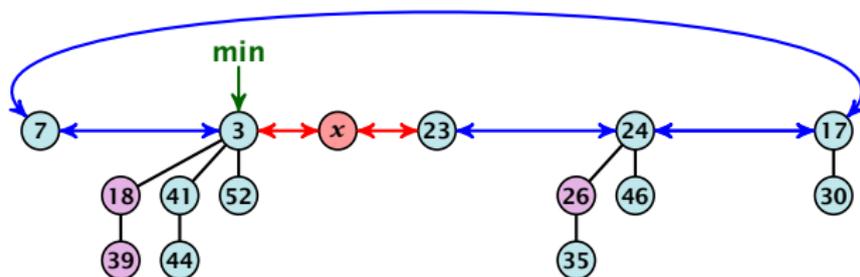
- ▶ Create a new tree containing x .
- ▶ Insert x into the root-list.
- ▶ Update min-pointer, if necessary.



8.3 Fibonacci Heaps

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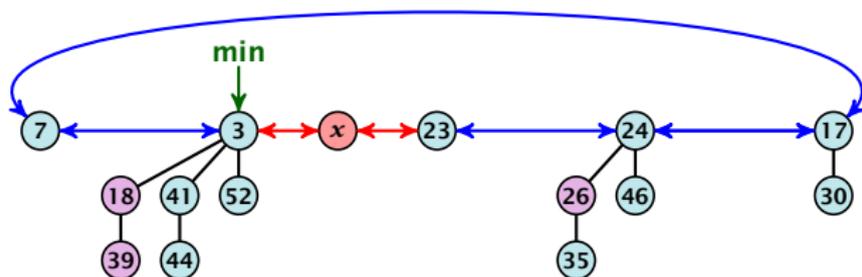
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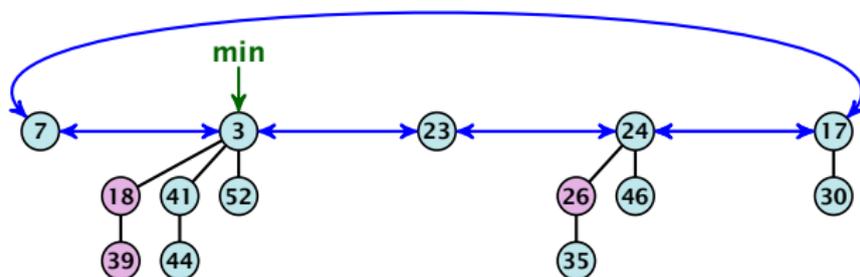


Running time:

- ▶ Actual cost $\mathcal{O}(1)$.
- ▶ Change in potential is $+1$.
- ▶ Amortized cost is $c + \mathcal{O}(1) = \mathcal{O}(1)$.

8.3 Fibonacci Heaps

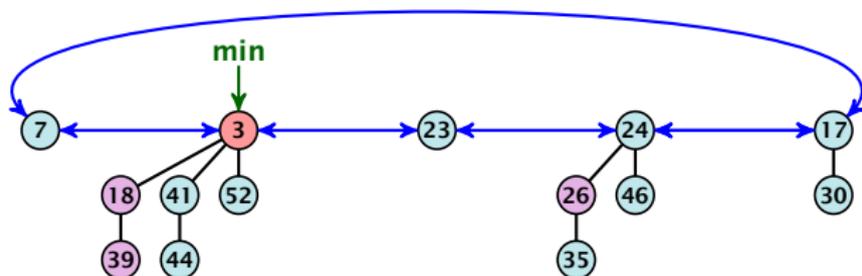
S. delete-min(x)



8.3 Fibonacci Heaps

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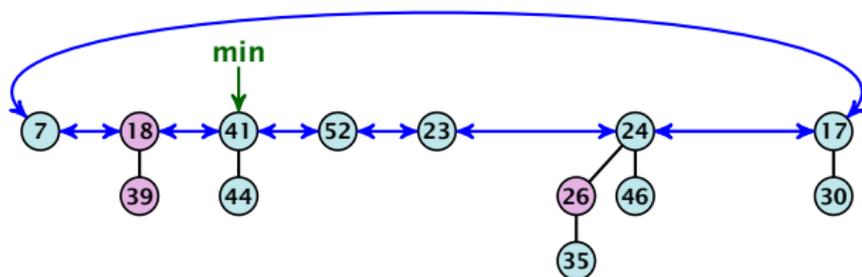
- ▶ Delete minimum; add child-trees to heap;
time: $D(\min) \cdot \mathcal{O}(1)$.



8.3 Fibonacci Heaps

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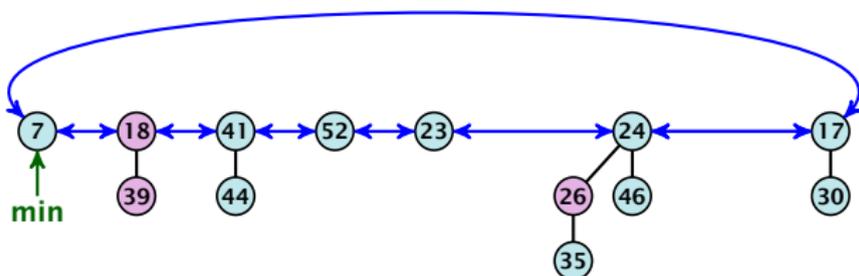
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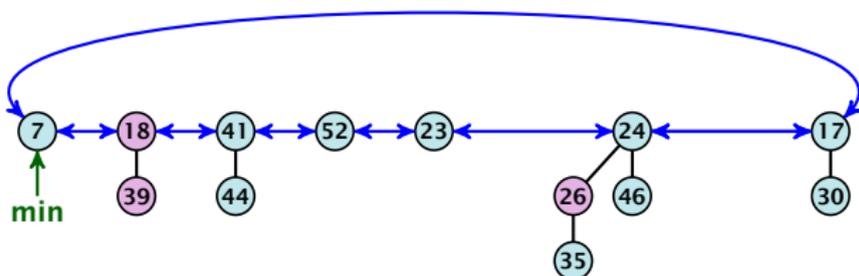
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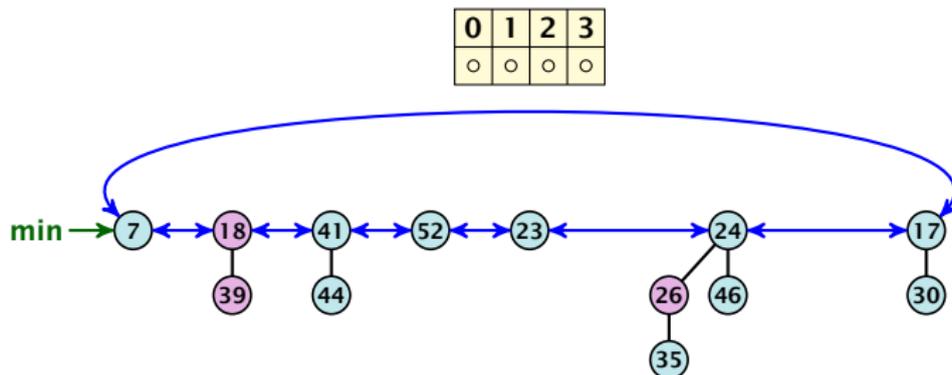
- ▶ Delete minimum; add child-trees to heap; time: $D(\min) \cdot \mathcal{O}(1)$.
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- ▶ Consolidate root-list so that no roots have the same degree. Time $t \cdot \mathcal{O}(1)$ (see next slide).

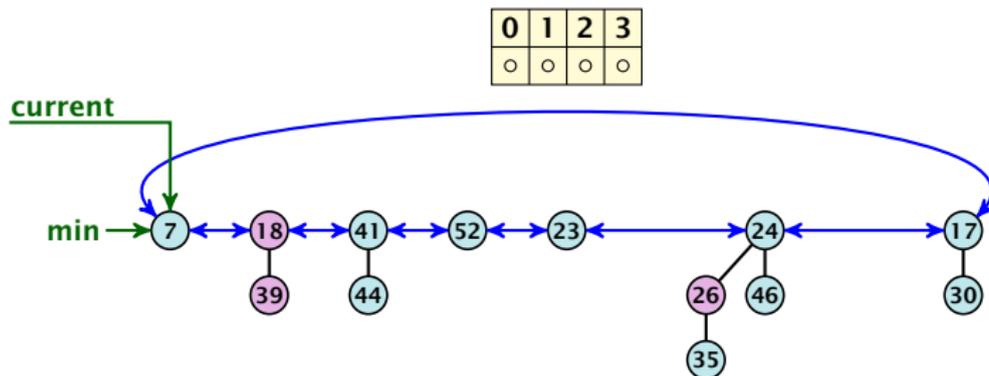
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Consolidate:



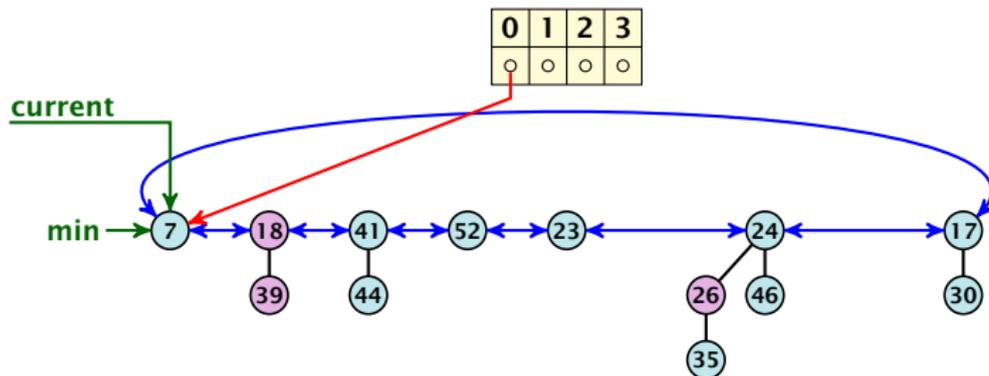
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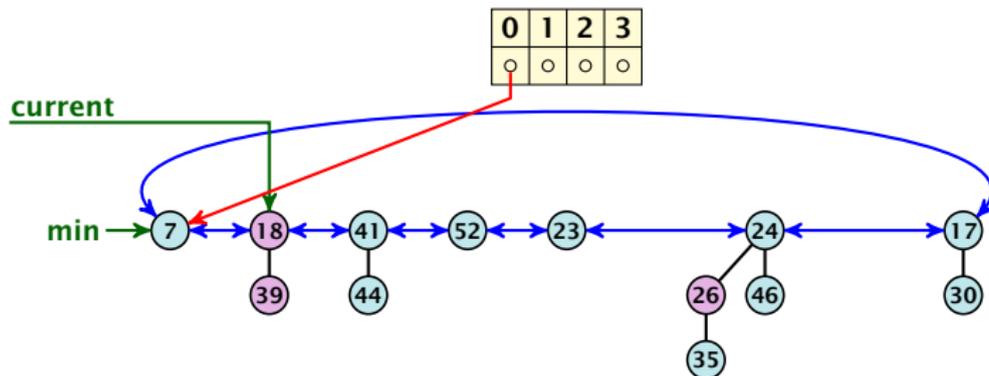
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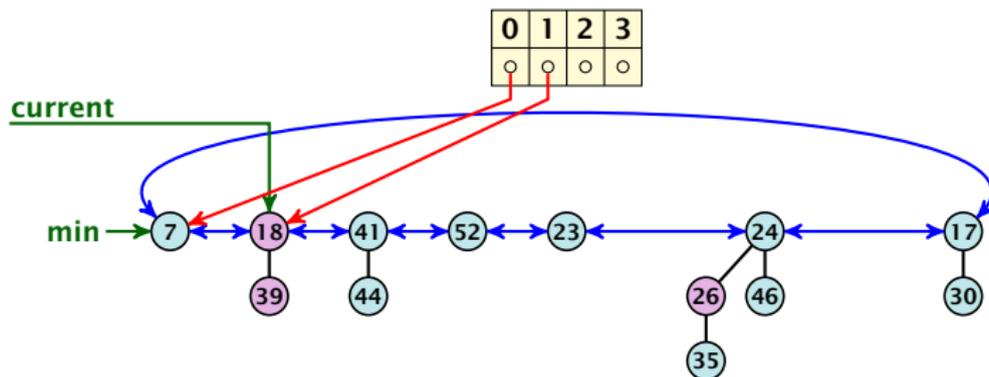
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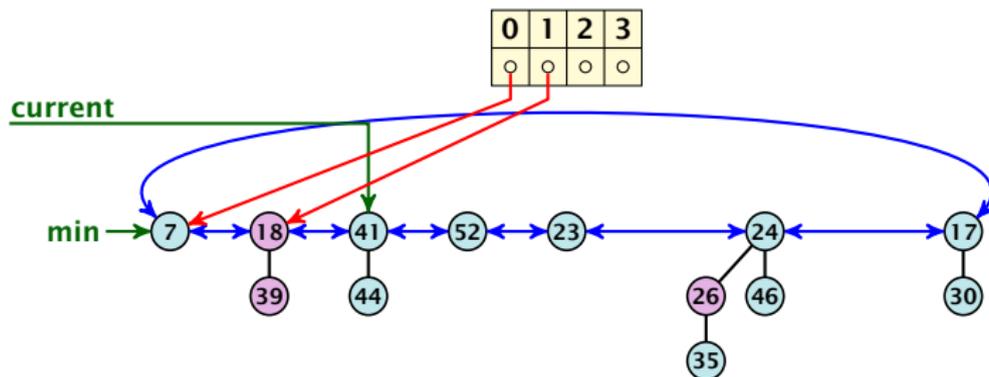
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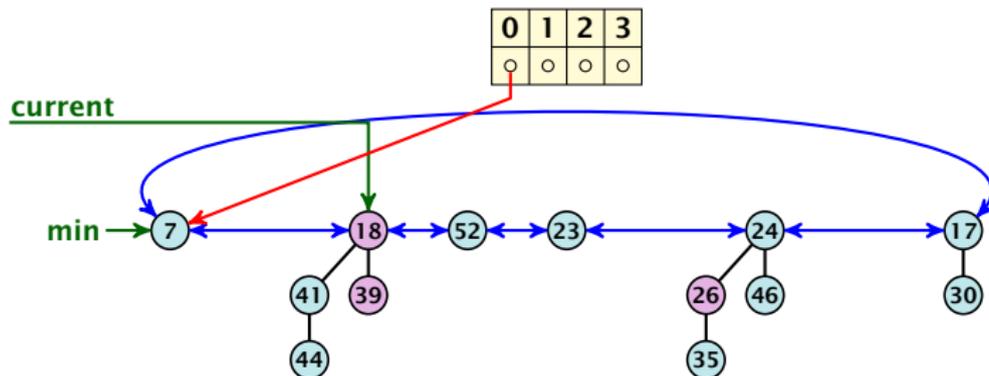
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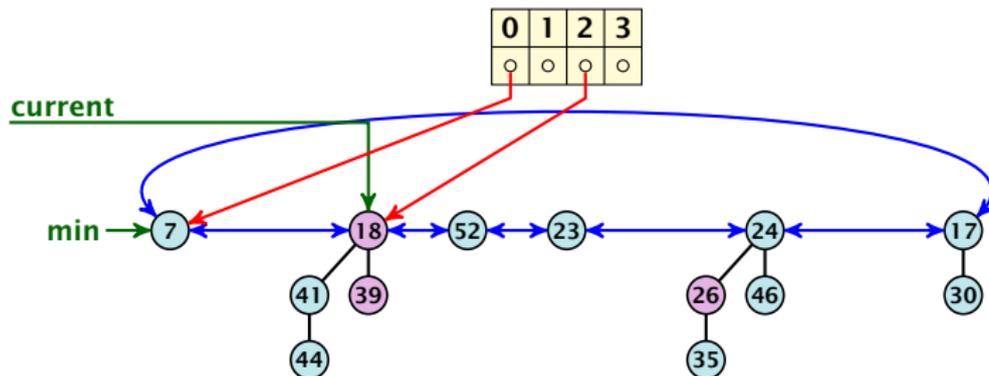
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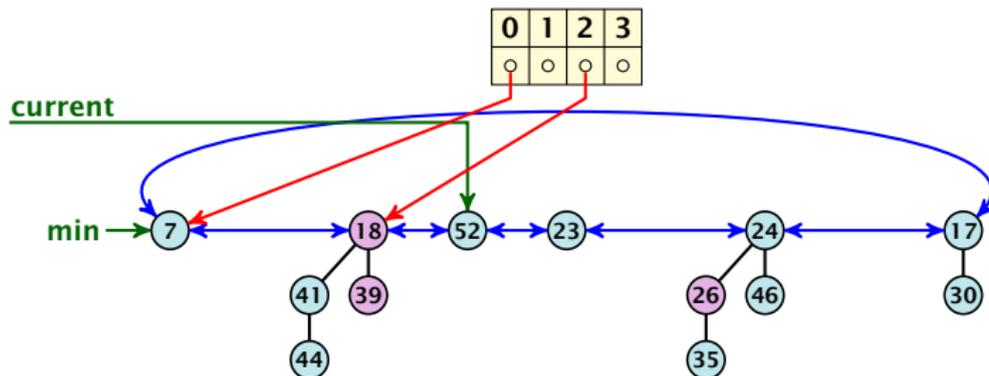
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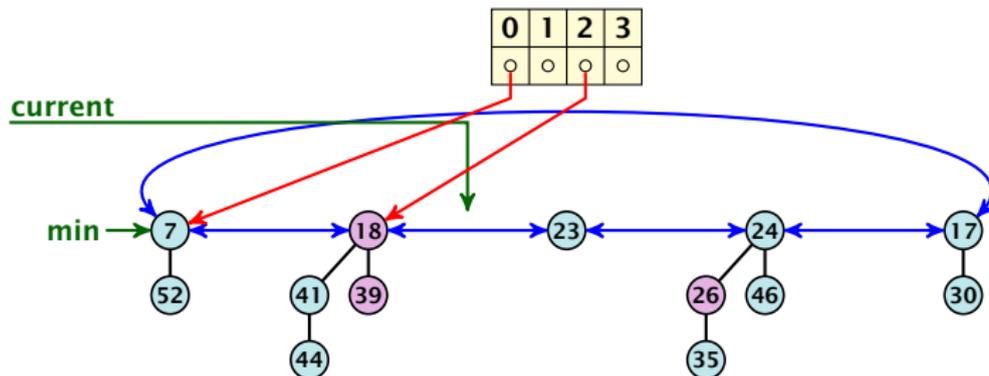
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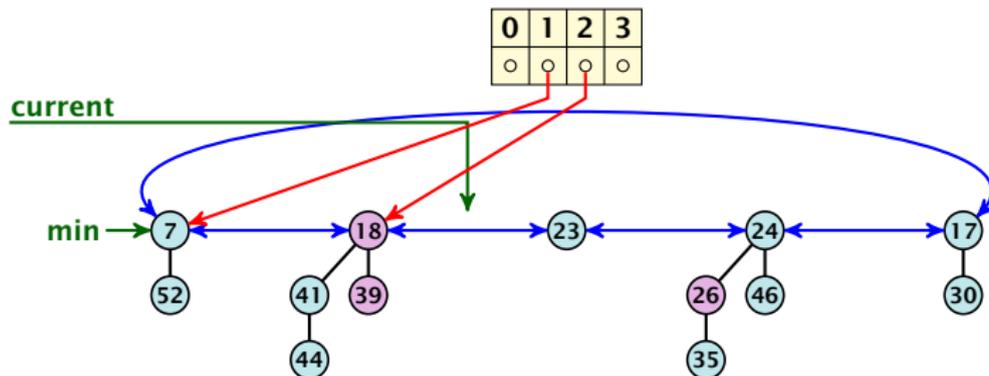
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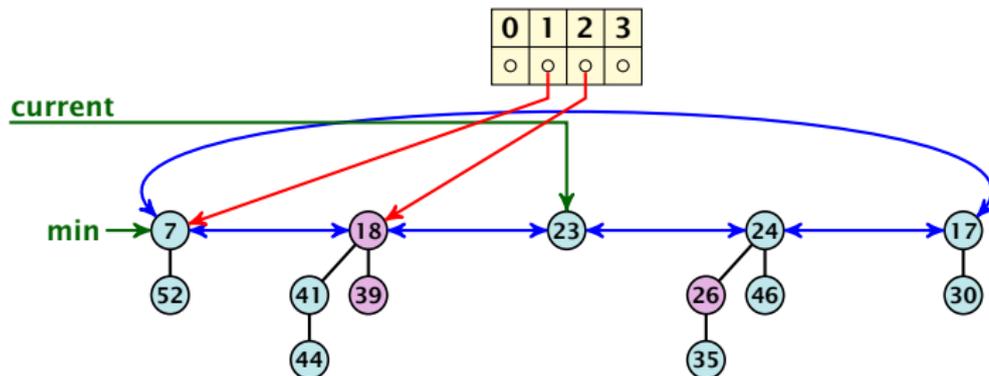
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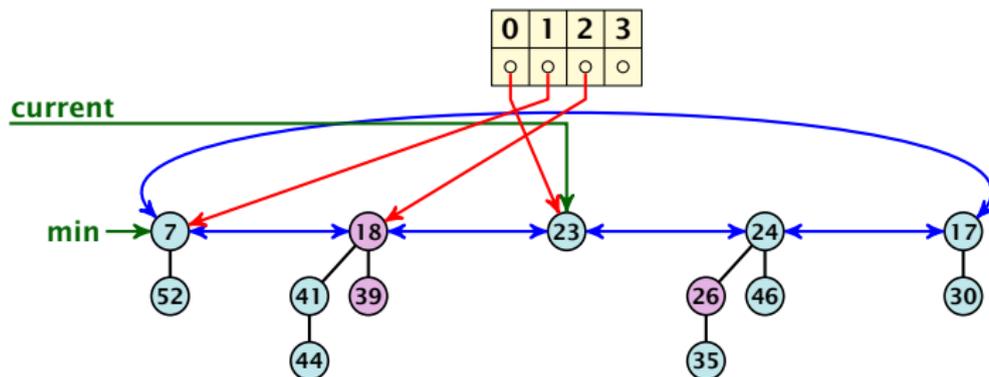
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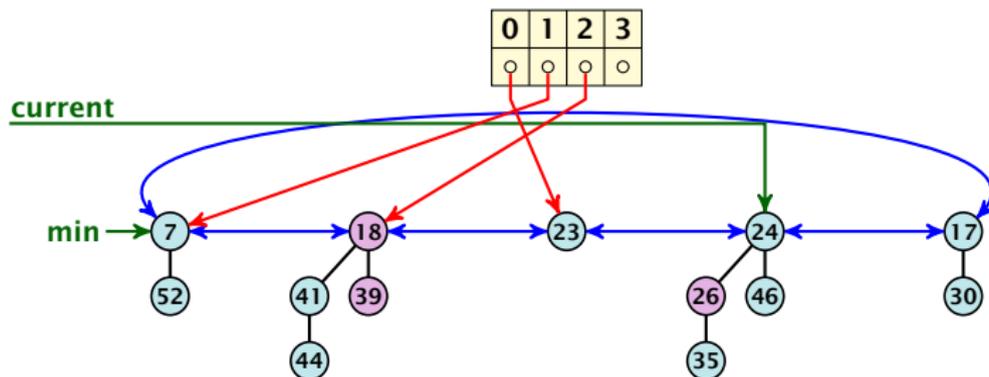
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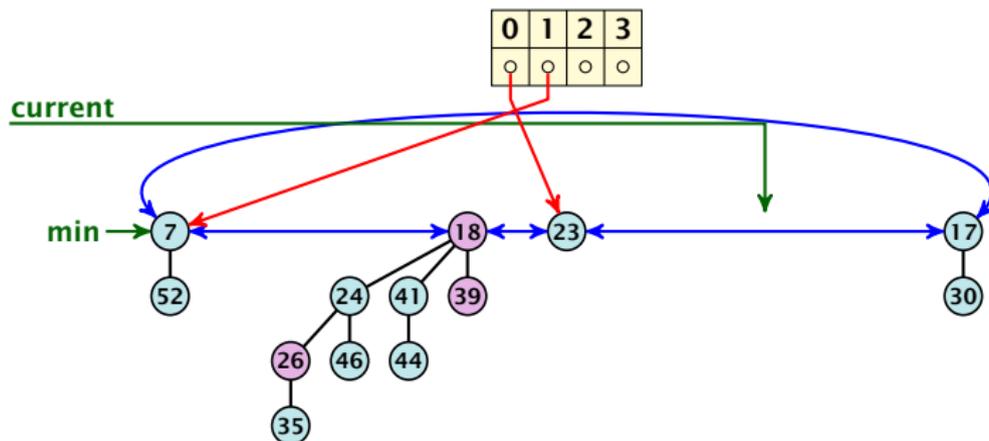
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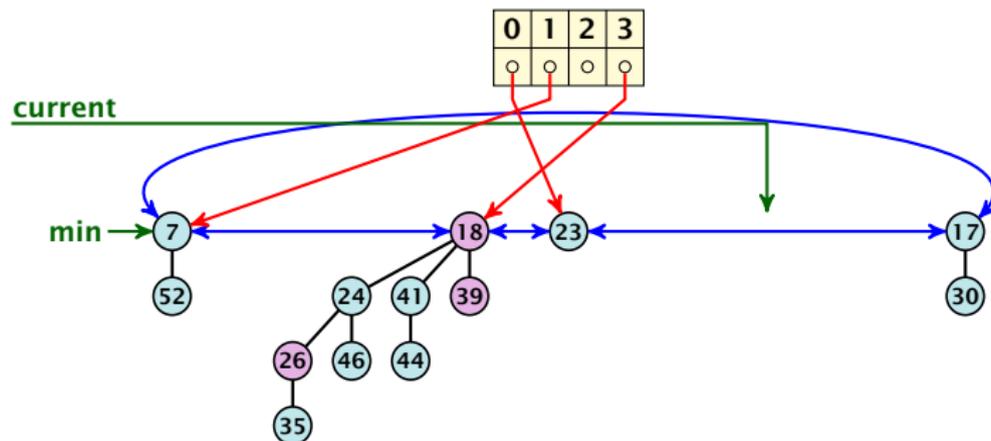
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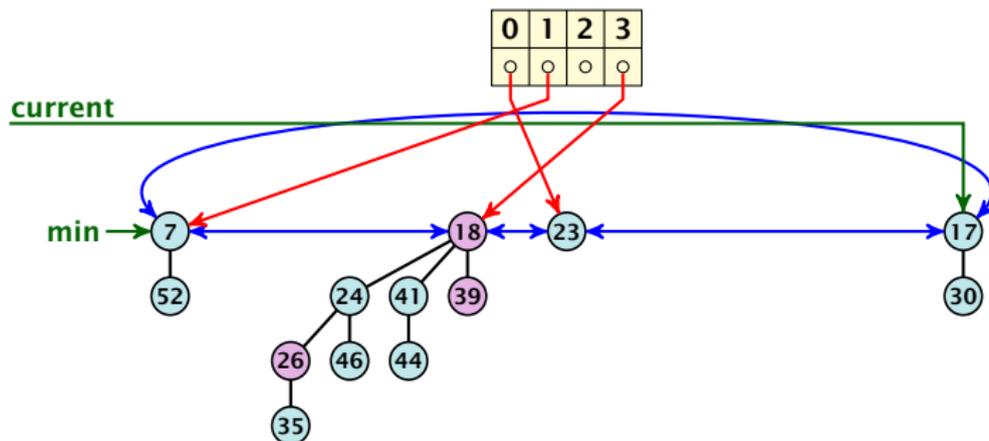
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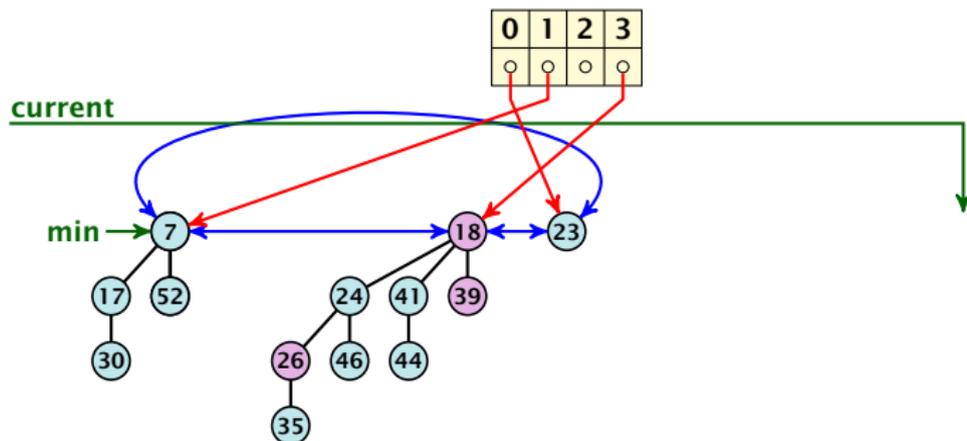
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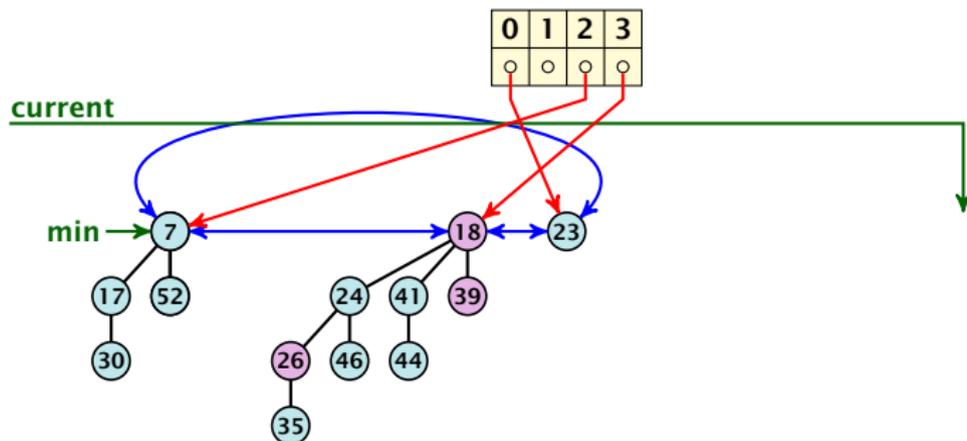
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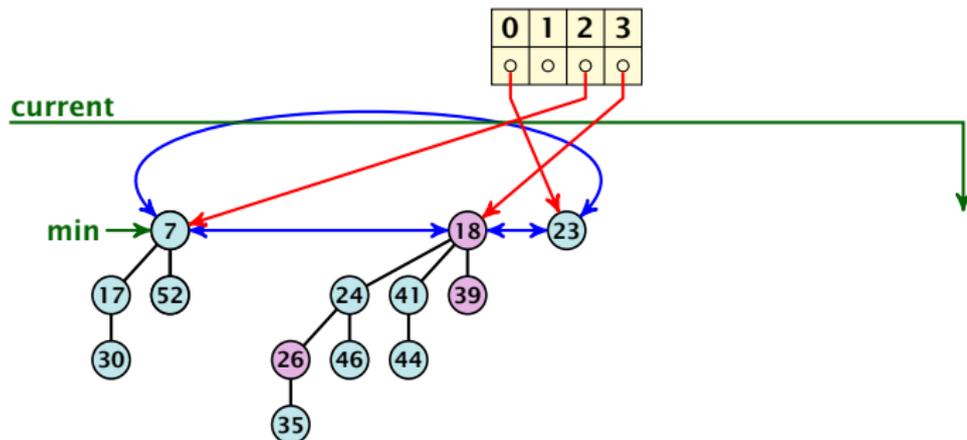
8.3 Fibonacci Heaps

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8.3 Fibonacci Heaps

Actual cost for delete-min()

- ▶ At most $D_n + t$ elements in root-list before consolidate.

Amortized cost for delete-min()

- ▶ $t' \leq D_n + 1$ as degrees are different after consolidating.
- ▶ Therefore $\Delta\Phi \leq D_n + 1 - t$;
- ▶ We can pay $c \cdot (t - D_n - 1)$ from the potential decrease.
- ▶ The amortized cost is

8.3 Fibonacci Heaps

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Hence, there exists c_1 s.t. actual cost is at most $c_1 \cdot (D_n + t)$.

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$$c_1 \cdot (D_n + t) - c \cdot (t - D_n - 1)$$

8.3 Fibonacci Heaps

Actual cost for delete-min()

- ▶ At most $D_n + t$ elements in root-list before consolidate.
- ▶ Actual cost for a delete-min is at most $\mathcal{O}(1) \cdot (D_n + t)$.
Hence, there exists c_1 s.t. actual cost is at most $c_1 \cdot (D_n + t)$.

Amortized cost for delete-min()

- ▶ $t' \leq D_n + 1$ as degrees are different after consolidating.
- ▶ Therefore $\Delta\Phi \leq D_n + 1 - t$;
- ▶ We can pay $c \cdot (t - D_n - 1)$ from the potential decrease.
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$$\begin{aligned}c_1 \cdot (D_n + t) - c \cdot (t - D_n - 1) \\ \leq (c_1 + c)D_n + (c_1 - c)t + c\end{aligned}$$

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for $c \geq c_1$.

8.3 Fibonacci Heaps

If the input trees of the consolidation procedure are binomial trees (for example only singleton vertices) then the output will be a set of distinct binomial trees, and, hence, the Fibonacci heap will be (more or less) a Binomial heap right after the consolidation.

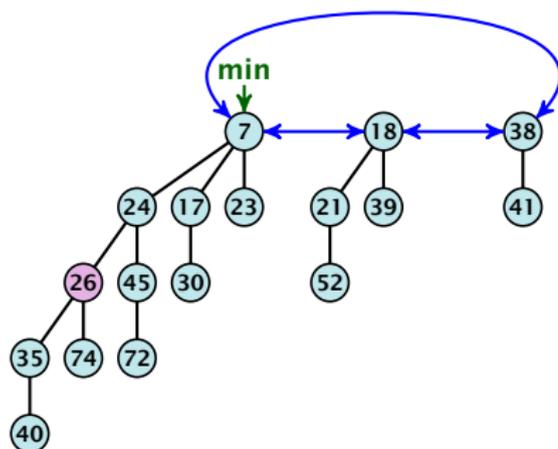
If we do not have delete or decrease-key operations then
 $D_n \leq \log n$.

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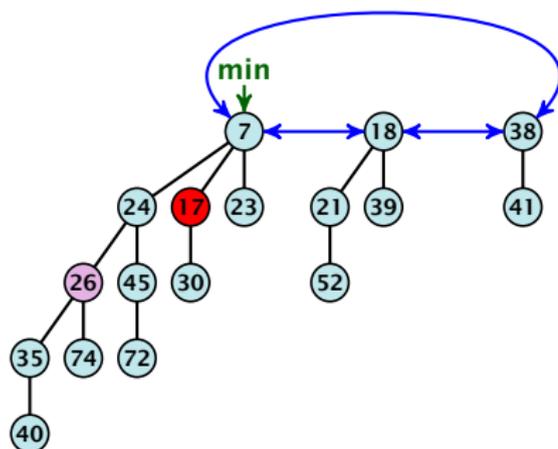
Fibonacci Heaps: decrease-key(handle h, v)



Case 1: decrease-key does not violate heap-property

- ▶ Just decrease the key-value of element referenced by h . Nothing else to do.

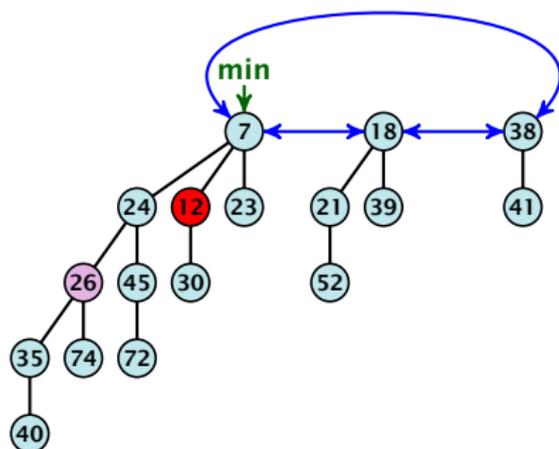
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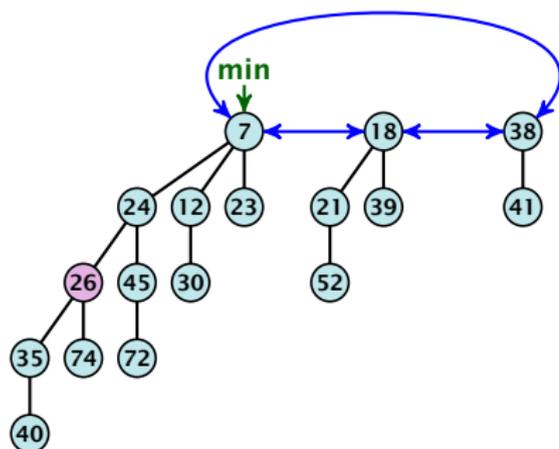
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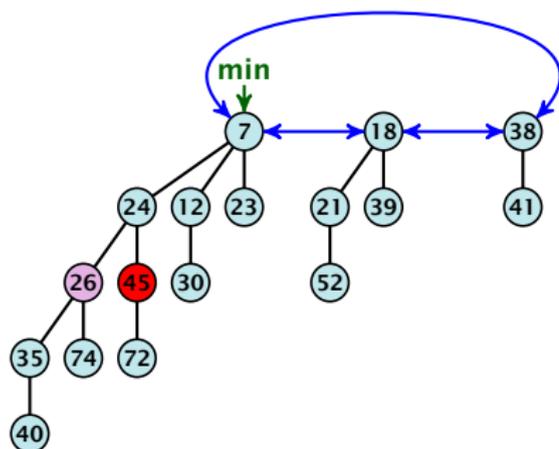
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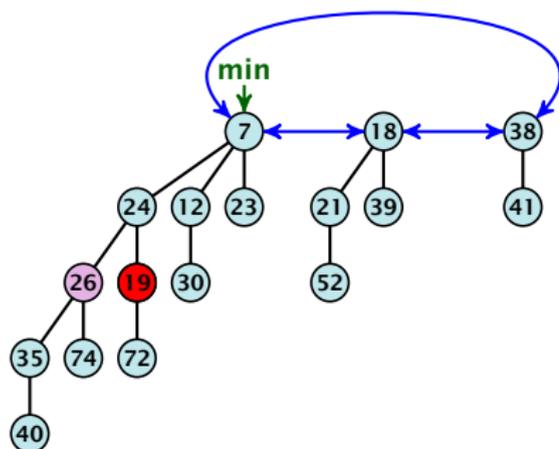
Fibonacci Heaps: decrease-key(handle h, v)



Case 2: heap-property is violated, but parent is not marked

- ▶ Decrease key-value of element x reference by h .
- ▶ If the heap-property is violated, cut the parent edge of x , and make x into a root.
- ▶ Adjust min-pointers, if necessary.
- ▶ Mark the (previous) parent of x .

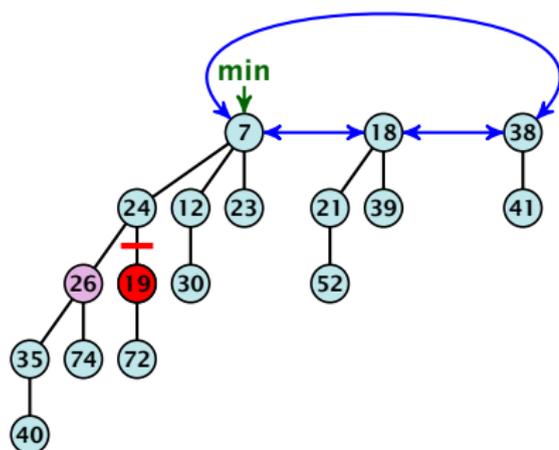
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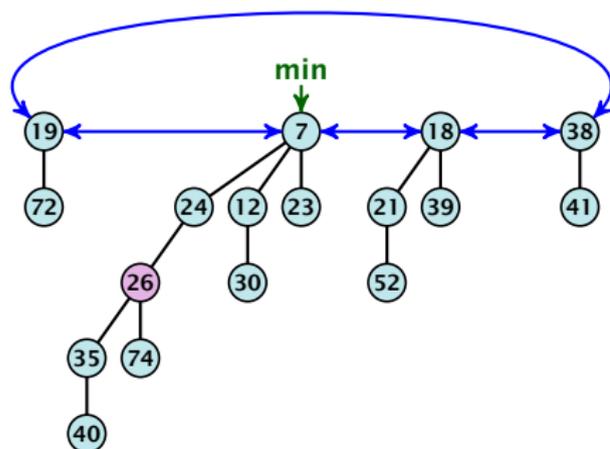
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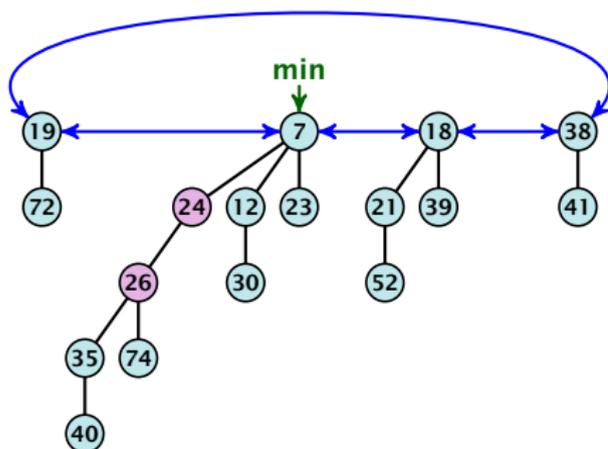
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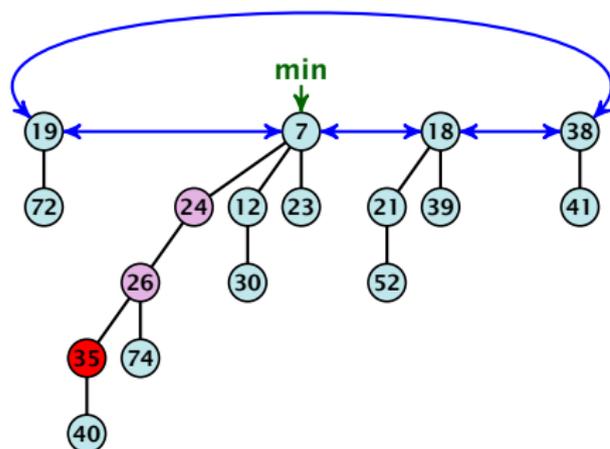
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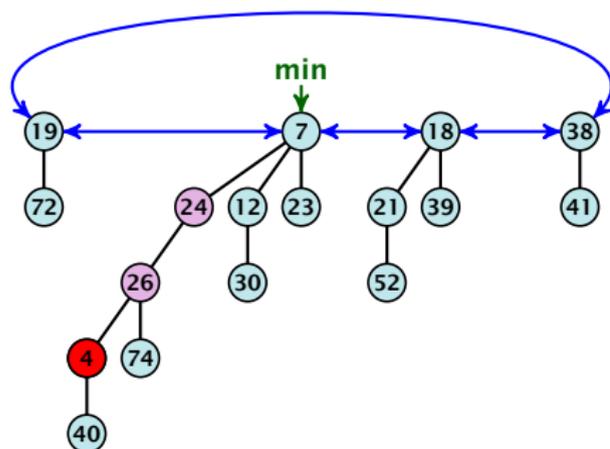
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Case 3: heap-property is violated, and parent is marked

- ▶ Decrease key-value of element x reference by h .
- ▶ Cut the parent edge of x , and make x into a root.
- ▶ Adjust min-pointers, if necessary.
- ▶ Continue cutting the parent until you arrive at an unmarked node.

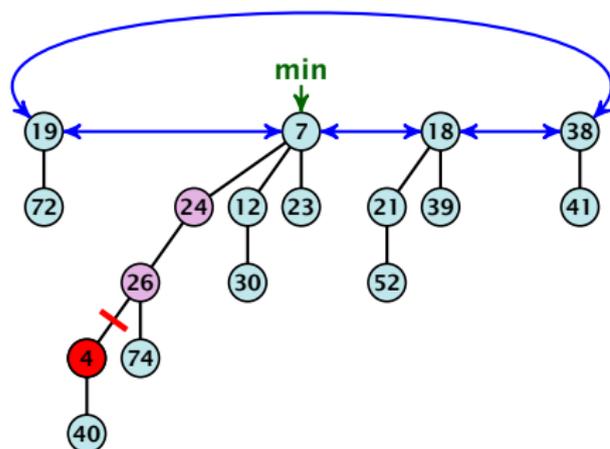
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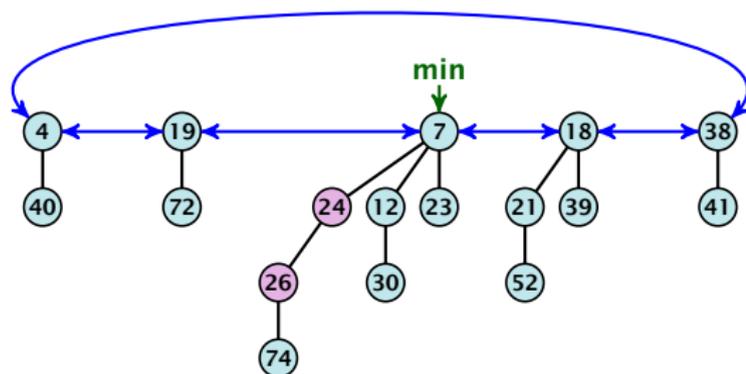
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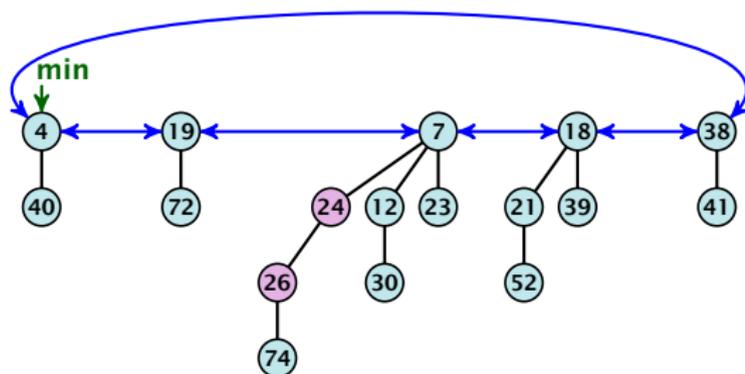
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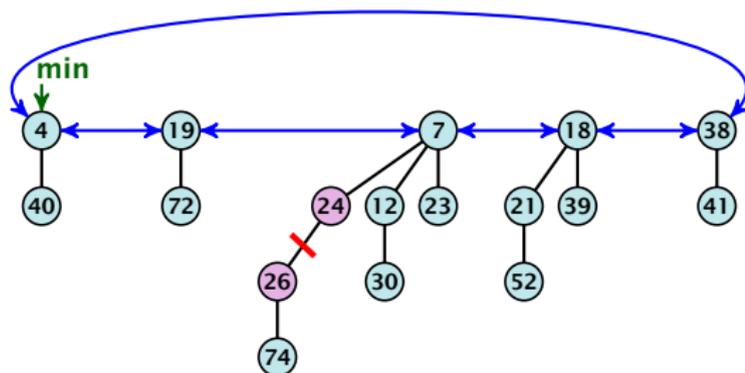
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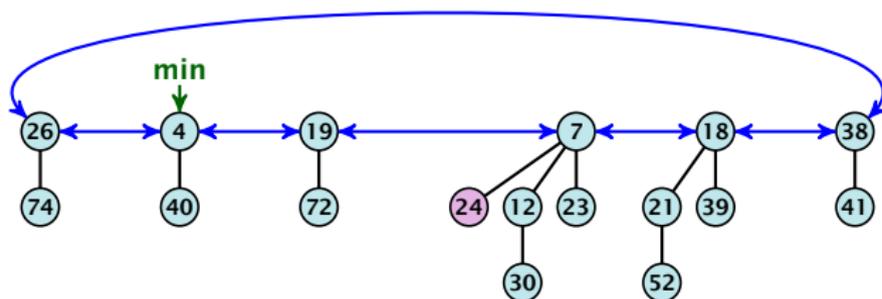
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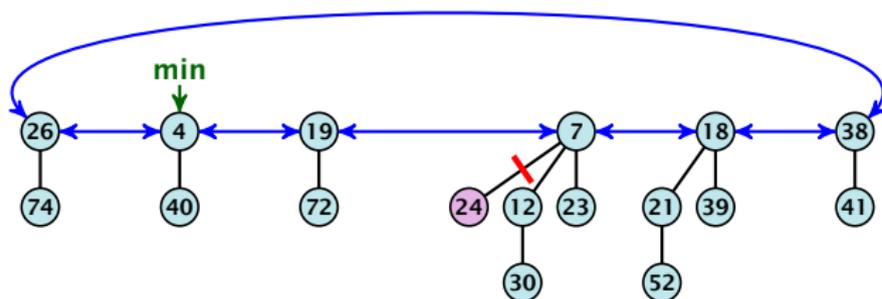
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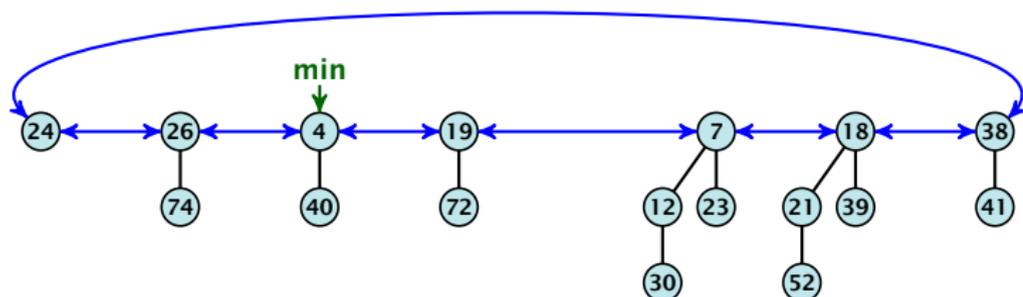
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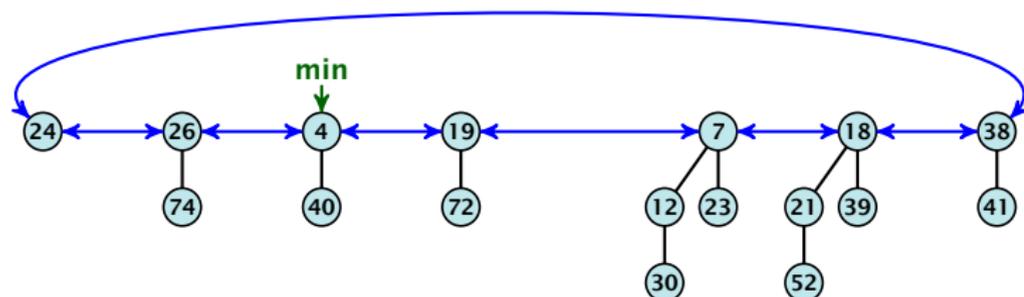
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- ▶ Decrease key-value of element x reference by h .
- ▶ Cut the parent edge of x , and make x into a root.
- ▶ Adjust min-pointers, if necessary.

- ▶ Execute the following:

```
 $p \leftarrow \text{parent}[x];$   
while ( $p$  is marked)  
     $pp \leftarrow \text{parent}[p];$   
    cut of  $p$ ; make it into a root; unmark it;  
     $p \leftarrow pp;$   
if  $p$  is unmarked and not a root mark it;
```

Fibonacci Heaps: decrease-key(handle h, v)

Actual cost:

- ▶ Constant cost for decreasing the value.
- ▶ Constant cost for each of ℓ cuts.
- ▶ Hence, cost is at most $c_2 \cdot (\ell + 1)$, for some constant c_2 .

Amortized cost:

- ▶ $\ell = O(\log n)$, as every cut creates one new root.
- ▶ $\ell = O(\log n) + 1 = O(\log n)$, since all but the first cut mark a node; the first cut may mark a node.
- ▶ Hence, $c_1 \cdot \ell + c_2 \cdot (\ell + 1) = O(\log n)$.

▶ Amortized cost is at most $O(\log n)$.

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Amortized cost:

- ▶ $\ell = O(\log V)$, as every cut creates one new root.
- ▶ $\ell = O(\log V)$, since all but the first cut will make a node the first cut may make a node.
- ▶ $\ell = O(\log V)$.
- ▶ Amortized cost is $O(\log V)$.

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Amortized cost:

- ▶ $\ell \leq \ell_0 + \ell_1 + \ell_2 + \dots + \ell_{\ell-1}$, as every cut creates one new root.
- ▶ $\ell_0 \leq \ell - 1$, $\ell_1 \leq \ell - 2$, since all but the first cut create a root, the first cut may not.
- ▶ $\ell_2 \leq \ell - 2$, $\ell_3 \leq \ell - 3$, \dots
- ▶ $\ell_0 + \ell_1 + \ell_2 + \dots + \ell_{\ell-1} \leq \ell^2$.

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- ▶ $t' = t + \ell$, as every cut creates one new root.
- ▶ $m' \leq m - (\ell - 1) + 1 = m - \ell + 2$, since all but the first cut marks a node; the last cut may mark a node.
- ▶ $\Delta\Phi \leq \ell + 2(-\ell + 2) = 4 - \ell$
- ▶ Amortized cost is at most

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$$c_2(\ell + 1) + c(4 - \ell) \leq (c_2 - c)\ell + 4c = \mathcal{O}(1),$$

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Delete node

H. delete(x):

- ▶ decrease value of x to $-\infty$.
- ▶ delete-min.

Amortized cost: $\mathcal{O}(D(n))$

- ▶ $\mathcal{O}(1)$ for decrease-key.
- ▶ $\mathcal{O}(D(n))$ for delete-min.

8.3 Fibonacci Heaps

Lemma 33

Let x be a node with degree k and let y_1, \dots, y_k denote the children of x in the order that they were linked to x . Then

$$\text{degree}(y_i) \geq \begin{cases} 0 & \text{if } i = 1 \\ i - 2 & \text{if } i \geq 2 \end{cases}$$

8.3 Fibonacci Heaps

Proof

- ▶ When y_i was linked to x , at least y_1, \dots, y_{i-1} were already linked to x .
- ▶ Hence, at this time $\text{degree}(x) \geq i - 1$, and therefore also $\text{degree}(y_i) \geq i - 1$ as the algorithm links nodes of equal degree only.
- ▶ Since, then y_i has lost at most one child.
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- ▶ s_k monotonically increases with k
- ▶ $s_0 = 1$ and $s_1 = 2$.

Let x be a degree k node of size s_k and let y_1, \dots, y_k be its children.

$$s_k = 2 + \sum_{i=2}^k \text{size}(y_i)$$

8.3 Fibonacci Heaps

- ▶ Let s_k be the minimum possible size of a sub-tree rooted at a node of degree k that can occur in a Fibonacci heap.
- ▶ s_k monotonically increases with k
- ▶ $s_0 = 1$ and $s_1 = 2$.

Let x be a degree k node of size s_k and let y_1, \dots, y_k be its children.

$$\begin{aligned} s_k &= 2 + \sum_{i=2}^k \text{size}(y_i) \\ &\geq 2 + \sum_{i=2}^k s_{i-2} \end{aligned}$$

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8.3 Fibonacci Heaps

Definition 34

Consider the following non-standard Fibonacci type sequence:

$$F_k = \begin{cases} 1 & \text{if } k = 0 \\ 2 & \text{if } k = 1 \\ F_{k-1} + F_{k-2} & \text{if } k \geq 2 \end{cases}$$

Facts:

1. $F_k \geq \phi^k$.
2. For $k \geq 2$: $F_k = 2 + \sum_{i=0}^{k-2} F_i$.

The above facts can be easily proved by induction. From this it follows that $s_k \geq F_k \geq \phi^k$, which gives that the maximum degree in a Fibonacci heap is logarithmic.