# Mincost Flow

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- $G = (V, E)$ is an oriented graph.
- $u : E \to \mathbb{R}_0^+ \cup \{\infty\}$ is the capacity function.
- $c : E \to \mathbb{R}$ is the cost function (note that $c(e)$ may be negative).
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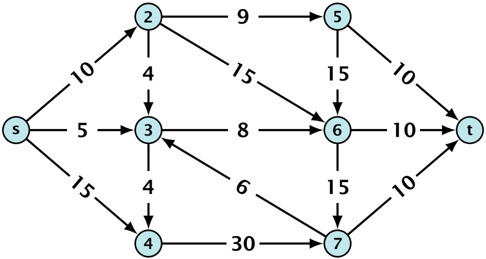
# Solve Maxflow Using Mincost Flow



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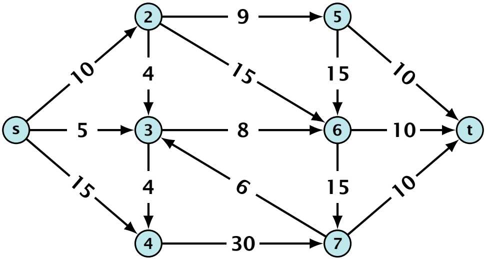


- Given a flow network for a standard maxflow problem.

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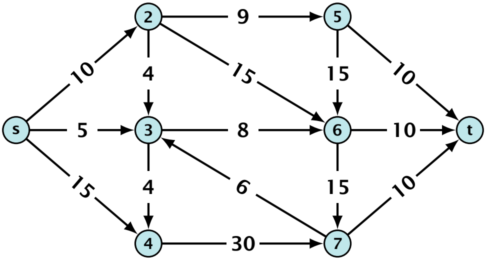


- Given a flow network for a standard maxflow problem.
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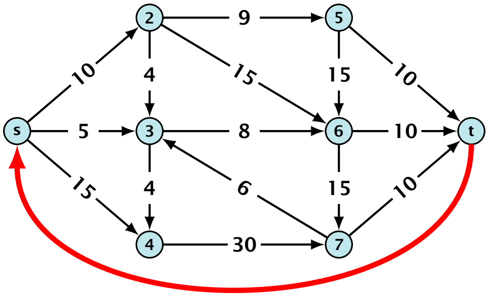


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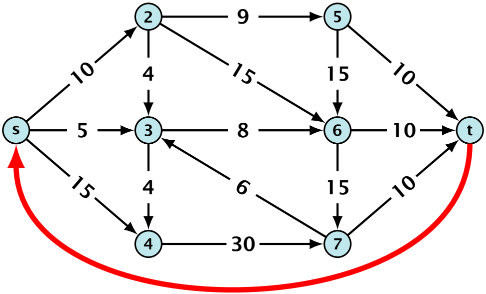


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- Add an edge from $t$ to $s$ with infinite capacity and cost $-1$.
- Then, $\text{val}(f^*) = -\text{cost}(f_{\text{min}})$, where $f^*$ is a maxflow, and $f_{\text{min}}$ is a mincost-flow.

# Solve Maxflow Using Mincost Flow

**Solve decision version of maxflow:**

- Given a flow network for a standard maxflow problem, and a value $k$.
- Set $b(v) = 0$ for every node apart from $s$ or $t$. Set $b(s) = -k$ and $b(t) = k$.
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- There exists a maxflow of value $k$ if and only if the mincost-flow problem is feasible.

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# Generalization

**Our model:**

$$\begin{array}{lll} \text{min} & \sum_e c(e)f(e) & \\ \text{s.t.} & \forall e \in E: & 0 \le f(e) \le u(e) \\ & \forall v \in V: & f(v) = b(v) \end{array}$$

where $b : V \to \mathbb{R}$, $\sum_v b(v) = 0$; $u : E \to \mathbb{R}_0^+ \cup \{\infty\}$; $c : E \to \mathbb{R}$;

**A more general model?**

$$\begin{array}{lll} \text{min} & \sum_e c(e)f(e) & \\ \text{s.t.} & \forall e \in E: & \ell(e) \le f(e) \le u(e) \\ & \forall v \in V: & a(v) \le f(v) \le b(v) \end{array}$$

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Add new node $r$.

Add edge $(r, v)$ for all $v \in V$.

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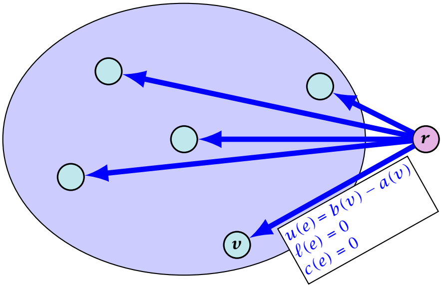

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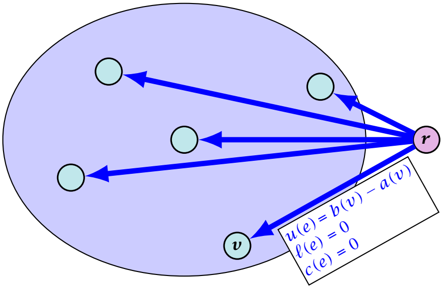

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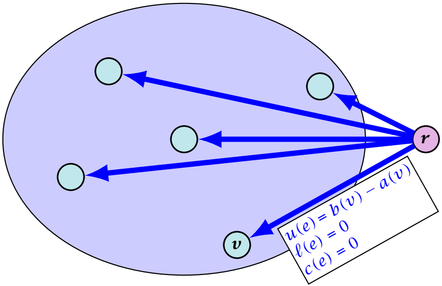


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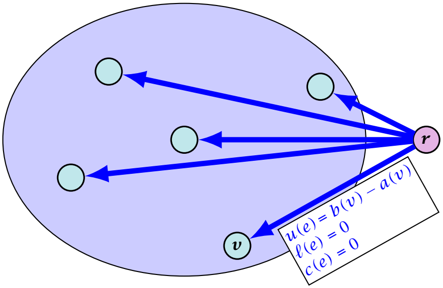

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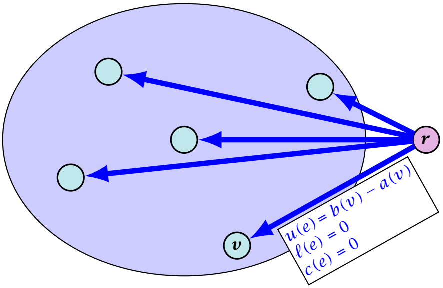


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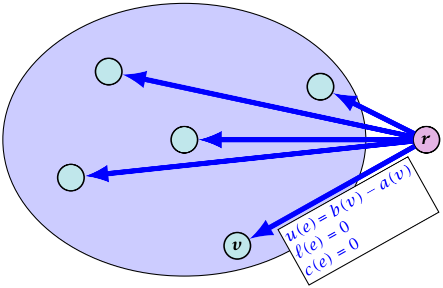

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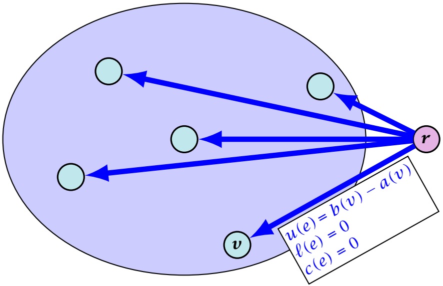

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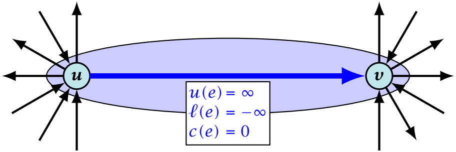
**We can assume that either $\ell(e) \neq -\infty$ or $u(e) \neq \infty$:**




If $c(e) = 0$ we can simply contract the edge/identify nodes $u$ and $v$

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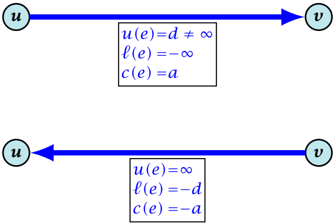


Replace the edge by an edge in opposite direction.

# Reduction IV

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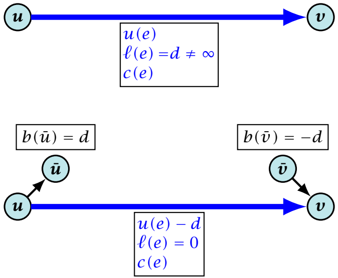


The added edges have infinite capacity and cost $c(e)/2$.

# Applications

**Caterer Problem**

- She needs to supply $r_i$ napkins on $N$ successive days.
- She can buy new napkins at $p$ cents each.
- She can launder them at a fast laundry that takes $m$ days and cost $f$ cents a napkin.
- She can use a slow laundry that takes $k > m$ days and costs $s$ cents each.
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# Residual Graph

The residual graph for a mincost flow is exactly defined as the residual graph for standard flows, with the only exception that one needs to define a cost for the residual edge.

For a flow of $z$ from $u$ to $v$ the residual edge $(v, u)$ has capacity $z$ and a cost of $-c((u, v))$.

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A circulation is feasible if it fulfills capacity constraints, i.e., $f(e) \le u(e)$ for every edge of $G$.

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## Lemma 85

*A given flow is a mincost-flow if and only if the corresponding residual graph $G_f$ does not have a feasible circulation of negative cost.*

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- $\Leftarrow$ Let $f$ be a non-mincost flow, and let $f^*$ be a min-cost flow. We need to show that the residual graph has a feasible circulation with negative cost.

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*A given flow is a mincost-flow if and only if the corresponding residual graph $G_f$ does not have a feasible circulation of negative cost.*

- ⇒ Suppose that $g$ is a feasible circulation of negative cost in the residual graph.

  Then $f + g$ is a feasible flow with cost $\text{cost}(f) + \text{cost}(g) < \text{cost}(f)$. Hence, $f$ is not minimum cost.

- ⇐ Let $f$ be a non-mincost flow, and let $f^*$ be a min-cost flow. We need to show that the residual graph has a feasible circulation with negative cost.

  Clearly $f^* - f$ is a circulation of negative cost. One can also easily see that it is feasible for the residual graph.

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## Lemma 86

*A graph (without zero-capacity edges) has a feasible circulation of negative cost if and only if it has a negative cycle w.r.t. edge-weights* $c : E \to \mathbb{R}$.

**Proof.**

- Suppose that we have a negative cost circulation.
- Find directed path only using edges that have non-zero flow.
- If this path has negative cost you are done.
- Otherwise send flow in opposite direction along the cycle until the bottleneck edge(s) does not carry any flow.
- You still have a circulation with negative cost.
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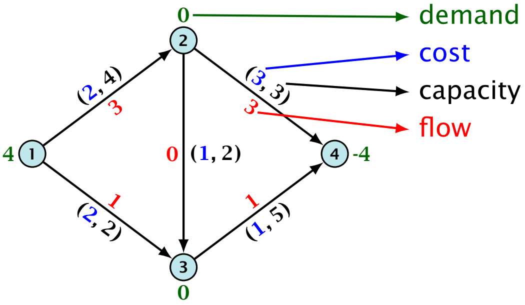
# 15 Mincost Flow

**Algorithm 51** CycleCanceling($G = (V, E), c, u, b$)

1: establish a feasible flow $f$ in $G$
2: **while** $G_f$ contains negative cycle **do**
3: use Bellman-Ford to find a negative circuit $Z$
4: $\delta \leftarrow \min\{u_f(e) \mid e \in Z\}$
5: augment $\delta$ units along $Z$ and update $G_f$

# 15 Mincost Flow



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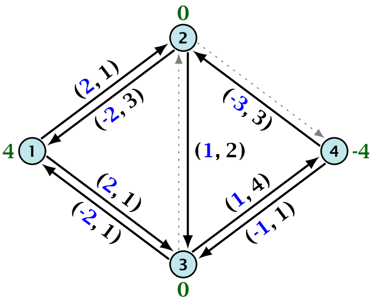

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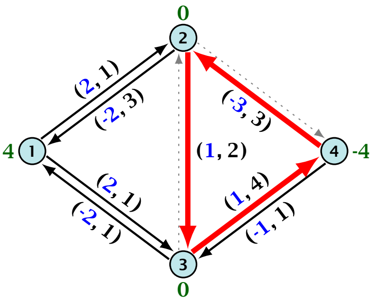

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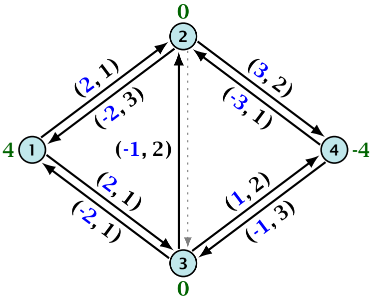

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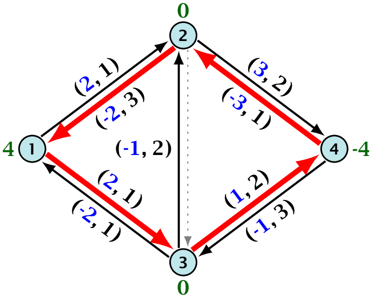

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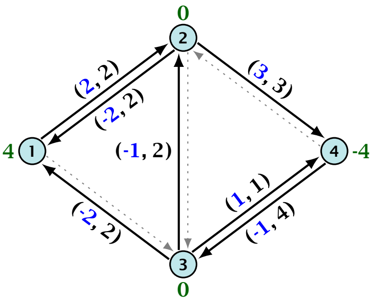

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## Lemma 87

*The improving cycle algorithm runs in time* $\mathcal{O}(nm^2CU)$*, for integer capacities and costs, when for all edges* $e$*,* $|c(e)| \le C$ *and* $|u(e)| \le U$*.*