Sensor Fusion

The Kalman Filter and its Extensions

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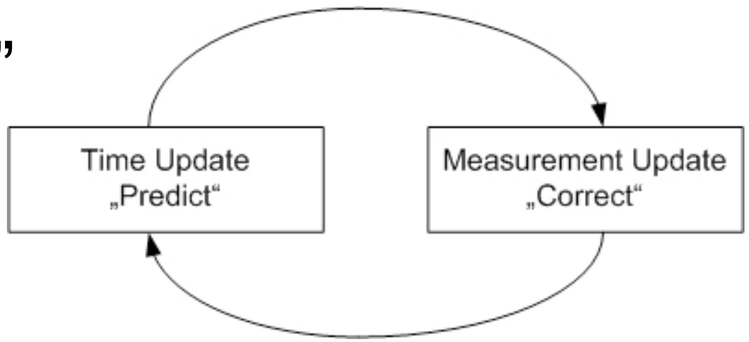
Introduction

- Task
  - Fusion of different data sets
- Approaches
  - Simple
  - Stochastic approach
    - No perfect model
    - Disturbances
    - Imperfect or incomplete data

Introduction

- Kalman Filter
  - Optimal linear recursive estimator
  - Incorporates all available information
    - Knowledge about system and measurement device
    - Statistical description of noise and error
    - Initial conditions
  - “prediction-correction-cycle”
    - Time update
    - Measurement update
  - Simple, robust and popular



Overview

- Stochastic Basics
- Discrete Kalman Filter
- Extended Kalman Filter
- Sensor Fusion
  - DKF and EKF
  - SCAAT
  - FKF
- Conclusion

Stochastic Basics

- Probability and Random Variables
  - Probability $p(A) = \frac{\text{possible outcomes favoring } A}{\text{total number of possible outcomes}}$
  - Random Variable X: Sample Space → Numbers
  - Cumm. distribution function $F_X(x) = p(-\infty, x]$
  - Probability density function $f_X(x) = \frac{\partial}{\partial x} F_X(x)$

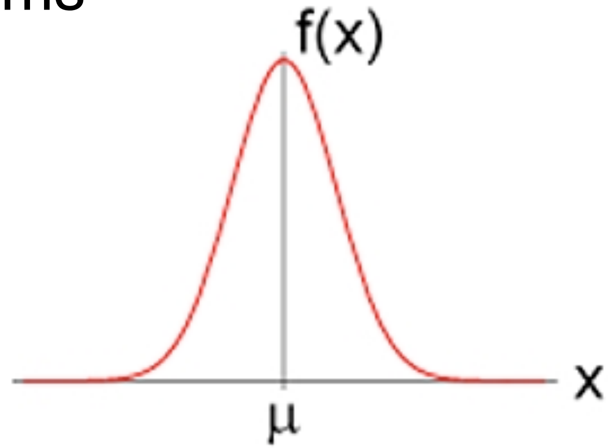
    $$p_X[a,b] = \int_a^b f_X(x)dx$$

- Mean and Covariance
  - Mean (discrete, continuous) $E[X] = \sum_{i=1}^{n} p_i x_i \qquad E[X] = \int_{-\infty}^{\infty} x f_X(x)dx$
  - Variance $Var(X) = E[(X - E[X])^2] = E[X^2] - E[X]^2$
  - Std. deviation $\sigma_X = \sqrt{Var(X)}$

Stochastic Basics

- Gaussian distribution
  - Popular for modelling random systems
  - Normally distributed $X \sim N(\mu, \sigma)$
  - Probability density function

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$



- White noise
  - Autocorrelation $R_X(\tau) = E[X(t)X(t+\tau)]$
  - White noise is uncorrelated, independent

$$R_X(\tau) = \begin{cases} a\ , \ if\ \hat{o} = 0 \\ 0\ , \ otherwise \end{cases}$$

The Discrete Kalman Filter

- Process and Measurement Models
  - Models $x_k = Ax_{k-1} + Bu_k + w_{k-1}$
    $z_k = Hx_k + v_k$
  - Noise $p(w) \sim N(0,Q)$
    $p(v) \sim N(0,R)$
- Origins of the Filter
  - state estimates, errors and covariances
    $\hat{x}_k^-, \hat{x}_k$ $\quad e_k^- = x_k - \hat{x}_k^-, e_k = x_k - \hat{x}_k$ $\quad P_k^- = E[e_k^- e_k^{-T}], P_k = E[e_k e_k^T]$
  - Computational origin $\hat{x}_k = \hat{x}_k^- + K(z_k - H\hat{x}_k^-)$
  - Probabilistic origin $p(x_k | z_k) \sim N(\hat{x}_k, P_k)$

The Discrete Kalman Filter

- Discrete Kalman Filter Cycle
  - Time update $\hat{x}_k^- = A\hat{x}_{k-1} + Bu_k$

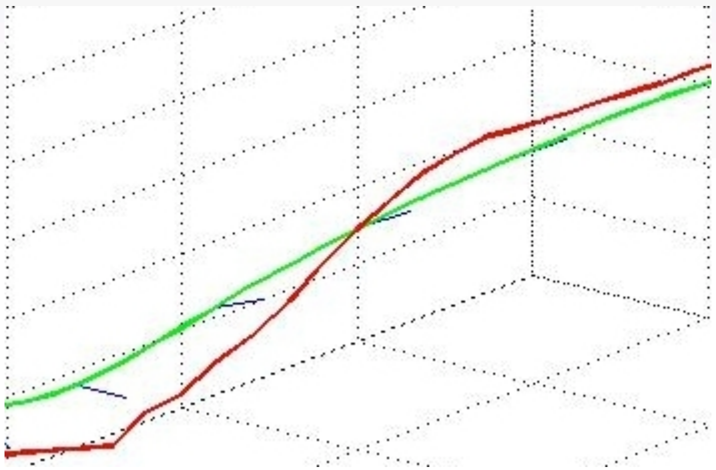
    $$P_k^- = AP_{k-1}A^T + Q$$
  - Measurement update $K_k = P_k^- H^T (HP_k^- H^T + R)^{-1}$

    $$\hat{x}_k = \hat{x}_k^- + K_k(z_k - H\hat{x}_k^-)$$

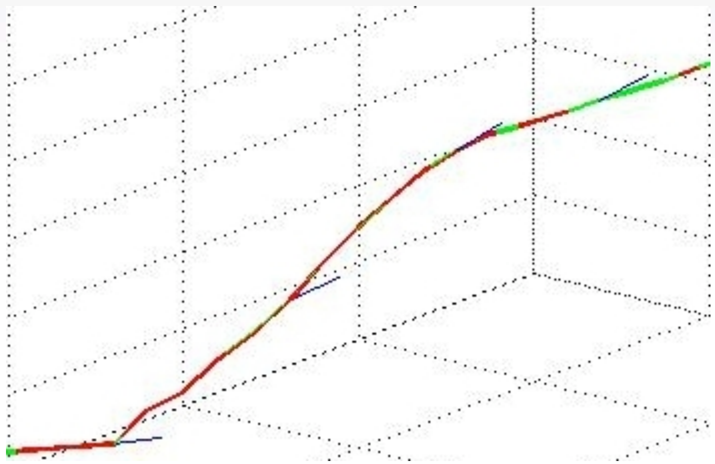
    $$P_k = (I - K_k H)P_k^-$$
  - Influence of Q and R
    - Process noise covariance Q:
      large → close track of changes in data
    - Measurement noise covariance R:
      large → measurements are considered not very accurate

The Discrete Kalman Filter

- Influence of Q and R



Q small, R large



Q large, R small

The Discrete Kalman Filter

- Assumptions
  - All underlying models are linear
    - Often adequate
    - More complete theory
  - Gaussian probability distribution
    - “natural”
    - Completely determined by μ and σ
  - White (independent) noise
    - Identical to wideband noise in bandpass
    - Mathematics are vastly simplified

The Discrete Kalman Filter

- Optimality
  - Filter minimizes the estimated error covariance

$$P_k = E[e_k e_k^T] = E[(x_k - \hat{x}_k)(x_k - \hat{x}_k)^T]$$

  - Based on computation of Kalman gain K

$$\hat{x}_k = \hat{x}_k^- + K_k(z_k - H\hat{x}_k^-) = \hat{x}_k^- + K_k((Hx_k + v_k) - H\hat{x}_k^-)$$

$$\Rightarrow P_k = E[((I - K_k H)(x_k - \hat{x}_k^-) - K_k v_k)((I - K_k H)(x_k - \hat{x}_k^-) - K_k v_k)^T]$$

$$= (I - K_k H)E[e_k^- e_k^{-T}](I - K_k H)^T + K_k E[v_k v_k^T]K_k^T$$

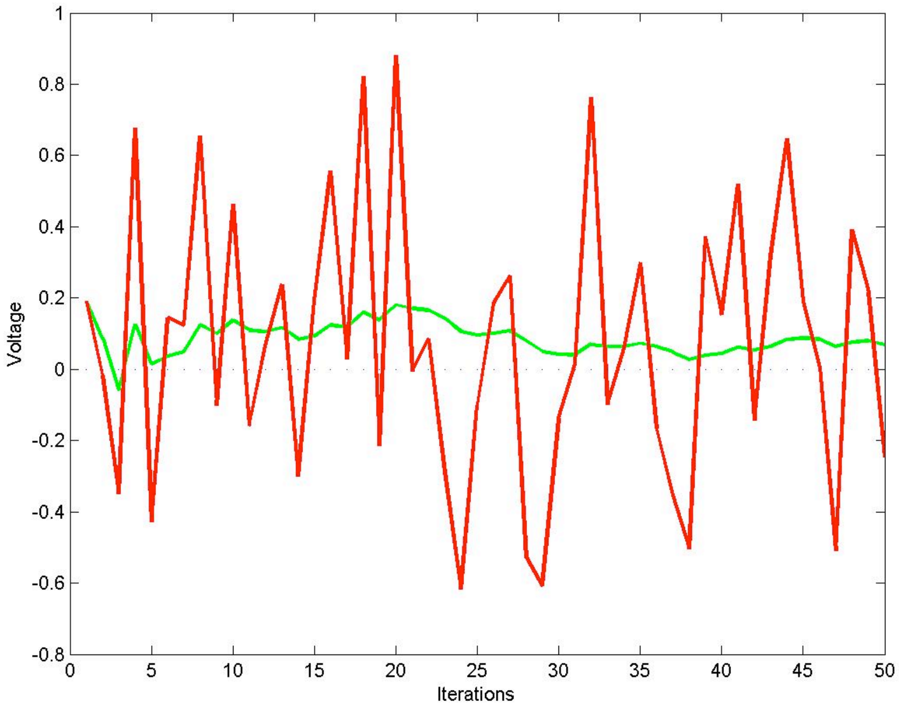
$$= (I - K_k H)P_k^-(I - K_k H)^T + K_k R_k K_k^T$$

diagonal of P contains mean squared errors → minimize trace

$$\frac{\partial T[P_k]}{\partial K_k} = -2(HP_k^-)^T + 2K_k(HP_k^- H^T + R) = 0 \quad \Rightarrow K = \frac{P_k^- H^T}{HP_k^- H^T + R}$$

The Discrete Kalman Filter

- Examples
  - 1D voltage measurement
    - Models
      $x_k = x_{k-1} \qquad z_k = x_k$
    - Noise covariances
      $Q = 10^{-5} \qquad R = 10^{-2}$
    - Measurements
      mean m → $z \sim N(m, 0.1)$
    - Results
      red=measurements
      green=predicted states



The Discrete Kalman Filter

- Examples
  - 3D position measurement
    - State vector $x = (x, y, z, \dot{x}, \dot{y}, \dot{z}, \ddot{x}, \ddot{y}, \ddot{z})^T$
    - Models

$$x(t + \Delta t) = A(\Delta t)x(t) + w$$

$$\begin{pmatrix} z_x \\ z_y \\ z_z \end{pmatrix} = z(t + \Delta t) = Hx(t) + v$$

$$A = \begin{bmatrix} I & \Delta t \cdot I & \frac{\Delta t^2}{2} \cdot I \\ 0 & I & \Delta t \cdot I \\ 0 & 0 & I \end{bmatrix}$$

$$H = \begin{bmatrix} 100000000 \\ 010000000 \\ 001000000 \end{bmatrix}$$

    - Filter cycle
      - Compute Δt since previous estimate
      - Compute state transition matrix A(Δt)
      - Do the prediction and correction steps
    - Determination of Q and R

The Extended Kalman Filter

- Non-Linearity
  - Assumptions of the DKF do not always hold
  - EKF linearizes about the current mean and covariance
- EKF Models
  - Non-linear equations $x_k = f(x_{k-1}, u_{k-1}, w_{k-1}) \quad z_k = h(x_k, v_k)$
  - Noise values unknown $\tilde{x}_k = f(\hat{x}_{k-1}, u_{k-1}, 0) \quad \tilde{z}_k = h(\tilde{x}_k, 0)$
  - Linearization

$$x_k \approx \tilde{x}_k + A(x_k - \hat{x}_{k-1}) + W w_{k-1} \qquad z_k \approx \tilde{z}_k + H(x_k - \tilde{x}_k) + V v_k$$

Jacobians

$$A_k = \frac{\partial f}{\partial x}(\hat{x}_k, u_k, 0) \qquad W_k = \frac{\partial f}{\partial w}(\hat{x}_k, u_k, 0)$$

$$H_k = \frac{\partial h}{\partial x}(\tilde{x}_k, 0) \qquad V_k = \frac{\partial h}{\partial v}(\tilde{x}_k, 0)$$

The Extended Kalman Filter

- Extended Kalman Filter Cycle
  - Time update

$$\hat{x}_k^- = f(\hat{x}_{k-1}, u_k, 0)$$

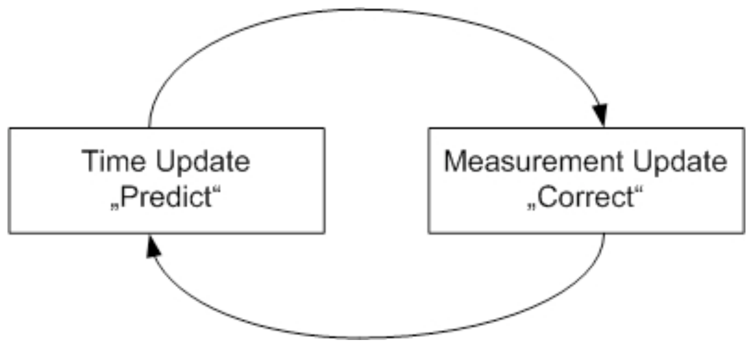
$$P_k^- = A_k P_{k-1} A_k^T + W_k Q_{k-1} W_k^T$$

  - Measurement update

$$K_k = P_k^- H_k^T (H_k P_k^- H_k^T + V_k R_k V_k^T)^{-1}$$

$$\hat{x}_k = \hat{x}_k^- + K_k (z_k - h(\hat{x}_k^-, 0))$$

$$P_k = (I - K_k H) P_k^-$$



The Extended Kalman Filter

- Example
  - 3D position and orientation tracking with quaternions
    - State vector $x = (x, y, z, \dot{x}, \dot{y}, \dot{z}, \ddot{x}, \ddot{y}, \ddot{z}, r_x, r_y, r_z, r_w, \omega_1, \omega_2, \omega_3, \dot{\omega}_1, \dot{\omega}_2, \dot{\omega}_3)^T$

      $x = (p^T, \dot{p}^T, \ddot{p}^T, r^T, \omega^T, \dot{\omega}^T)^T$
    - Models

$$\begin{pmatrix} p_k \\ \dot{p}_k \\ \ddot{p}_k \end{pmatrix} = \begin{bmatrix} I & \Delta t \cdot I & \frac{\Delta t^2}{2} \cdot I \\ 0 & I & \Delta t \cdot I \\ 0 & 0 & I \end{bmatrix} \begin{pmatrix} p_{k-1} \\ \dot{p}_{k-1} \\ \ddot{p}_{k-1} \end{pmatrix} \qquad r_k = r_{k-1} \otimes d_{k-1} = r_{k-1} \otimes e^{\Delta t \cdot \omega_{k-1} + \frac{1}{2} \Delta t^2 \cdot \dot{\omega}_{k-1}}$$

$$z_k = h(x) = \begin{pmatrix} p \\ normalize(r) \end{pmatrix}$$

    - Filter cycle: equations as presented
      Jacobians need to be computed

Kalman Filter Discussion

- Kalman Filter

  stable, robust and popular optimal estimator

- DKF

  (+) optimal linear estimator

  applicable to many system processes

  (-) three assumptions

- EKF

  (+) faces non-linearity problem

  (-) unreliable for non Gaussian distributions

Sensor Fusion with KFs

- Discrete and Extended Kalman Filter
  - One Filter
    - Multiple sensors summed up in a single filter
    - Updates when enough information is gathered
    - → UNC hybrid landmark-magnetic tracker
  - Separate Filters
    - Separate filters for each sensor
    - Optimal adjusting
    - → Azuma: head location prediction

Sensor Fusion with KFs

- Single Constraint at a Time (SCAAT)
  - Introduction
    - Multiple seq. measurements for a single update
    - Problems
      - Simultaneity assumption
      - System depends on sufficient data sets
  - SCAAT idea
    - Single-constraint-at-a-time
      - Each measurement provides some information about the current state
      - Incremental improvement of previous estimates

Sensor Fusion with KFs

- Single Constraint at a Time (SCAAT)
  - State vector and models
    - State vector $x = (x, y, z, \dot{x}, \dot{y}, \dot{z}, \psi, \theta, \varphi, \dot{\psi}, \dot{\theta}, \dot{\varphi})^T$
    - Process model $A(\Delta t): \quad x(t + \Delta t) = x(t) + \dot{x}(t) \cdot \Delta t$

      $$\dot{x}(t + \Delta t) = \dot{x}(t)$$

    - Measurement model $z_\sigma(t) = h_\sigma(x(t), b_t, c_t) + v_\sigma(t)$

      $$H_\sigma(x(t), b_t, c_t)[i, j] = \frac{\partial}{\partial x(j)} h_\sigma(x(t), b_t, c_t)[i]$$

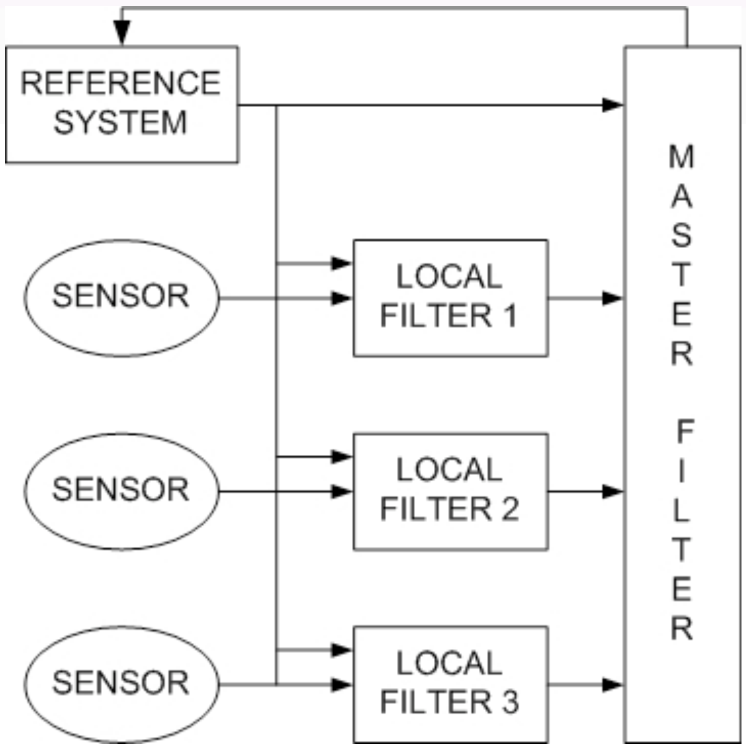
      for each sensors σ a corresponding measurement vector
      b and c are tracker source and sensor parameters
    - Ideal SCAAT application
      only a single source and sensor pair for each update $|z_\sigma| = 1$

Sensor Fusion with KFs

- Single Constraint at a Time (SCAAT)
  - Algorithm
    - Compute Δt since previous estimate
    - Predict state and error covariance
    - Predict measurement and compute Jacobian
      $\hat{z} = h_\sigma(\hat{x}^-, b_t, c_t) \quad H = H_\sigma(\hat{x}^-, b_t, c_t)$
    - Compute Kalman gain $K = P^- H^T (HP^- H^T + R_\sigma(t))^{-1}$
    - Correct state estimate and error covariance
      $\hat{x}(t) = \hat{x}^- + K(z_\sigma(t) - \hat{z}) \quad P(t) = (I - KH)P^-$
  - Discussion
    - SCAAT integrates individual (incomplete) measurements
    - Faster, more accurate, no simultaneity assumption

Sensor Fusion with KFs

- The Federated Kalman Filter
  - Introduction
    - Computational load problems in multisensor systems
    - Decentralization and reduced rate at master filter
  - FKF idea
    - Decentr. approach with local filter and a master filter
    - Local data compression through pre-filtering
    - Optimal or suboptimal accuracy via selectable master filter rate



Sensor Fusion with KFs

- The Federated Kalman Filter
  - Filter Structure
    - Models $x_k = Ax_{k-1} + Gw \qquad \hat{z}_i = H_i x + v_i$
    - Composite global filter

$$x = [x_1 \ldots x_N]^T \qquad P = \begin{bmatrix} P_{11} \ldots P_{1N} \\ \ldots \\ P_{N1} \ldots P_{NN} \end{bmatrix}$$

    - Global cost index $\psi = \sum_{i=1}^{N} \| (\hat{x}_i - x_i) \|^2_{P_{ii}^{-1}}$
    - Globally optimal solution if local estimates are uncorrelated

$$\hat{x}_m = P_{mm}[P_{11}^{-1}\hat{x}_1 + \ldots + P_{NN}^{-1}\hat{x}_N]$$

$$P_{mm} = [P_{11}^{-1} + \ldots + P_{NN}^{-1}]^{-1}$$

    - Elimination of cross-correlations through upper bounds for covariances Q and P: γ_i as bounding variable

Sensor Fusion with KFs

- The Federated Kalman Filter
  - Algorithm
    - Set initial local covariances to γ_i x common system value
    - Local filters process own measurements via locally optimal KF
    - Master filter combines local filter solutions after each cycle update via the equations

      $$\hat{x}_m = P_{mm}[P_{11}^{-1}\hat{x}_1 + \ldots + P_{NN}^{-1}\hat{x}_N]$$

      $$P_{mm} = [P_{11}^{-1} + \ldots + P_{NN}^{-1}]^{-1}$$

    - Master filter resets local filter states to master value and local covariances to γ_i x master value
  - Discussion
    - Highly fault tolerant, rate-reduced, decentralized filtering approach

Conclusion

- Kalman Filter
  - DKF: optimal linear estimator with three assumptions
  - EKF: faces non-linear models, linearizes about μ and σ
- Sensor Fusion
  - KF - Direct fusion: easy and common
  - KF - Separate filters: faces complexity, ignores possible correlations
  - SCAAT: integrates single measurements, more accurate and faster
  - FKF: decentralized system with pre-filtering, high fault tolerance and globally optimal/suboptimal estimation

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