Classification of algebraic surfaces up to fourth degree

Vinogradova Anna

Bases of algebraic surfaces

The algebraic surfaces are described by algebraic equations:

- In a general view → $F(x^n, y^n, z^n) = 0$
- Surface of n-th order →

$$a_m x^n + a_{m-1} y^n + a_{m-2} z^n + \dots a_0 = 0,$$

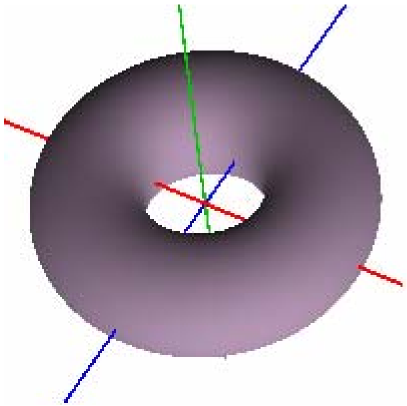
where m - number of factor of the algebraic equation received depending on the order of the equation.

In process of increase in the order of the equation the number of factors of the equation and every possible forms of surfaces grows.

Degree of the algebraic equation	*Quantity of factors*
1	4
2	10
3	20
4	35
5	56
6	84
7	120
8	165
9	220
10	286

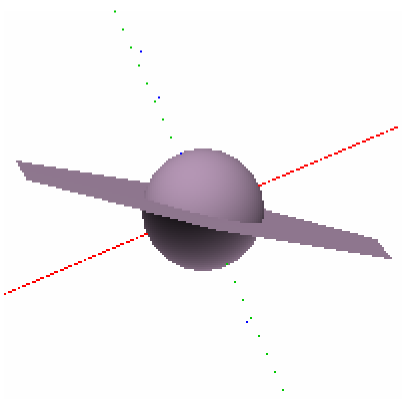
Types of surfaces

The type of a surface is certain by conditions at which numerical values of factors of the equation define the concrete geometrical form of a surface.



Examples:

- $x^4 + y^4 + z^4 + 2x^2y^2 + 2x^2z^2 + 2y^2z^2 – 10x^2 – 10y^2 + 3z^2 + 9 = 0$



- $x^3 + y^3 + z^3 + x^2y + x^2z + y^2x + y^2z + z^2x + z^2y -9x – 9y – 9z = 0$

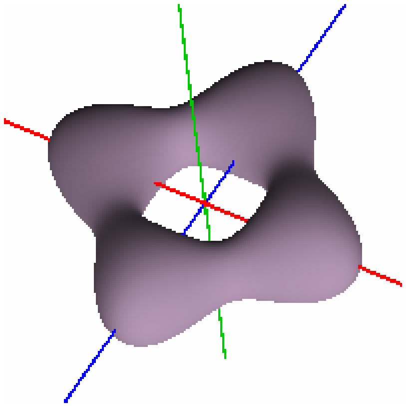
note: x - a red axis

y - a dark blue axis

z - a green axis

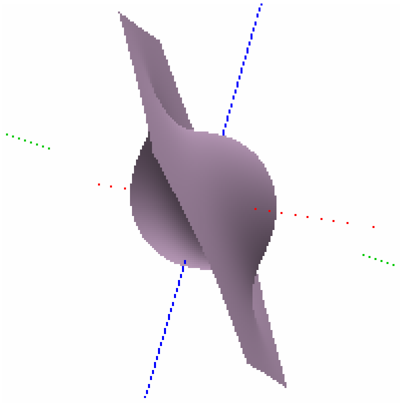
Subtypes of surfaces

Subtypes of a surface - the changed surface keeps the basic properties of type of a surface.



Examples:

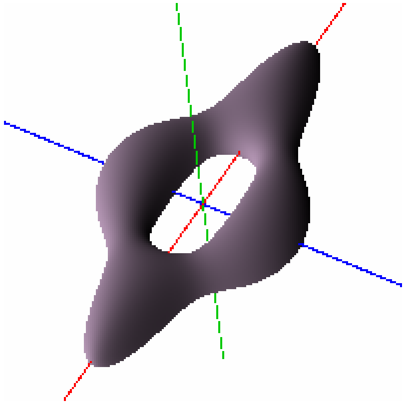
- $x^4 + y^4 + z^4 + 6x^2y^2 + 2x^2z^2 + 2y^2z^2 - 10x^2 - 10y^2 + 3z^2 + 9 = 0$



- $x^3 + y^3 + z^3 + x^2y + x^2z + y^2x + y^2z + z^2x + z^2y - 20x - 9y - 9z = 0$

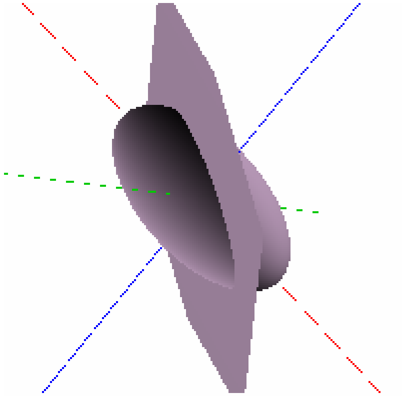
Kinds of surfaces

Kind of a surface - type or a subtype of the surface proportionally changed in sizes concerning three axes of system of coordinates (scaling), turned or moved to space.



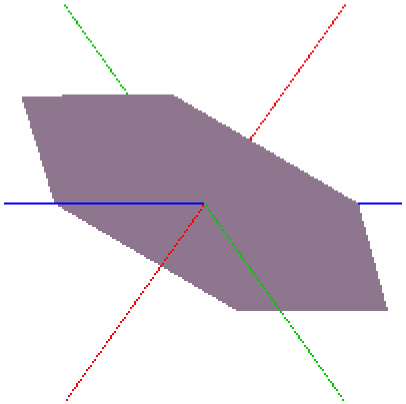
Examples:

- $x^4/2^4 + y^4 + z^4 + 6/2^2x^2y^2 + 2/2^2x^2z^2 + 2y^2z^2 - 10x^2 - 10/2^2y^2 + 3z^2 + 9 = 0$



- $x^3/2^3 + y^3 + z^3 + x^2y/2^2 + x^2z/2^2 + y^2x/2 + y^2z + z^2x/2 + z^2y - 9x/2 - 9y - 9z = 0$

Classification of types and subtypes of 1-st and 2-nd orders



The analysis of the algebraic equations of 1-st order has shown, that we have only one type of a surface - a plane.

Examples:

- **$x + y + z = 0$**

The analysis of the algebraic equations of 2-nd order shows: there were various types of surfaces and so-called imaginary surfaces when change of one factor leads to absence of the decision of the algebraic equation.

An example of an imaginary surface:

$x^2 + y^2 + z^2 = -1$

Classification on types of surfaces of 2-nd order starts with invariance of factors of the equations of 2-nd order:

- Two planes (1)

$x^2 - 9 = 0$

- Spherical (2)

$x^2 - y^2 + z^2 - 9 = 0$

- Cylindrical (3)

$-x^2 - y^2 + 9 = 0$

- Conic (4)

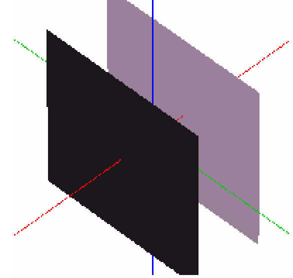
$x^2 - y^2 - z^2 = 0$

- Hyperbolic (5)

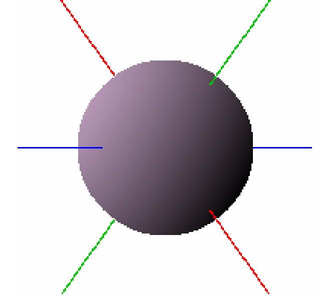
$0.25x^2 + 0.25y^2 – 0.25z^2 + 1 = 0$

- Parabolic (6)

$0.5x^2 + 0.5y^2 - z = 0$



(1)



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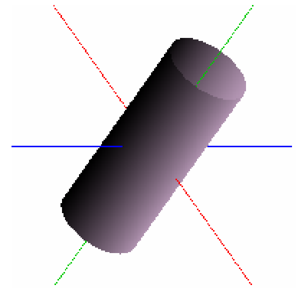

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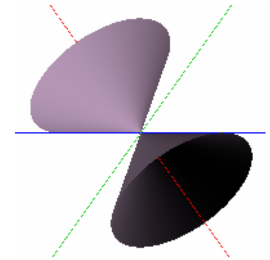

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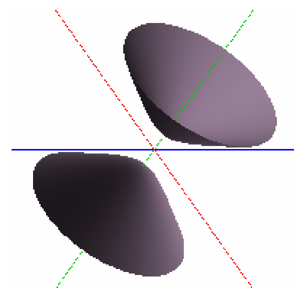

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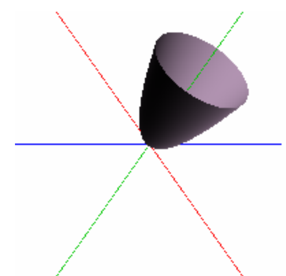
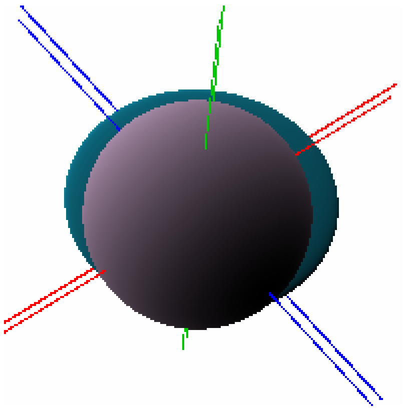


(6)

Classification on subtypes of surfaces of 2-nd order it is carried out, forming the geometrical form of a surface by change of the sizes of an initial surface on one or two axes of system of coordinates. For example, stretching sphere on one of axes of system of coordinates, we receive two-axes ellipsoid.



For surfaces of 2-nd order 6 types and 22 subtypes are certain.

Reception of kinds of surfaces

For reception of kinds of surfaces of any orders it is necessary to use transformation of factors of their equations under formulas:

► *Scaling on three axes*

$x_m = x * K_m$

$y_m = y * K_m$ where K_m - number of scaling

$z_m = z * K_m$

► *Displacement in space*

$x_d = x + dx$

$y_d = y + dy$

$z_d = z + dz$

where *dx*, *dy*, *dz* - sizes of carry of a surface on corresponding axes

► *Rotation in space*

Around of an axis x

$x_r = x$; $y_r = y \cos r - z \sin r$; $z_r = y \sin r + z \cos r$

Around of an axis *y*

$y_r = y$; $y_r = x \cos r - z \sin r$; $z_r = x \sin r + z \cos r$

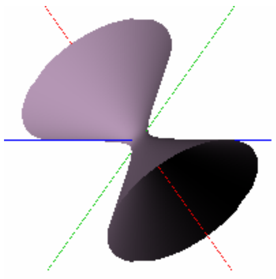
Around of an axis *z*

$z_r = z$; $x_r = x \cos r - y \sin r$; $y_r = x \sin r + y \cos r$

Using the given formulas it is possible to receive any amount of kinds of surfaces from types and subtypes.

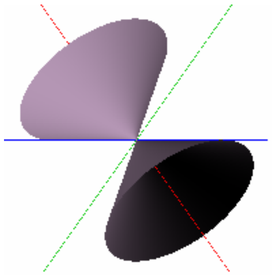
Searching method of factors values of the equation (-1,0,1)

The analysis of surfaces of 2-nd order shows, that the equations of surfaces which axes of symmetry are located or pass through the beginning of system of coordinates, have the values of factors equal on occasion or -1 or 0 or 1. Such equations can be named resulted.



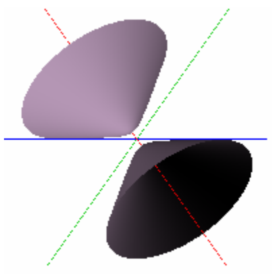
To demonstrate this principle consider a shape of the second order:

One-sheeted hyperboloid: $x^2 - y^2 - z^2 + 1 = 0$



That we set a free coefficient to 0 and get a new shape:

Elliptic cone: $x^2 - y^2 - z^2 = 0$



That we set a free coefficient to -1 and get a new shape once again:

Two-sheeted hyperboloid: $x^2 - y^2 - z^2 - 1 = 0$

Equation of the **4-th** order → contains **35** factors
Each factor can accept values → **-1, 0, 1**

Consequently, it would be $\mathbf{3^{35}}$ different cases!
But:

- for surfaces of the fourth order all factors with $\boldsymbol{a_{35}}$ on $\boldsymbol{a_{20}}$ should not be equal to zero simultaneously, differently it will not be the equation of the fourth order
- it is possible to assume, that changes in factors at variables where a total degree of variables below the order of a surface, will lead to creation only subtypes or kinds of surfaces
- the factor $\mathbf{a_0}$ is the equalizing factor and enables to define imaginary surfaces (when the equation has no decisions)

Thus, we have reduced to search of 16 factors for a surface of **4-th** order (with $\boldsymbol{a_{35}}$on $\boldsymbol{a_{20}}$, $\boldsymbol{a_0}$).

The number of possible types of surfaces of **4-th** order, including imaginary surfaces and excepting zero variants, can be **4 782 969**.

The similar algorithm applied to surfaces of 3-rd order gives **59 049** variants.