Polynomials in graph theory

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Outline

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We are going to color maps on an island (or on a sphere).
Countries are planar regions.
In case of proper coloring 2 neighboring countries must have different colors.

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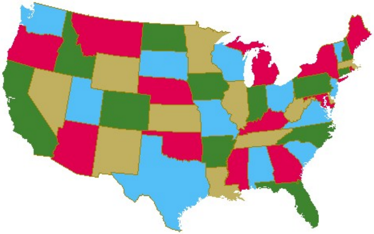

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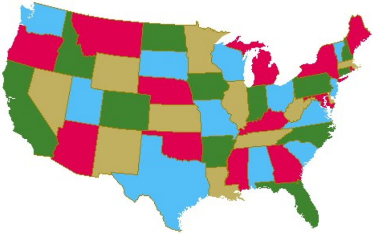

Terminological remark:

proper coloring	*vs*	**coloring**
coloring	*vs*	*assignment of colors*

Introduction to Coloring Problem

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It can be proved that countries may be colored in 6 colors.

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Line graph:
its vertices correspond to edges of initial graph,
2 vertices are connected by edge iff 2 corresponding edges of initial graph are incidental

Introduction to Coloring Problem

1852, Guthrie: The Four Color Conjecture (**4CC**).
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Searching for simplified reformulations where proofs can be checked by a human being.
Polynomials are the main instrument of counting colorings.

Outline

Algebraic methods of counting graph colorings

$\chi_G(p)$ is the number of (proper) colorings of (vertices of) graph G in p colors.

Algebraic methods: the theorem

$$M_G(p) = \prod_{(v_k, v_l) \in E} N_p(x_k, x_l)$$

$$N_p(y, z) = p - 1 - y^{p-1}z - y^{p-2}z^2 - \ldots - yz^{p-1}$$

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Theorem

For any graph $G = (V, E)$, $|V| = m$, $|E| = n \quad \forall p \in \mathbb{N}$

$$\chi_G(p) = p^{m-n}(\mathcal{R}_p M_p(G))(0, \ldots, 0).$$

Algebraic methods: Proof of the theorem

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$\mu : V \to C = \{c_0, \ldots, c_{p-1}\}$ is a coloring of the graph G.

Algebraic methods: Proof of the theorem (cont.)

Using interpolation theorem we get:

$$\mathcal{R}_p M_p(G) = \sum_{\mu} (\mathcal{R}_p M_p(G))(\mu(v_1), \ldots, \mu(v_m)) P_\mu$$

(the sum is taken through all p^m colorings),

$$P_\mu = \prod_{k=1}^{m} S_p(x_k, \mu(v_k)),$$

$$S_p(x, c_q) = \prod_{0 \le l \le p-1,\, l \ne q} \frac{x - c_l}{c_q - c_l}.$$

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For arguments from C values of $M_p(G)$ and $\mathcal{R}_p M_p(G)$ are the same, so we have

$$\mathcal{R}_p M_p(G) = \sum_{\mu} M_p(G)(\mu(v_1), \ldots, \mu(v_m)) P_\mu$$

Algebraic methods: Proof of the theorem (cont.)

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$M_p(G)(\mu(v_1), \ldots, \mu(v_m)) = p^n$ for a proper coloring,
else $M_p(G)(\mu(v_1), \ldots, \mu(v_m)) = 0$. We came to

$$\mathcal{R}_p M_p(G) = p^n \sum_{\mu} P_\mu$$

where sum is taken through $\chi_G(p)$ proper colorings.

Algebraic methods: Proof of the theorem (cont.)

Substituting $x_1 = \ldots = x_m = 0$:

$$P_\mu(0, \ldots, 0) = \prod_{k=1}^{m} S_p(0, \mu(v_k)),$$

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So $S_p(0, c_q) = S_p(0)$ is independent of c_q.

Algebraic methods: Proof of the theorem (cont.)

$$(\mathcal{R}_p M_p(G))(0,\ldots,0) = p^n S_p(0)^m \chi_G(p)$$

If we substitute any specific graph (e.g. K_1 that has 1 vertex and 0 edges) we get $S_p(0)$:
$m = 1,\ n = 0,\ (\mathcal{R}_p M_p(K_1))(0,\ldots,0) = 1,\ \chi_{K_1}(p) = p.$

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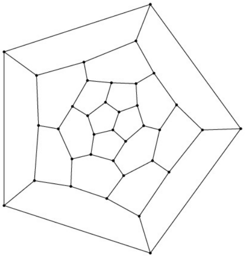
$m = 1,\ n = 0,\ (\mathcal{R}_p M_p(K_1))(0, \ldots, 0) = 1,\ \chi_{K_1}(p) = p.$

Therefore $S_p(0) = p^{-1}$ and $\chi_G(p) = p^{m-n}(\mathcal{R}_p M_p(G))(0, \ldots, 0).$

Algebraic methods: Colorings of 3-valent graphs



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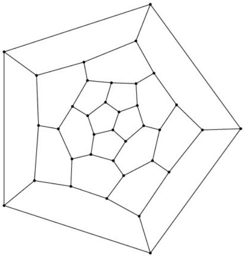

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3-valent graph T can be represented as
$T = \{\langle e_{i_1}, e_{j_1}, e_{k_1}\rangle, \ldots, \langle e_{i_{2n}}, e_{j_{2n}}, e_{k_{2n}}\rangle\}$ ($2n$ vertices, $3n$ edges).

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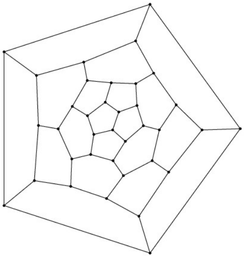

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$\lambda_G(p)$ is the number of (proper) colorings of edges of G in p colors.

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$$\mathcal{R}_3 M(G)(x_1, \dots, x_{3n}) = \sum_{d_1,\dots,d_{3n} \in \{0,1,2\}} c_{d_1,\dots,d_{3n}} x_1^{d_1} \dots x_{3n}^{d_{3n}}$$

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Theorem

For any planar 3-valent graph G

$$\lambda_G(3) = (\mathcal{R}_3 M(G))(0, \ldots, 0) = c_{0,\ldots,0}.$$

Algebraic methods: Proof of the theorem

Here coloring is defined as $\nu : E \to \{1, \omega, \omega^2\}$ where $\omega = \frac{-1+i\sqrt{3}}{2}$, the primitive cubic root of 1.

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$$M(G)(\nu(e_1), \ldots, \nu(e_{3n})) =$$

$$= \prod_{l=1}^{2n} (\nu(e_{i_l}) - \nu(e_{j_l}))(\nu(e_{j_l}) - \nu(e_{k_l}))(\nu(e_{k_l}) - \nu(e_{i_l}))$$

equals 0 if ν is not a proper coloring.

Algebraic methods: Proof of the theorem cont.

If the coloring ν is proper
then there are $1, \omega, \omega^2$ between $\nu(e_p), \nu(e_q), \nu(e_r)$,
so $L(\nu(e_p), \nu(e_q), \nu(e_r)) = \pm i3\sqrt{3}$,
and $M(G)(\nu(e_1), \ldots, \nu(e_{3n})) = \pm 3^{3n}$.

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Then $\mathcal{R}_3 M(G) = 3^{3n} \sum_\nu P_\nu$,
here are $\lambda_G(3)$ summands.
$(\mathcal{R}_3 M(G))(0, \ldots, 0) = 3^{3n}(S_3(0))^{3n}\lambda_G(3) = \lambda_G(3)$.

Outline

Orientations: Definitions 1

$G = (V, E)$, $f : V \to \mathbb{Z}$. G is f-**choosable** if
$\forall S : V \to 2^{\mathbb{Z}}$, $|S(v)| = f(v) \forall v$
there exists a proper coloring $c : V \to \mathbb{Z}$ such that $\forall v \; c(v) \in S(v)$.

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G is k-**choosable** ($k \in \mathbb{Z}$) if $f \equiv k$.
Minimal k for which G is k-choosable is referred to as a **choice number** of G.

Orientations: Definitions 2

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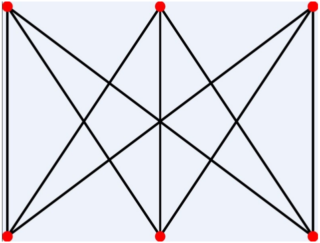

Figure: $S(u_i) = S(v_i) = \{1, 2, 3\} \setminus \{i\}$

But there is a conjecture claiming that $\forall G$ ch'$(G) = \chi'(G)$.

Orientations: Definitions 3. Main Theorem

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Theorem

$D = (V, E)$ being a digraph, $|V| = n$, $d_i = d_D^+(v_i)$, $f(i) = d_i + 1 \; \forall i \in \{1, \ldots, n\}$, $\quad EE(D) \neq EO(D) \Rightarrow$ D is f-choosable.

Orientations: Proof of the theorem

Lemma

Let $P(x_1, \ldots, x_n)$ be polynomial in n variables over $\mathbb{Z}$, for $1 \le i \le n$ the degree of P in x_i doesn't exceed d_i, $S_i \subset \mathbb{Z} : |S_i| = d_i + 1$. If $\forall (x_1, \ldots, x_n) \in S_1 \times \ldots \times S_n \; P(x_1, \ldots, x_n) = 0$ then $P \equiv 0$.

(Exercise — proof by induction.)

Orientations: Proof of the theorem (cont.)

Graph polynomial of undirected graph G:

$$f_G(x_1, \ldots, x_n) = \prod_{i<j, v_i v_j \in E} (x_i - x_j)$$

Orientations: Proof of the theorem (cont.)

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Orientation is **even** if it has even number of decreasing edges, else it's **odd**.

Orientations: Proof of the theorem (cont.)

$DE(d_1, \ldots, d_n)$ and $DO(d_1, \ldots, d_n)$ are sets of even and odd orientations;
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Then evidently holds

Lemma

$$f_G(x_1, \ldots, x_n) = \sum_{d_1, \ldots, d_n \geq 0} \left(|DE(d_1, \ldots, d_n)| - |DO(d_1, \ldots, d_n)|\right) \prod_{i=1}^{n} x_i^{d_i}$$

Orientations: Proof of the theorem (cont.)

Let us further take $D_1, D_2 \in DE(d_1, \ldots, d_n) \cup DO(d_1, \ldots, d_n)$. $D_1 \oplus D_2$ denotes set of edges in D_1 that have the opposite direction in D_2.

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Mapping $D_2 \mapsto D_1 \oplus D_2$ is a bijection between $DE(d_1, \dots, d_n) \cup DO(d_1, \dots, d_n)$ and set of Eulerian subgraphs of D_1.

Orientations: Proof of the theorem (cont.)

If D_1 is even then it maps even orientations to even subgraphs and odd ones to odd ones.

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Thus we get

$$||DE(d_1, \ldots, d_n)| - |DO(d_1, \ldots, d_n)|| = |EE(D_1) - EO(D_1)|$$

(it's the coefficient of the monomial $\prod x_i^{d_i}$ in f_G).

Orientations: Proof of the theorem (cont.)

Recall the statement of the theorem. Suppose there is no such coloring.

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Then $\forall (x_1, \ldots, x_n) \in S_1 \times \ldots \times S_n \; f_G(x_1, \ldots, x_n) = 0$.

Orientations: Proof of the theorem (cont.)

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Let $Q_i(x_i)$ be

$$Q_i(x_i) = \prod_{s \in S_i} (x_i - s) = x_i^{d_i+1} - \sum_{j=0}^{d_i} q_{ij} x_i^j.$$

Orientations: Proof of the theorem (cont.)

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$\forall (x_1, \dots, x_n) \in S_1 \times \dots \times S_n\ \tilde{f_G}(x_1, \dots, x_n) = 0$ and by first Lemma $\tilde{f_G} \equiv 0$.

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But coefficient of $\prod_{i=1}^{n} x_i^{d_i}$ in f_G is nonzero, and it remains the same in $\tilde{f_G}$ due to homogeneity of f_G. We come to a contradiction.

Orientations: Corollaries 1

Corollary

If undirected graph G has orientation D satisfying $EE(D) \neq EO(D)$ in which maximal outdegree is d then G is $(d+1)$-colorable.

(Evident)

Orientations: Corollaries

Definition

Set of vertices $S \subset V$ is called **independent** if vertices in S can be colored in the same color.

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Corollary

If undirected graph G with vertices $V = \{v_1, \ldots, v_n\}$ has orientation D satisfying $EE(D) \neq EO(D)$,
$d_1 \geq \ldots \geq d_n$ is ordered sequence of outdegrees of its vertices then $\forall k : 0 \leq k < n$ G has an independent set of size at least $\left\lceil \frac{n-k}{d_{k+1}+1} \right\rceil$.

(Exercise)

Outline

Probabilistic restatement: introduction

G is a 3-valent biconnected undirected graph with $2n$ vertices, $3n$ edges.

Its undirected line graph F_G has $3n$ vertices (each of degree 4) and $6n$ edges.

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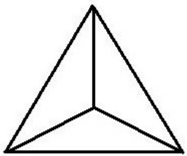

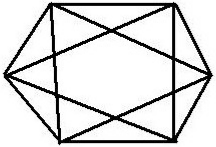

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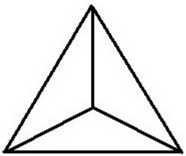

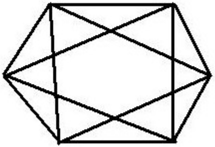

We assign the same probability to each of 2^{6n} orientations that can be attached to F_G.

Probabilistic restatement: introduction

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Event A_G: 2 randomly chosen orientations have the same parity.
Event B_G: 2 randomly chosen orientations are equivalent modulo 3.

Probabilistic restatement: the theorem

Theorem

For any biconnected planar 3-valent graph G having $2n$ vertices

$$P\left(B_G|A_G\right) - P(B_G) = \left(\frac{27}{4096}\right)^n \cdot \frac{\chi_G(4)}{4}.$$

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So 4CC is equivalent to the statement
that there is a positive correlation between A_G and B_G.

Probabilistic restatement: proof of the theorem

Let us examine 2 graph polynomials:
$M' = \prod_{e_i e_j \in L_G} (x_i - x_j)$ (here L_G is a set of edges of F_G),
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Let us count m_0, that is, free term of $\mathcal{R}_3(M'M'')$ in two different ways.

Probabilistic restatement: proof of the theorem

The first way.
$P(A_G) = \frac{1}{2} \Rightarrow P\left(B_G|A_G\right) - P(B_G) = 2P(A_G B_G) - P(B_G)$

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If they are equivalent modulo 3 and have the same parity then they contribute 2^{-12n} into probabilities and 1 into m_0.

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Finally we have $m_0 = 2^{12n}(P\left(B_G | A_G\right) - P(B_G))$.

Probabilistic restatement: proof cont.

The second way deals with **Tait colorings**. We have a map colored in 4 colors (say, $\alpha, \beta, \gamma, \delta$) and construct a coloring of edges by the following rule.

Probabilistic restatement: proof cont.

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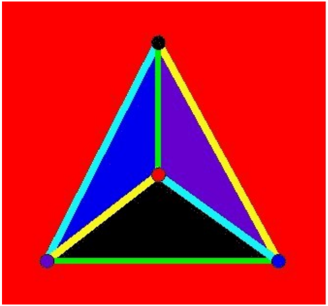
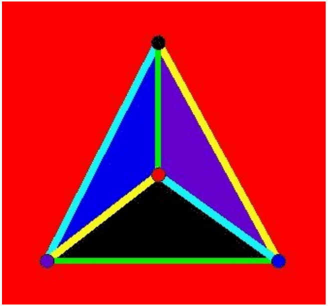
- An edge which separates α from β or γ from δ gets color 1.
- An edge which separates α from γ or β from δ gets color 2.
- An edge which separates α from δ or β from γ gets color 3.

Probabilistic restatement: proof cont.



α	dark blue
β	purple
γ	black
δ	red
1	green
2	yellow
3	sky blue

Probabilistic restatement: proof cont.



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For any vertex three edges incidental to it have 3 different colors.

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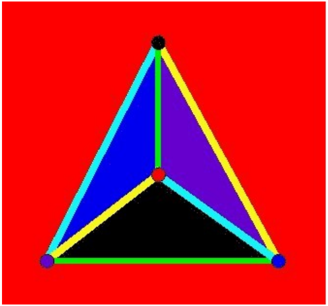

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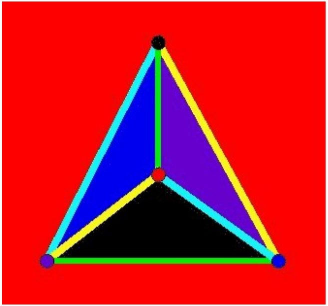

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Thus number of Tait colorings is $\frac{\chi_G(4)}{4}$.

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Applying interpolation formula we see

$$\mathcal{R}_3 M = \sum_{\mu} (\mathcal{R}_3 M)(\omega^{\mu(v_1)}, \ldots, \omega^{\mu(v_{3n})}) P_\mu = \sum_{\mu} M(\omega^{\mu(v_1)}, \ldots, \omega^{\mu(v_{3n})}) P_\mu$$

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$$P_\mu = \prod_{k=1}^{3n} \frac{(x_k - \omega^{\mu(v_k)+1})(x_k - \omega^{\mu(v_k)+2})}{(\omega^{\mu(v_k)} - \omega^{\mu(v_k)+1})(\omega^{\mu(v_k)} - \omega^{\mu(v_k)+2})}$$

Probabilistic restatement: proof cont.

We break $12n$ factors in M into $2n$ groups, each group looking like $(x_i - x_j)(x_j - x_k)(x_i - x_k)(x_i^2 - x_j^2)(x_j^2 - x_k^2)(x_i^2 - x_k^2)$ ($i < j < k$, edges e_i, e_j, e_k are incidental to the same vertex).

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$$2^{12n}(P(B_G|A_G) - P(B_G)) = 3^{3n} \cdot \frac{\chi_G(4)}{4}$$

We have proved the theorem.

Outline

Arithmetical restatement: The Main Theorem Formulation

Theorem

$\exists p, q \in \mathbb{N}$, $4q$ linear functions $A_k(m, c_1, \ldots, c_p), B_k(m, c_1, \ldots, c_p), C_k(m, c_1, \ldots, c_p), D_k(m, c_1, \ldots, c_p)$, $k \in \{1, \ldots, q\}$ such that 4CC is equivalent to the following statement:

$$\forall m, n \; \exists c_1, \ldots, c_p \; E(n, m, c_1, \ldots, c_p) \not\equiv 0 \mod 7,$$

$$E(n, m, c_1, \ldots, c_p) = \begin{pmatrix} A_k(m, c_1, \ldots, c_p) + 7^n B_k(m, c_1, \ldots, c_p) \\ C_k(m, c_1, \ldots, c_p) + 7^n D_k(m, c_1, \ldots, c_p) \end{pmatrix}.$$

Arithmetical restatement: Value of The Main Theorem

Having $E(n, m, c_1, \ldots, c_p)$
we can take $G(m, n)$ whose values are never divisible by 7,
arbitrary integer-valued functions $F(n, m, c_1, \ldots, c_p)$,

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and

$$\sum_{c_1} \ldots \sum_{c_p} E(n, m, c_1, \ldots, c_p) F(n, m, c_1, \ldots, c_p) = G(m, n)$$

implies the 4CC.

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- Then we introduce **internal** and **external** (as a whole – **ranked**) edges ($G = \langle V, E_I, E_X \rangle$). Ends of internal edge should have the same color, ends of external edge should be colored differently. Now the 4CC sounds as follows:

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> If a planar graph with ranked edges can be colored in some number of colors (more precisely in 6 colors – it's always possible) then it can be colored in 4 colors.

Arithmetical restatement: Reformulations

- Then we say: we have a graph with vertices (V) colored somehow and edges (E).
  2 colorings are **equivalent** if they induce the same division of E in 2 groups (internal and external).
  For every coloring we are searching for the equivalent one consisting of 4 colors.

Arithmetical restatement: Reformulations

- The next term to introduce is **spiral graph**.
  It has infinitely many vertices k.
  2 vertices i, j are connected by edge iff $|i - j| = 1$ or $|i - j| = n$.
  We color this construct in colors from $\{0, \ldots, 6\}$ so that finitely many vertices have color greater than 0.

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  4-coloring λ for a given coloring μ is made paying attention to 3 properties:

  1. $\lambda(k) = 0 \iff \mu(k) = 0$
  2. $\lambda(k) = \lambda(k+1) \iff \mu(k) = \mu(k+1)$
  3. $\lambda(k) = \lambda(k+n) \iff \mu(k) = \mu(k+n)$

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- We can represent our colorings as a natural number in **base-7 notation**: $m = \sum_{k=0}^{\infty} \mu(k) 7^k, \ l = \sum_{k=0}^{\infty} \lambda(k) 7^k$.
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  Our requirements to λ imply that:

  1. there are no 7-digits '5' and '6' in l
  2. the k-th digit of l equals 0 $\iff$ the k-th digit of m equals 0
  3. and 2 more (for $(k+1)$ and $(k+n)$)

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  and know that Bool(c_{ij}), $c_{ij} \perp c_{i'j'}$ if $\langle i,j \rangle \neq \langle i',j' \rangle$.
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  1. $7c_{ij} \perp c_{ij'},\ j \neq j'$
  2. $7c_{ij} \perp c_{i'j},\ i \neq i'$
  3. $7^n c_{ij} \perp c_{ij'},\ j \neq j'$
  4. $7^n c_{ij} \perp c_{i'j},\ i \neq i'$

Arithmetical restatement: Reformulations

Theorem

(E. E. Kummer)
A prime number p comes into the factorization of the binomial coefficient

$$\binom{a+b}{a}$$

with the exponent equal to the number of carries performed while computing sum $a+b$ in base-p positional notation.

(Exercise)

Arithmetical restatement: Reformulations

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All 7-digits of a are less or equal to 3 iff

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The third corollary:

$$\mathrm{Bool}(a)\&\mathrm{Bool}(b) \Rightarrow$$

$$\left[a \perp b \iff \binom{2(a+b)}{a+b}\binom{4(a+b)}{2(a+b)} \not\equiv 0 \mod 7\right].$$

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The last corollary:

$$\mathrm{Bool}(a)\&\mathrm{Bool}(b) \Rightarrow \left[a \perp b \iff \begin{pmatrix} 4(a+b) \\ 2(a+b) \end{pmatrix} \not\equiv 0 \mod 7 \right].$$

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The last reformulation:
$\forall n, m \in \mathbb{Z}_+ \; \exists c_{ij} \in \mathbb{Z}_+, \; i \in \{1, \ldots, 6\}, j \in \{1, \ldots, 4\}$:
none of 986 binomial coefficients is divisible by 7:

$$\binom{2c_{ij}}{c_{ij}}, \binom{4c_{ij}}{2c_{ij}}, \binom{4(c_{i'j'} + c_{i''j''})}{2(c_{i'j'} + c_{i''j''})}, \langle i', j' \rangle \neq \langle i'', j'' \rangle,$$

$$\binom{4(7c_{ij} + c_{ij'})}{2(7c_{ij} + c_{ij'})}, j \neq j', \binom{4(7c_{ij} + c_{i'j})}{2(7c_{ij} + c_{i'j})}, i \neq i',$$

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The last reformulation cont.:
$\forall n, m \in \mathbb{Z}_+ \ \exists c_{ij} \in \mathbb{Z}_+, \ i \in \{1, \ldots, 6\}, j \in \{1, \ldots, 4\}$:
none of 986 binomial coefficients is divisible by 7:

$$\binom{4(7^n c_{ij} + c_{ij'})}{2(7^n c_{ij} + c_{ij'})}, j \neq j', \binom{4(7^n c_{ij} + c_{i'j})}{2(7^n c_{ij} + c_{i'j})}, i \neq i',$$

$$\binom{m}{C}, \binom{C}{m},$$

$$\text{where } C = \sum_{i=1}^{6} \left(\sum_{j=1}^{4} c_{ij} \right).$$

Outline

The Tutte polynomial: definitions

Let $G = (V, E)$ be a (multi)graph (may have loops and multiple edges).
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1. **cut** : $G - e$, where $e \in E$ (delete the edge e)
2. **fuse** : $G \setminus e$, where $e \in E$ (delete the edge e and join vertices incident to e)

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For any spanning subgraph F we write $k\langle F\rangle, r\langle F\rangle, n\langle F\rangle$. Then

$$S(G; x, y) = \sum_{F \subset E(G)} x^{r\langle E\rangle - r\langle F\rangle} y^{n\langle F\rangle} = \sum_{F \subset E(G)} x^{k\langle F\rangle - k\langle E\rangle} y^{n\langle F\rangle}$$

is called **rank-generating polynomial**.

The Tutte polynomial: theorem 1

Theorem

$$S(G;x,y) = \begin{cases} (x+1)S(G-e;x,y), & \text{e is a bridge,} \\ (y+1)S(G-e;x,y) & \text{e is a loop,} \\ S(G-e;x,y) + S(G\setminus e;x,y), & \text{otherwise.} \end{cases}$$

Furthermore, $S(E_n;x,y) = 1$ *for an empty graph* E_n *with* n *vertices.*

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Furthermore, $S(E_n;x,y) = 1$ for an empty graph E_n with n vertices.

This can be easily proved if we form two groups of F's (subsets of $E(G)$): those which include e (the edge to be eliminated) and those which do not — and investigate simple properties of rank and nullity.
(Exercise)

The Tutte polynomial: main definition

The Tutte polynomial is defined as follows:

$$T_G(x,y) = S(G; x-1, y-1) = \sum_{F \subset E(G)} (x-1)^{r\langle E\rangle - r\langle F\rangle}(y-1)^{n\langle F\rangle}$$

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Of course the appropriate statement holds:

$$T_G(x,y) = \begin{cases} xT_{G-e}(x,y), & \text{e is a bridge,} \\ yT_{G-e}(x,y) & \text{e is a loop,} \\ T_{G-e}(x,y) + T_{G\setminus e}(x,y), & \text{otherwise.} \end{cases}$$

The Tutte polynomial: theorem 2

Theorem

There is a unique map U from the set of multigraphs to the ring of polynomials over $\mathbb{Z}$ of variables $x, y, \alpha, \sigma, \tau$ such that $U(E_n) = U(E_n; x, y, \alpha, \sigma, \tau) = \alpha^n$ and

$$U(G) = \begin{cases} xU_{G-e}(x,y), & \text{e is a bridge,} \\ yU_{G-e}(x,y) & \text{e is a loop,} \\ \sigma U_{G-e}(x,y) + \tau U_{G\setminus e}(x,y), & \text{otherwise.} \end{cases}$$

Furthermore,

$$U(G) = \alpha^{k(G)}\sigma^{n(G)}\tau^{r(G)}T_G(\alpha x/\tau, y/\sigma).$$

The Tutte polynomial: theorem 2, proof sketch

$U(G)$ is a polynomial of σ and τ because
$deg_x T_G(x,y) = r(G),\ deg_y T_G(x,y) = n(G)$.

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The uniqueness of U is implied by constructive definition.

It can be checked simply that $U(E_n) = \alpha^n$ and recurrent equalities hold for $U(G) = \alpha^{k(G)}\sigma^{n(G)}\tau^{r(G)}T_G(\alpha x/\tau, y/\sigma)$.

The Tutte polynomial: corollary

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If $p_G(x)$ is the number of proper colorings of vertices of graph G in x colors then $p_G(x)$ is called the **chromatic function** of G.

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Corollary

$$p_G(x) = (-1)^{r(G)} x^{k(G)} T_G(1 - x, 0)$$

So the chromatic function is actually the **chromatic polynomial**.

The Tutte polynomial: corollary, proof sketch

$$p_{E_n}(x) = x^n$$

and $\forall e \in E(G)$

$$p_G(x) = \begin{cases} \frac{x-1}{x} p_{G-e}(x), & \text{e is a bridge,} \\ 0 & \text{e is a loop,} \\ p_{G-e}(x) - p_{G\setminus e}(x), & \text{otherwise.} \end{cases}$$

$$\Rightarrow p_G(x) = U(G; \frac{x-1}{x}, 0, x, 1, -1) = x^{k(G)}(-1)^{r(G)} T_G(1-x, 0)$$

List of exercises

1. Colorings of 3-valent graph,

$$M(G)(\nu(e_1), \ldots, \nu(e_{3n})) = \pm 3^{3n},$$

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2. Let $P(x_1, \ldots, x_n)$ be polynomial in n variables over $\mathbb{Z}$, for $1 \leq i \leq n$ the degree of P in x_i doesn't exceed d_i,
   $S_i \subset \mathbb{Z} : \ |S_i| = d_i + 1.$
   If $\forall (x_1, \ldots, x_n) \in S_1 \times \ldots \times S_n \ P(x_1, \ldots, x_n) = 0$ then $P \equiv 0.$

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3. If undirected graph G with vertices $V = \{v_1, \ldots, v_n\}$ has orientation D satisfying $EE(D) \neq EO(D)$,
   $d_1 \geq \ldots \geq d_n$ is ordered sequence of outdegrees of its vertices then $\forall k : 0 \leq k < n \ \ G$ has an independent set of size at least $\left\lceil \frac{n-k}{d_{k+1}+1} \right\rceil$.

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4. (Kummer theorem)
   A prime number p comes into the factorization of the binomial coefficient $\binom{a+b}{a}$ with the exponent equal to the number of carries performed while computing sum $a+b$ in base-p positional notation.

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5. $$S(G;x,y) = \begin{cases} (x+1)S(G-e;x,y), & \text{e is a bridge,} \\ (y+1)S(G-e;x,y) & \text{e is a loop,} \\ S(G-e;x,y) + S(G \setminus e;x,y), & \text{otherwise.} \end{cases}$$

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6. $$p_G(x) = \begin{cases} \frac{x-1}{x} p_{G-e}(x), & \text{e is a bridge,} \\ 0 & \text{e is a loop,} \\ p_{G-e}(x) - p_{G\setminus e}(x), & \text{otherwise.} \end{cases}$$

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Thank you for your attention!
Danke für Ihre Aufmerksamkeit!

Any questions?