Robust Constrained Model Predictive Control using Linear Matrix Inequalities*

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Overview

- Motivation
  - Model Predictive Control (MPC)
  - Problem with uncertainties -- Robust MPC (RMPC)

- Linear Matrix Inequalities (LMI) Approach for RMPC
  - Robust unconstrained MPC
  - Robust constrained MPC

- Numerical Example -- Angular Positioning System

- Conclusions

Model Predictive Control (MPC) (I)

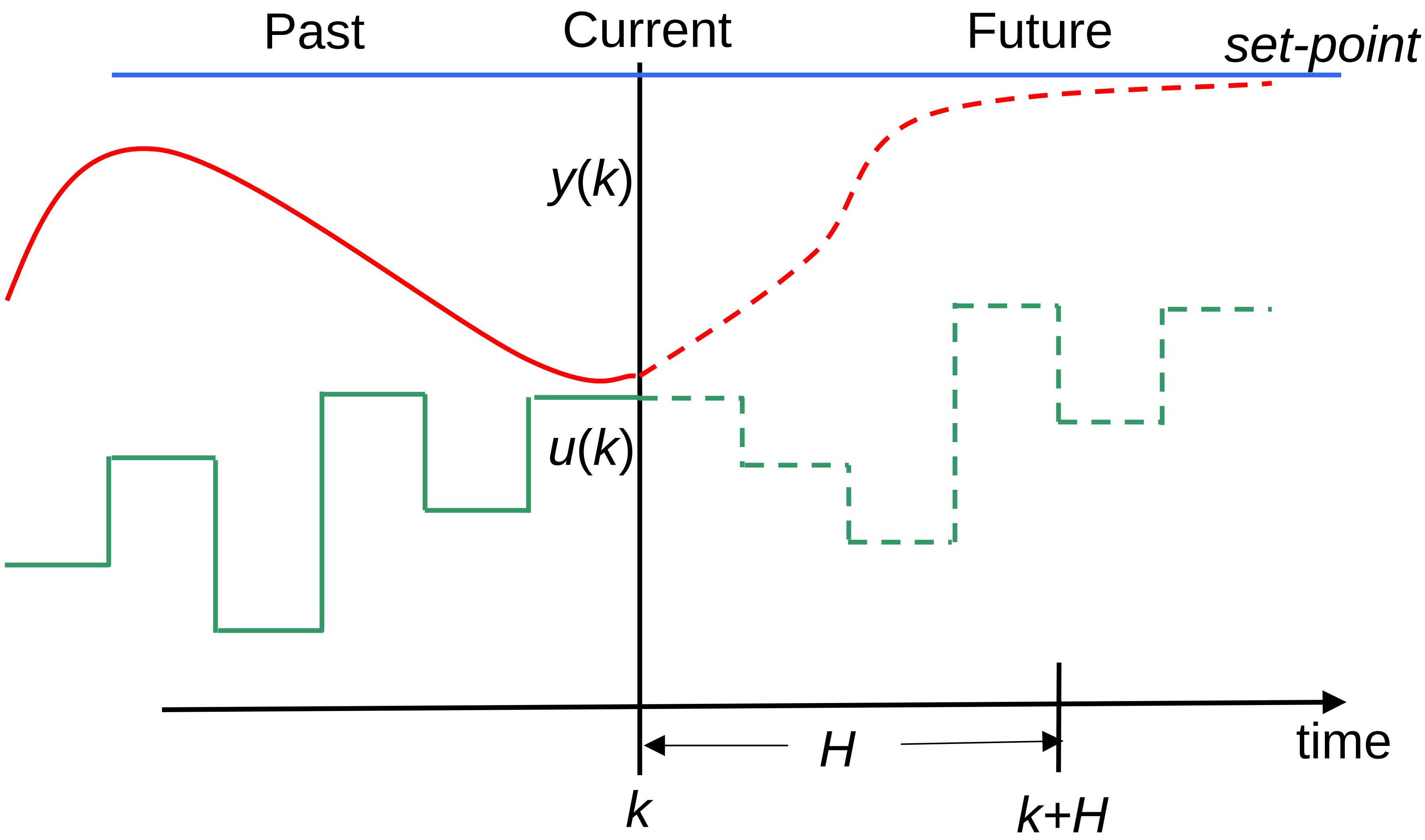

Fig. 1. Scheme of Model Predictive Control (MPC)

Model Predictive Control (MPC) (I)

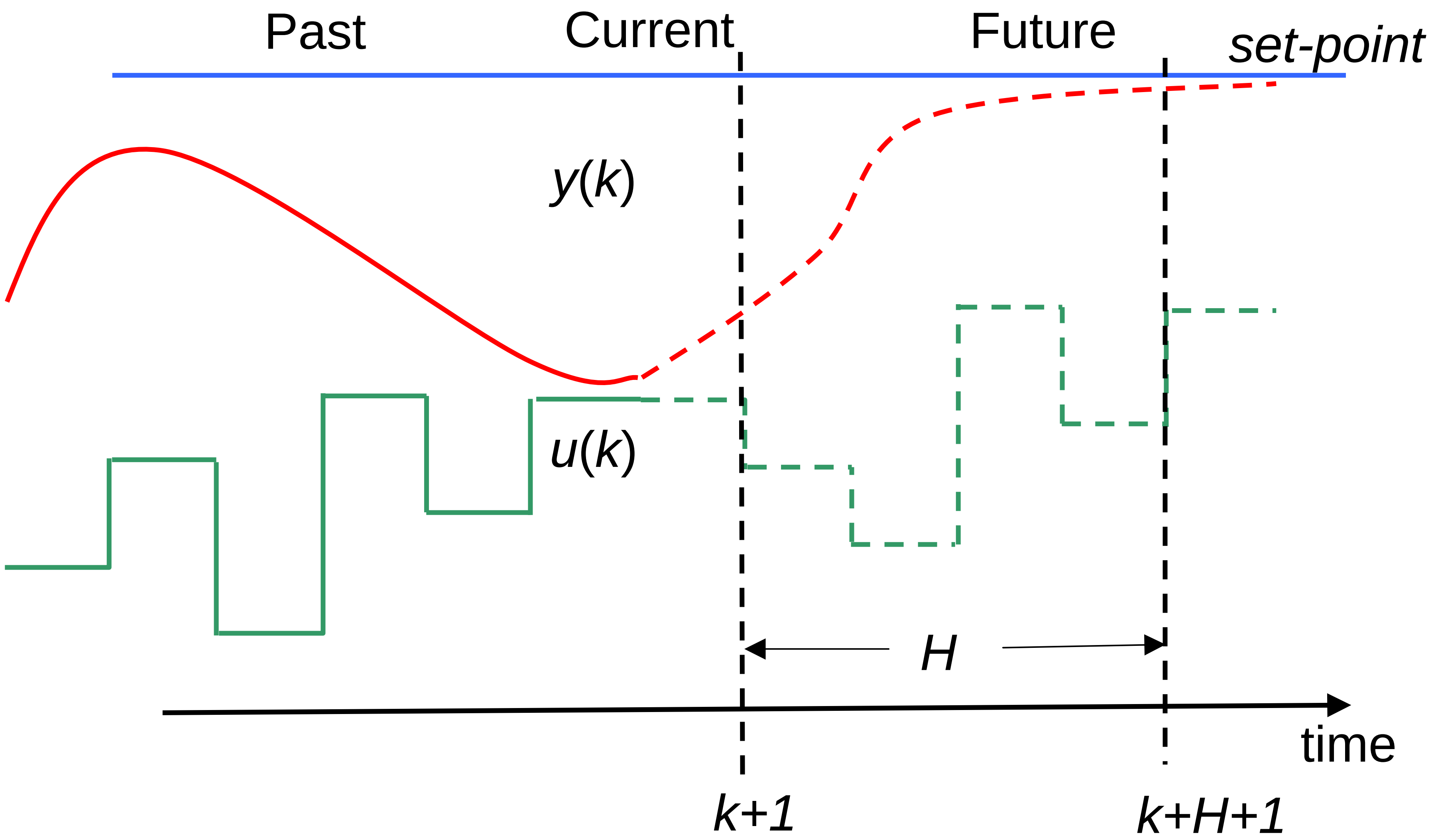

Fig. 1. Scheme of Model Predictive Control (MPC)

Model Predictive Control (MPC) (II)

- Linear discretized model:

$$x(k+1) = Ax(k) + Bu(k)$$
$$y(k) = Cx(k)$$

where,

- $u(k) \in \Re^{n_u}$ the control input
- $x(k) \in \Re^{n_x}$ the state of the plant
- $y(k) \in \Re^{n_y}$ the plant output
- k current time
- A, B, and C are system matrices with no uncertainties

- Cost function:

$$\min_{u(k+i), i=0,1,\ldots,H} J(k)$$

subject to constraints on the control inputs $u(k+i)$, states $x(k+i)$, and outputs $y(k+i)$, i is the time index, and H is the time horizon.

Model Predictive Control (MPC) (III)

- Quadratic cost function:

$$J(k) = \sum_{i=0}^{H} \left(x(k+i)^T Q_1 x(k+i) + u(k+i)^T R u(k+i) \right)$$

where $Q_1 > 0, R > 0$ symmetric weighting matrices

- Advantages:
  - Capable of dealing with constraints
  - Easily deals with multivariable case
  - Easy-to-implement control law
  - Compensates small disturbances and small model inaccuracies

Problem Statement

- Primary disadvantage of current design techniques for model predictive control (MPC):

  Inability to deal explicitly with plant model uncertainty

- Selected approaches to robustness of MPC:
  - Analysis of robustness properties of MPC [Garcia and Morari], [Zafifiou]
  - Particle filters [Blackmore]
  - MPC with explicit uncertainty description [Campo and Morari], [Allwright and Papavasiliou], [Zheng and Morari]

  Modify the on-line constrained minimization problem to a min-max problem (minimizing the worst-case value of the objective function, where the worst case is taken over the set of uncertain models)

Models for Uncertain Systems

- Linear time-varying system

$$x(k+1) = A(k)x(k) + B(k)u(k)$$
$$y(k) = Cx(k)$$

where,

- the control input: $u(k) \in \Re^{n_u}$
- the state of the plant: $x(k) \in \Re^{n_x}$
- the plant output: $y(k) \in \Re^{n_y}$

with uncertaintis on system matrices *A*(*k*) and *B*(*k*)

$$[A(k) \quad B(k)] \in \Omega \qquad \Omega = Co\{[A_1 \quad B_1], \ [A_2 \quad B_2], \ \ldots, \ [A_L \quad B_L]\}$$

Ω : prespecified set (polytope)

where *Co*: the convex hull

L: number of vertices

- Polytopic system model:
  - input/output data sets
  - Jacobian matrix of a nonlinear discrete time-varying system

Min-Max Approach for RMPC

$$\min J(k) = \min \sum_{i=0}^{H} \left(x(k+i)^T Q_1 x(k+i) + u(k+i)^T R u(k+i) \right)$$

Min-max approach: modify the minimization of the cost function to a minimization of the *worst-case* (maximization over Ω) cost function.

$$\Rightarrow \quad \min_{u(k+i),\, i=0,1,\ldots,H} \left(\max_{[A(k+i)\;\; B(k+i)] \in \Omega,\, i \geq 0} J(k) \right)$$

Question: How to deal with the inner maximization problem? $\left(\max_{[A(k+i)\;\; B(k+i)] \in \Omega,\, i \geq 0} J(k) \right)$

Derive a upper bound of max $J(k)$, then minimize this upper bound with a constant state-feedback control law:

$$u(k+i) = Fx(k+i),\; i \geq 0$$

Derivation of the Upper Bound

- Given quadratic function $V(x) = x^T P x, P > 0 \quad V(0) = 0$
- Suppose V satisfies the following inequality:

$$V\big(x(k+i+1)\big) - V\big(x(k+i)\big) \le -[x(k+i)^T Q_1 x(k+i) + u(k+i)^T R u(k+i)] \,*$$

for $x(\infty) = 0 \Rightarrow V\big(x(\infty)\big) = 0$

Summing (*) from $i = 0$ to $i = \infty \Rightarrow -V\big(x(k)\big) \le -J(k)$

$$\max_{[A(k+i) \quad B(k+i)] \in \Omega, i \ge 0} J(k) \le V\big(x(k)\big)$$

- Substitute the original optimization problem:

$$\min_{u(k+i), i = 0,1,\ldots,H} \left(\max_{[A(k+i) \quad B(k+i)] \in \Omega, i \ge 0} J(k) \right)$$

Implicitly depends on the uncertainties

$$\Rightarrow \min_{u(k+i), i = 0,1,\ldots H} V\big(x(k)\big)$$

Linear Matrix Inequalities (LMIs) (I)

- A linear matrix inequality or LMI is a matrix inequality of the form:

$$M(x) = M_0 + \sum_{r=1}^{m} s_r M_r > 0$$

where, $s \in \Re^m$ is the variable, $M_r = M_r^T \in \Re^{n\times n}$ are given.

- Multiple LMIs $M_1(x) > 0, \ldots, M_n(x) > 0$ can be expressed as the single LMI:

$$diag\left(M_1(x), \ldots, M_n(x)\right) > 0$$

- Convex quadratic inequalities are converted to LMI form using Schur complements.

$$\begin{bmatrix} Q(x) & S(x) \\ S(x)^T & R(x) \end{bmatrix} > 0 \Leftrightarrow Q(x) > 0, R(x) - S(x)^T Q(x)^{-1} S(x) > 0$$

$$\Leftrightarrow R(x) > 0, Q(x) - S(x) R(x)^{-1} S(x)^T > 0$$

where $Q(x) = Q(x)^T, R(x) = R(x)^T, S(x)$ depends affinely on x

Linear Matrix Inequalities (LMIs) (II)

- Example of Schur Complement:

$$c(x)^T P(x)^{-1} c(x) < 1 \Leftrightarrow \left(1 - c(x)^T P(x)^{-1} c(x)\right) > 0 \Leftrightarrow \begin{bmatrix} P(x) & c(x) \\ c(x)^T & 1 \end{bmatrix} > 0$$

with $c(x) \in \Re^n$ and $P(x) = P(x) \in \Re^{n \times n}$

- LMI-based optimization problem can be formulated as:

$$\min c^T x$$
$$s.t. \;\; M(x) > 0$$

where, *M* is a symmetric matrix that depends affinely on the optimization variable *x*, and *c* is a real vector of appropriate size.

Linear Matrix Inequality (III)

- Advantages:
  - LMI problems are tractable and can be solved in polynomial time
  - Robust control problems can be recasted in to LMI formulations

- Main concept of LMI approach for RMPC:

  At each time instant, an LMI optimization problem (as opposed to conventional linear or quadratic programs) is solved that incorporates input and output constraints and a description of the plant uncertainty and guarantees certain robustness properties.

$$\min c^T x$$
$$s.t.\ M(x) > 0$$

Robust Unconstrained MPC using LMIs (I)

Substitution of the original optimization problem:

$$\min_{u(k+i),i=0,1,\dots,H} \left(\max_{[A(k+i) \;\; B(k+i)]\in\Omega,i\geq 0} J(k) \right)$$

$$\Rightarrow \min_{u(k+i),i=0,1,\dots H} V(x(k))$$

Implicitely depends on the uncertainties

$$\min_{u(k+i),i=0,1,\dots H} V(x(k)) \quad \overset{?}{\Longleftrightarrow} \quad \min \gamma$$

with $V(x(k)) = x(k)^T P x(k) \leq \gamma$

Robust Unconstrained MPC using LMIs (II)

Theorem 1: Given $F = YQ^{-1}$,

where, $Q > 0$ and Y is obtained from the solution of the following linear minimization problem.

$$\min_{\gamma, Q, Y} \gamma$$

$$s.t. \begin{bmatrix} 1 & x(k) \\ x(k) & Q \end{bmatrix} \geq 0$$

$$\begin{bmatrix} Q & QA_j^T + Y^T B_j^T & QQ_1^{1/2} & Y^T R^{1/2} \\ A_j Q + B_j Y & Q & 0 & 0 \\ Q_1^{1/2} Q & 0 & \gamma I & 0 \\ R^{1/2} Y & 0 & 0 & \gamma I \end{bmatrix} \geq 0$$

$j = 1, 2, \ldots, L.$ L : number of vertices of the convex hull

Proof of *Theorem 1* (I)

Defining $Q = \gamma P^{-1} > 0$ and Schur complement:

$$\begin{bmatrix} Q(x) & S(x) \\ S(x)^T & R(x) \end{bmatrix} > 0 \Leftrightarrow R(x) > 0, Q(x) - S(x)R(x)^{-1}S(x)^T > 0$$

$$\Rightarrow \quad V(x(k)) = x(k)^T P x(k) \le \gamma \Leftrightarrow 1 - x(k)^T \gamma^{-1} P x(k) > 0$$

$$\Leftrightarrow 1 - x(k)^T Q^{-1} x(k) \Leftrightarrow \begin{bmatrix} 1 & x(k) \\ x(k) & Q \end{bmatrix} \ge 0$$

$$\min_{\gamma, P} \gamma \quad s.t. V(x(k)) = x(k)^T P x(k) \le \gamma \quad \Leftrightarrow \quad \min_{\gamma, Q} \gamma \quad s.t. \begin{bmatrix} 1 & x(k) \\ x(k) & Q \end{bmatrix} \ge 0$$

Proof of *Theorem 1* (II)

$$V\big(x(k+i+1)\big)-V\big(x(k+i)\big)\le -\big(x(k+i)^T Q_1 x(k+i)+u(k+i)^T R u(k+i)\big)$$

- Substituting by $V(x(k))=x(k)^T P x(k)$

  $V(x(k+i+1))=x(k+i+1)^T P x(k+i+1)$ and $u(k+i)=Fx(k+i)$

$$x(k+i)^T\begin{pmatrix}\big(A(k+i)+B(k+i)F\big)^T P\big(A(k+i)+B(k+i)F\big)\\ -P+F^T RF+Q_1\end{pmatrix}x(k+i)\le 0$$

That is satisfied, if

$$\Big(\big(A(k+i)+B(k+i)F\big)^T P\big(A(k+i)+B(k+i)F\big)-P+F^T RF+Q_1\Big)\le 0$$

- Substituting $P=\gamma Q^{-1}$, $Q>0$ and $Y=FQ$, then pre- and post-multiplying by Q

Proof of *Theorem 1* (III)

$$\Rightarrow \begin{bmatrix} Q & QA(k+i)^T + Y^T B(k+i)^T & QQ_1^{1/2} & Y^T R^{1/2} \\ A(k+i)Q + B(k+i)Y & Q & 0 & 0 \\ Q_1^{1/2} Q & 0 & \gamma I & 0 \\ R^{1/2} Y & 0 & 0 & \gamma I \end{bmatrix} \geq 0$$

Affine in $[A(k+i) \quad B(k+i)]$. Hence, it is satisfied for all

$$[A(k+i) \quad B(k+i)] \in \Omega = Co\{[A_1 \quad B_1], \ [A_2 \quad B_2], \ \ldots, \ [A_L \quad B_L]\}$$

if and only if there exist $Q > 0, Y = FQ$, and γ such that:

$$\begin{bmatrix} Q & QA_j^T + Y^T B_j^T & QQ_1^{1/2} & Y^T R^{1/2} \\ A_j Q + B_j Y & Q & 0 & 0 \\ Q_1^{1/2} Q & 0 & \gamma I & 0 \\ R^{1/2} Y & 0 & 0 & \gamma I \end{bmatrix} \geq 0$$

$$j = 1,2,\ldots,L.$$

The feedback matrix is then given by $F = YQ^{-1}$.

Proof of *Theorem 1* (IV)

Theorem 1: Given $F = YQ^{-1}$,

⇨ where, $Q > 0$ and Y is obtained from the solution of the following linear minimization problem.

$$\min_{\gamma, Q, Y} \gamma$$

$$s.t. \begin{bmatrix} 1 & x(k) \\ x(k) & Q \end{bmatrix} \geq 0$$

$$\begin{bmatrix} Q & QA_j^T + Y^T B_j^T & QQ_1^{1/2} & Y^T R^{1/2} \\ A_j Q + B_j Y & Q & 0 & 0 \\ Q_1^{1/2} Q & 0 & \gamma I & 0 \\ R^{1/2} Y & 0 & 0 & \gamma I \end{bmatrix} \geq 0$$

$j = 1, 2, \ldots, L.$ L : number of vertices of the convex hull

Varying State-Feedback Matrix (I)

- The feedback matrix *F: u*(*k*+*i*) = *Fx*(*k*+*i*) computed from *Theorem 1* is constant. But in the presence of uncertainty, *F* shows a strong dependence on the state of the system.

⇨ using a receding horizon approach and recomputing *F*(*k*+*i*) at each sampling time shows significant improvement in performance as opposed to using a static state feedback control law.

- Example: consider the ‚polytopic' system with:

$$A_1 = \begin{bmatrix} 0.9347 & 0.5194 \\ 0.3835 & 0.8310 \end{bmatrix}, A_2 = \begin{bmatrix} 0.0591 & 0.2641 \\ 1.7971 & 0.8717 \end{bmatrix}, B = \begin{bmatrix} -1.4462 \\ -0.7012 \end{bmatrix}$$

with weighting matrices in the cost function $Q_1 = R, R = I$

$$J(k) = \sum_{i=0}^{H} \left(x(k+i)^T Q_1 x(k+i) + u(k+i)^T R u(k+i) \right)$$

Varying State-Feedback Matrix (II)

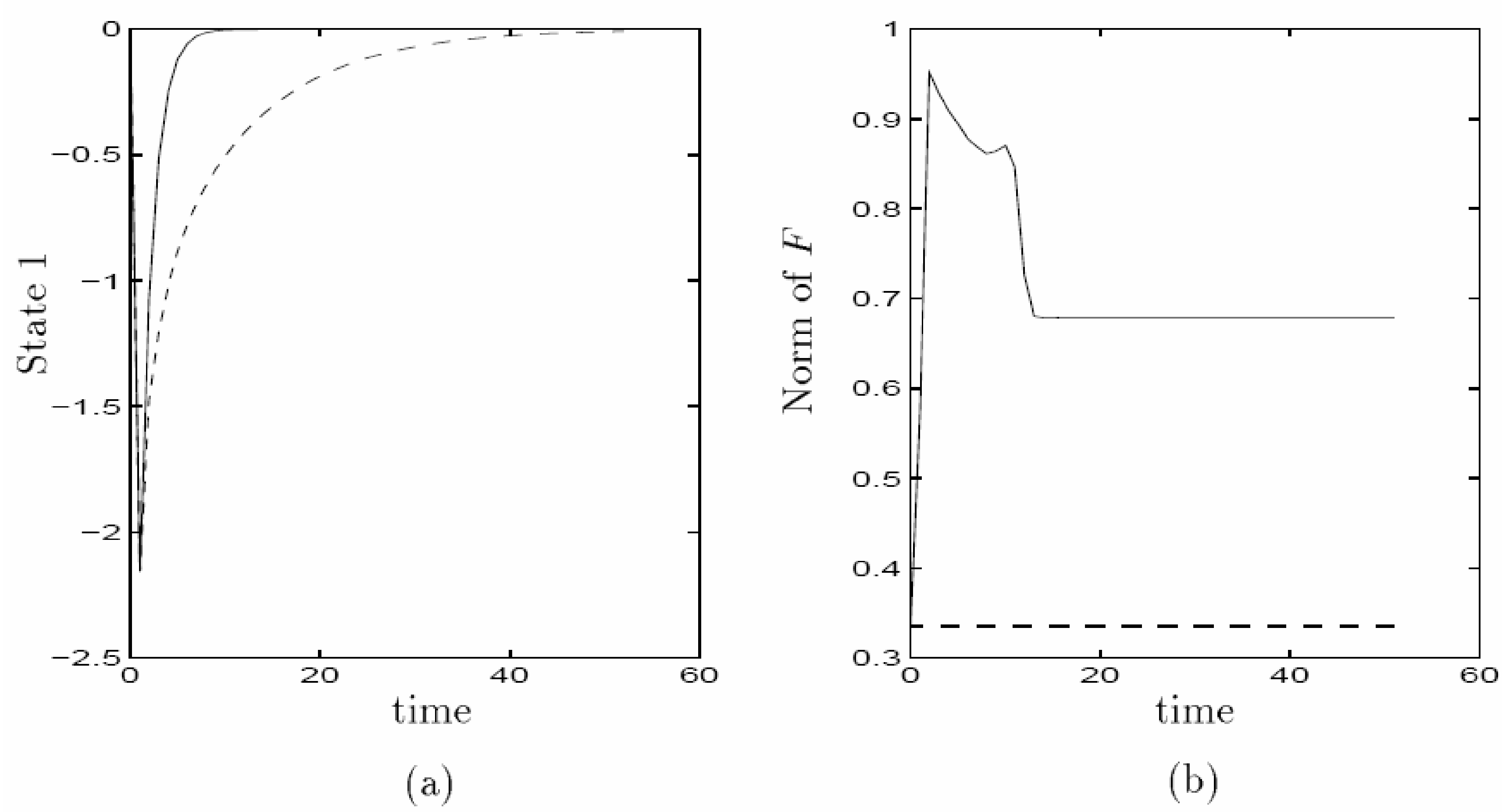

Fig. 2. (a) Unconstrained closed-loop responses and (b) norm of the feedback matrix $F(k)$: **solid line**, using receding horizon state feedback; **dashed lines**, using robust static state feedback.

Robust Constrained MPC using LMIs

Lemma 1. (Invariant ellipsoid): if

$$x(k)^T Q^{-1} x(k) \le 1 \Leftrightarrow x(k)^T P x(k) \le 1, P = \gamma Q^{-1}$$

then

$$\max_{[A(k+i)\ \ B(k+i)] \in \Omega, i \ge 0} x(k+i)^T Q^{-1} x(k+i) < 1, i \ge 1$$

$$\Leftrightarrow \max_{[A(k+i)\ \ B(k+i)] \in \Omega, i \ge 0} x(k+i)^T P x(k+i) < \gamma, i \ge 1$$

Thus

$$\Phi = \{x | x^T Q^{-1} x \le 1\} = \{x | x^T P x \le \gamma\}$$

is an invariant ellipsoid for the predicted states of the uncertain system.

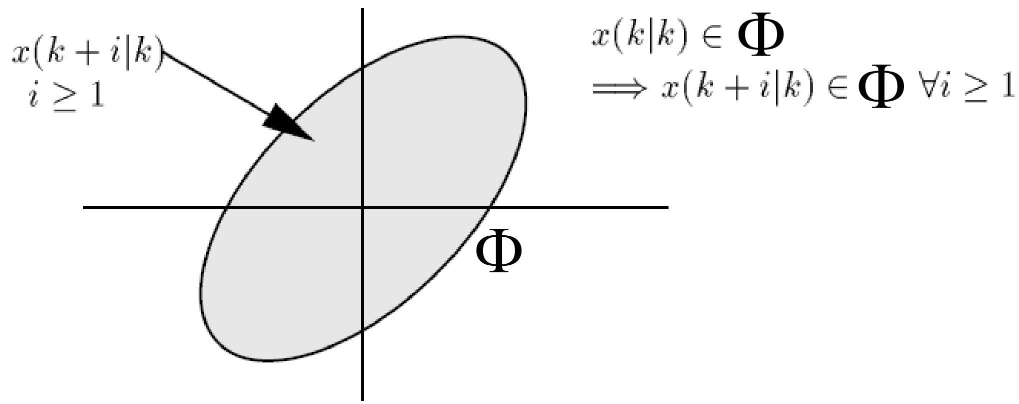

Fig. 3. Graphical representation of the state-invariant ellipsoid Φ in two dimensions

Proof of *Lemma 1*

Suppose: $V\big(x(k+i+1)\big)-V\big(x(k+i)\big)\leq$

$$-[x(k+i)^T Q_1 x(k+i)+u(k+i-1)^T Ru(k+i-1)]$$

since $Q_1 > 0, R > 0$

$$\Rightarrow x(k+i+1)^T Px(k+i+1) - x(k+i)^T Px(k+i) \leq$$

$$-x(k+i)^T Q_1 x(k+i) - u(k+i-1)^T Ru(k+i-1) < 0$$

$$\Rightarrow x(k+i+1)^T Px(k+i+1) < x(k+i)^T Px(k+i), i \geq 0,$$

$$x(k+i) \neq 0$$

Thus if $x(k)^T Px(k) < \gamma \Rightarrow x(k+1)^T Px(k+1) < \gamma$.

This argument can be continued for $x(k+2), x(k+3), \ldots$

Input Constraints

Given $\|u(k+i)\|_2 \leq u_{\max}, i \geq 0$

From [Boyd et al.] $\Rightarrow \max_{i \geq 0} \|u(k+i)\|_2 = \max_{i \geq 0} \|YQ^{-1}x(k+i)\|_2 \leq \max_{x \in \Phi} \|YQ^{-1}x\|_2$

$= \lambda_{\max}(Q^{-1/2}Y^T YQ^{-1/2})$ maximal value of the eigenvalue

and using Schur Complement

$$R(x) > 0, Q(x) - S(x)R(x)S(x)^T > 0 \Leftrightarrow \begin{bmatrix} Q(x) & S(x) \\ S(x)^T & R(x) \end{bmatrix} > 0$$

$\|u(k+i)\|_2 \leq u_{\max}$ is enforced at all times $i \geq 0$

if the LMI $\begin{bmatrix} u_{\max}^2 I & Y \\ Y^T & Q \end{bmatrix} \geq 0$ holds.

Output Constraints

At sampling time *k*, consider

$$\max_{[A(k+j)\ \ B(k+j)]\in\Omega,\, j\geq 0} \left\| y(k+i) \right\|_2 \leq y_{\max}, i \geq 1$$

If

$$\begin{bmatrix} Q & (A_j Q + B_j Y)^T C^T \\ C(A_j Q + B_j Y) & y_{\max}^2 I \end{bmatrix} \geq 0, j = 1,2,\ldots,L$$

then

$$\max_{[A(k+j)\ \ B(k+j)]\in\Omega,\, j\geq 0} \left\| y(k+i) \right\|_2 \leq y_{\max}, i \geq 1$$

Output Constraints as LMIs

At sampling time k, consider $\max_{[A(k+j)\ B(k+j)]\in\Omega, j\geq 0} \|y(k+i)\|_2 \leq y_{\max}, i \geq 1$

$$\max_{i\geq 0}\|y(k+i)\|_2 = \max_{i\geq 0}\left\|C\left(A(k+i)+B(k+i)F\right)x(k+i)\right\|_2$$

$$\leq \max_{z\in\Phi}\left\|C\left(A(k+i)+B(k+i)F\right)z\right\|_2, i \geq 0 = \lambda\left[C\left(A(k+i)+B(k+i)F\right)Q^{1/2}\right] \leq y_{\max}$$

$$\Leftrightarrow Q^{1/2}\left(A(k+i)+B(k+i)F\right)^T C^T C\left(A(k+i)+B(k+i)F\right)Q^{1/2} \leq y_{\max} I$$

$$\Leftrightarrow \begin{bmatrix} Q & (A(k+i)Q+B(k+i)Y)^T C^T \\ C(A(k+i)Q+B(k+i)Y) & y_{\max}^2 I \end{bmatrix} \geq 0, i \geq 0$$

Since the last inequality is affine in $\left[A(k+i) \quad B(k+i)\right] \in \Omega$

$$\begin{bmatrix} Q & (A_j Q+B_j Y)^T C^T \\ C(A_j Q+B_j Y) & y_{\max}^2 I \end{bmatrix} \geq 0, j = 1,2,\ldots,L$$

Problem Formulation (I)

Substitution of the original optimization problem:

$$\min_{u(k+i),i=0,1,\ldots,H} \left(\max_{[A(k+i)\;\; B(k+i)]\in\Omega, i\geq 0} J(k) \right)$$

$$\Rightarrow \quad \min_{u(k+i),i=0,1,\ldots H} V\left(x(k)\right)$$

Implicitely depends on the uncertainties

$$\min_{u(k+i),i=0,1,\ldots H} V\left(x(k)\right) \quad \Leftrightarrow \quad \min \gamma$$

with $V\left(x(k)\right) = x(k)^T P x(k) \leq \gamma$

Problem Formulation (II)

$$\min_{\gamma, Q, Y} \gamma$$

$$s.t. \begin{bmatrix} 1 & x(k) \\ x(k) & Q \end{bmatrix} \geq 0$$

$$\begin{bmatrix} Q & QA_j^T + Y^T B_j^T & QQ_1^{1/2} & Y^T R^{1/2} \\ A_j Q + B_j Y & Q & 0 & 0 \\ Q_1^{1/2} Q & 0 & \gamma I & 0 \\ R^{1/2} Y & 0 & 0 & \gamma I \end{bmatrix} \geq 0, \; j = 1,2,\ldots,L$$

Input constraints: $$\begin{bmatrix} u_{max}^2 I & Y \\ Y^T & Q \end{bmatrix} \geq 0$$

Output constraints: $$\begin{bmatrix} Q & (A_j Q + B_j Y)^T C^T \\ C(A_j Q + B_j Y) & y_{max}^2 I \end{bmatrix} \geq 0, \; j = 1,2,\ldots,L$$

Numerical Example

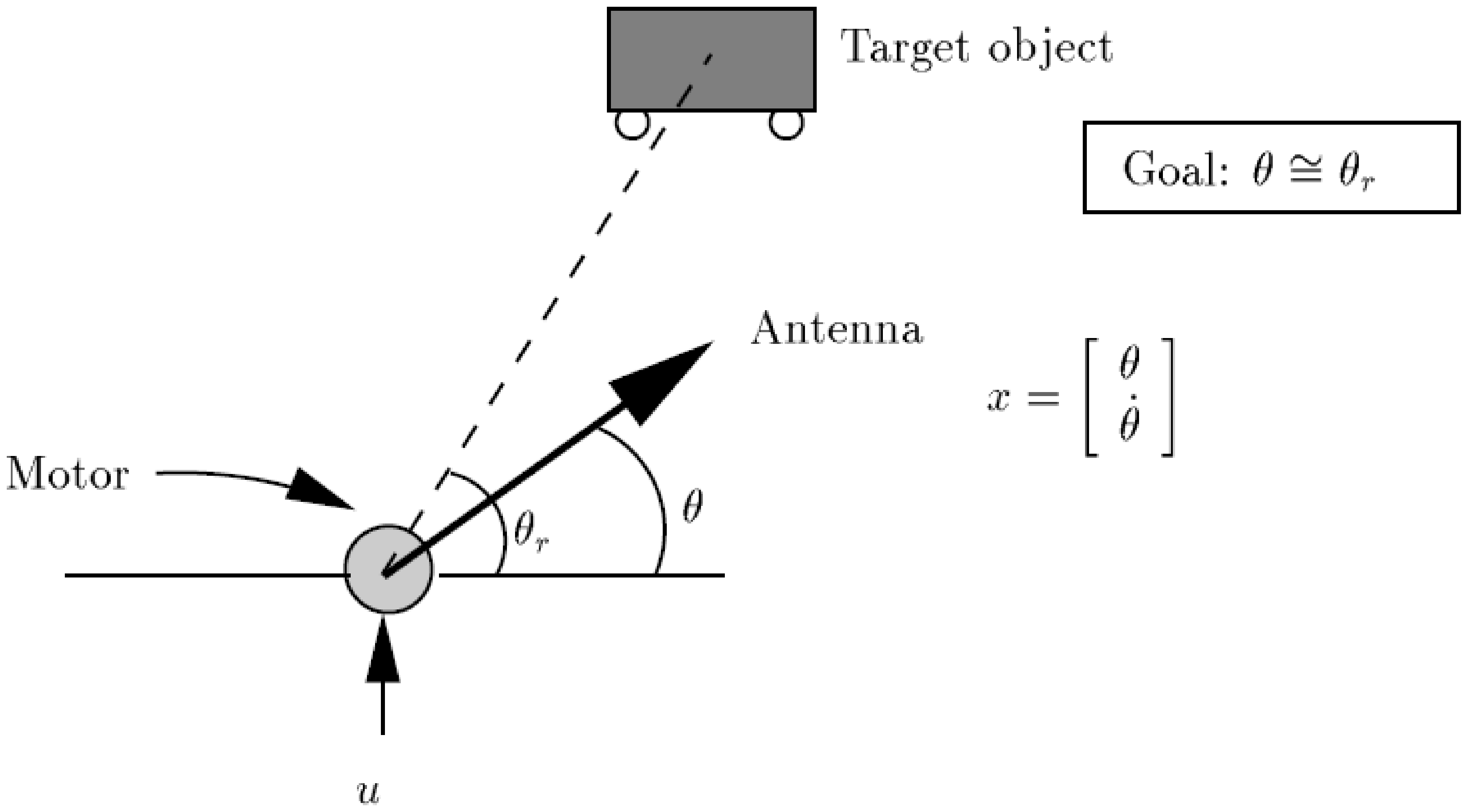

Fig. 4. Angular positioning system. [Kwakernaak et al.]

Solver: LMI Control Toolbox [Gahinet et al.] in MATLAB

System Dynamics

- System dynamics:

$$x(k+1)=\begin{bmatrix}\theta(k+1)\\ \dot{\theta}(k+1)\end{bmatrix}=\begin{bmatrix}1 & 0.1\\ 0 & 1-0.1\alpha(k)\end{bmatrix}x(k)+\begin{bmatrix}0\\ 0.1\kappa\end{bmatrix}u(k)$$

$$y(k)=\begin{bmatrix}1 & 0\end{bmatrix}x(k)$$

with $\kappa=0.787,\ 0.1\le\alpha(k)\le 10,\ x(0)=\begin{bmatrix}0.05\\ 0\end{bmatrix}$

$\alpha(k)$ is proportional to the coefficient of viscous friction

$$\Rightarrow A_1=\begin{bmatrix}1 & 0.1\\ 0 & 0.99\end{bmatrix},\ A_2=\begin{bmatrix}1 & 0.1\\ 0 & 0\end{bmatrix}\Rightarrow A(k)\in\Omega=Co\{A_1,A_2\}$$

Cost Function

- Cost function:

$$\min_{\substack{u(k+i)=Fx(k+i)\\ i\geq 0}} \max_{\substack{A(k+i)\in\Omega\\ i\geq 0}} \left(J(k) = \sum_{i=0}^{H} \left(y(k+i)^2 + Ru(k+i)^2 \right) \right),$$

$$R = 0.00002$$

$$s.t. \quad \left\| u(k+i) \right\|_2 \leq 2, \;\; i \geq 0$$

Simulation Results (I)

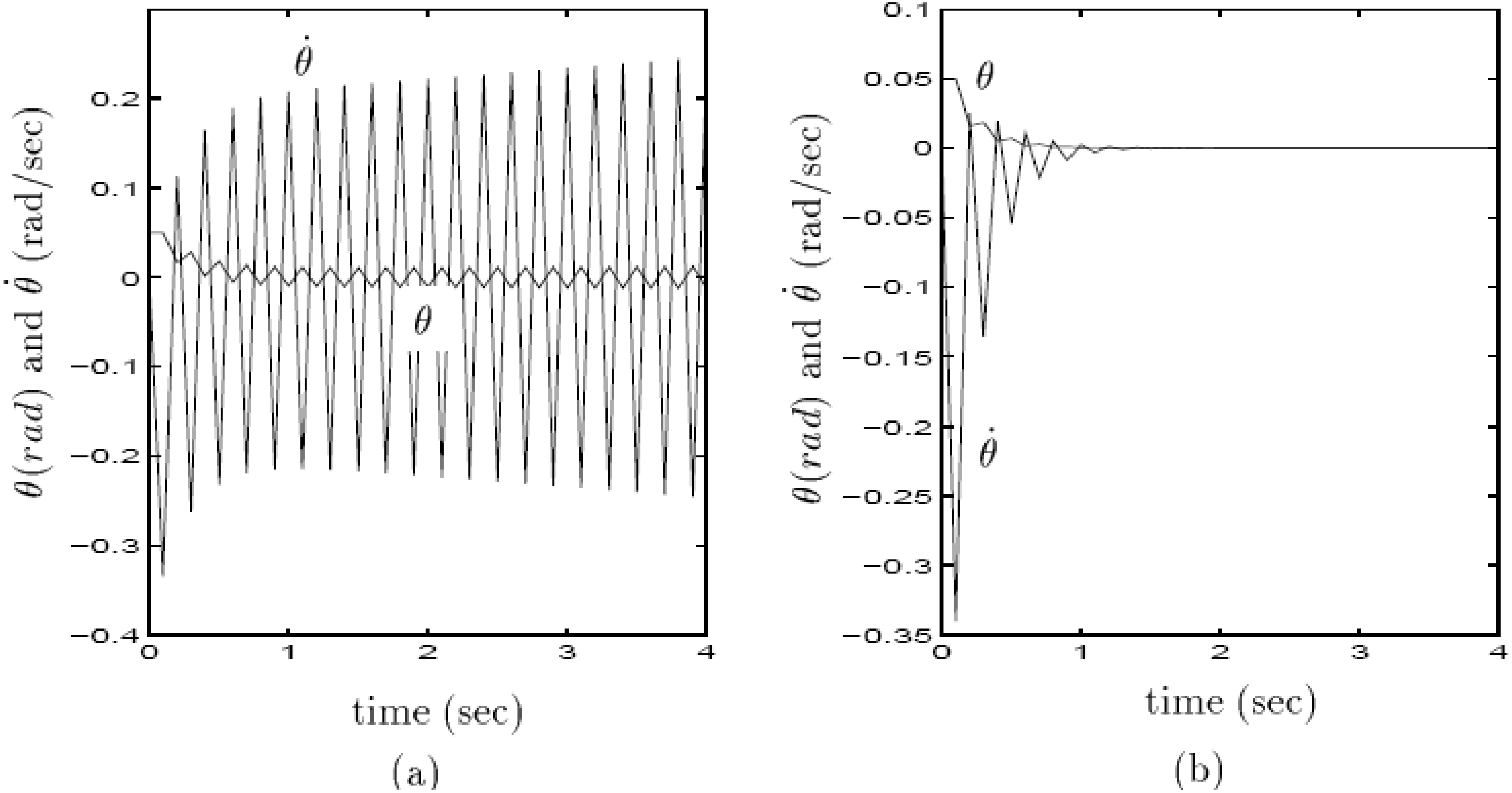

Fig. 5. Unconstrained closed-loop responses for the plant: (a) using standard MPC with $\alpha(k) = 1$; (b) using robust LMI-based MPC with random $\alpha(k)$.

Simulation Results (II)

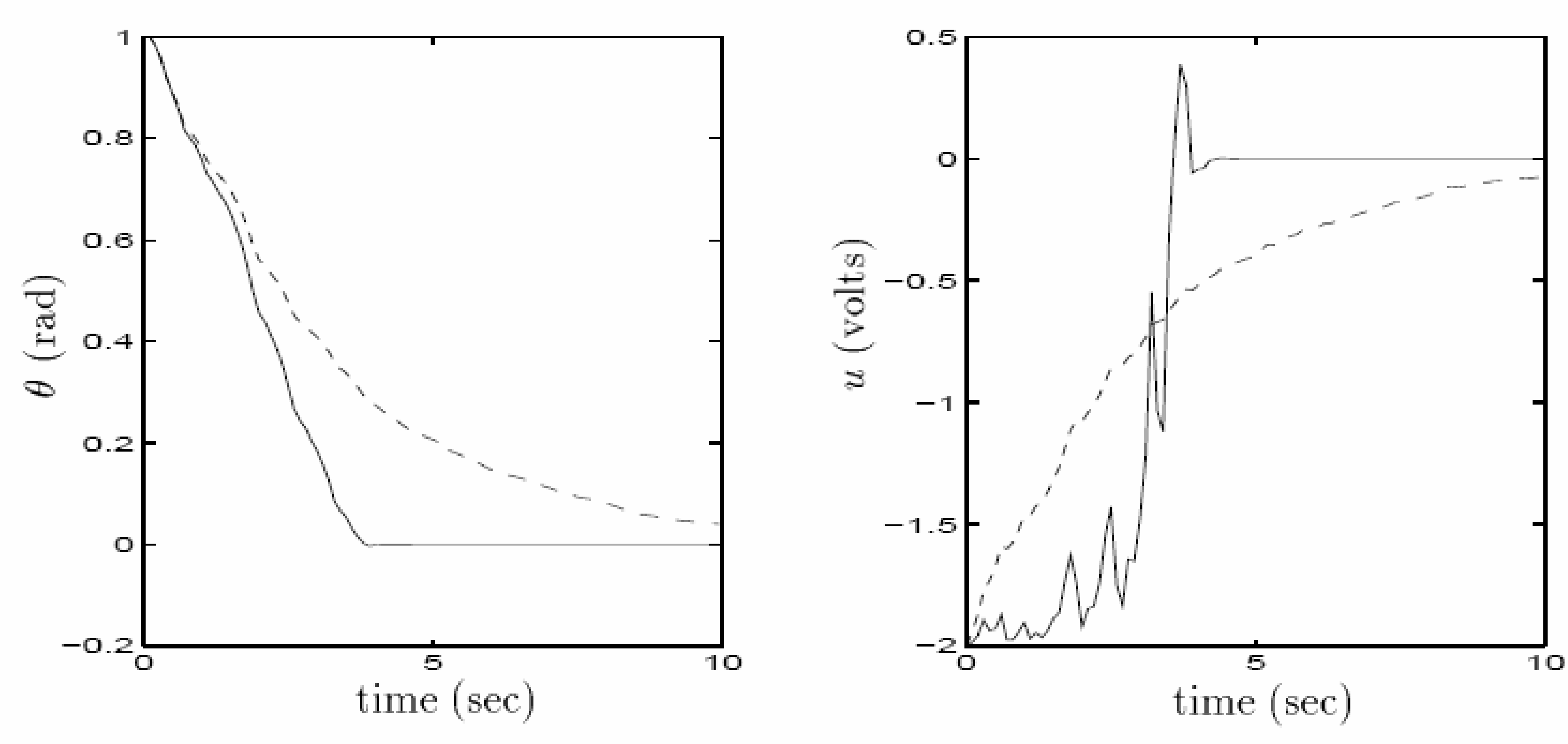

Fig. 6. Closed-loop responses for the time-varying system with input constraint: **solid lines**, using robust receding horizon state feedback *F*(*k*); **dashed lines**, using robust static state feedback *F*.

Conclusions

- A new theory for robust MPC synthesis (based on the assumption of full state feedback)
- On-line optimization involving an LMI-based linear objective minimization
- Extensions:
  - Models with additive uncertainties

    $$x(k+1) = Ax(k) + Bu(k) + Gw(k)$$

    where $w(k) \in W$: additive bounded uncertainties
  - Reference trajectory tracking
  - Delay systems
  - RMPC for hybrid systems

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